Measure Theory
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Lecture - 35
Completeness of L^p

So, in the last lecture we saw the definition of L p spaces. So, these were functions which were whose p th power was integrable and we also saw some inequalities Holder's inequality, and the Minkowski's inequality. I will briefly recall them before we go ahead, our aim today would be to prove that the L p spaces as a matrix space. So, remember the L p norm gives you a matrix space which makes it into a complete matrix space that is what we want to do now.

So, recall that the space of functions which we consider are equivalence classes in fact, functions which are equal almost everywhere or thought of as same functions in the space L p for most of the computation this is does not matter but this is a point one should remember all the time. So, let us start.

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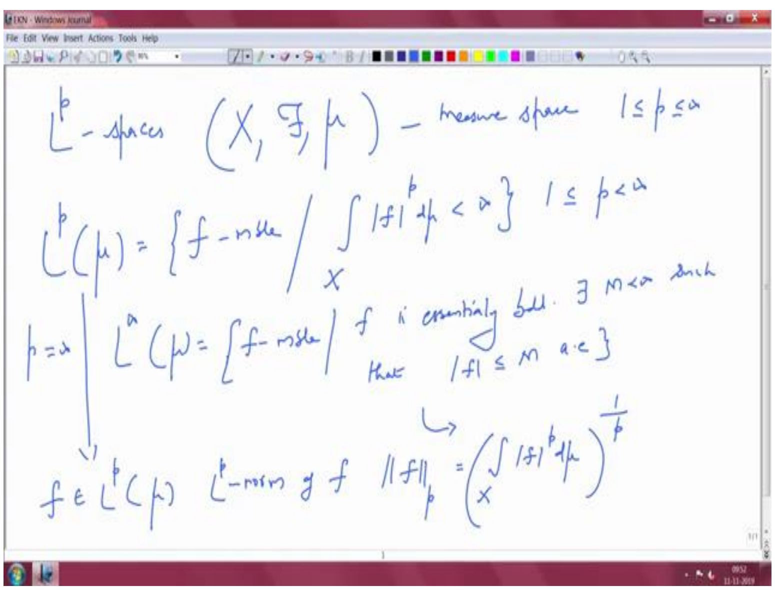

So, recall that we have L p spaces. So, as usual we have a space x, a sigma algebra f, and the measure mu. So, this is our measure space and on this we defined L p spaces. So, we generally denote it by L p of mu which is all measurable complex valued functions such that, the integral of mod f to the p d mu is finite.

So, we looked at only p between 1 and infinity and when p equal to infinity. So, this is, this definition for p strictly less than infinity when p is equal to infinity we had L infinity of mu. So, this was all those functions which are essentially bounded. So, that is about which I did not use last time.

So, let me define this f is essentially bounded, essentially bounded which means that, that is there exists sum m strictly less than infinity such that, such that mod f is less than or equal to m almost everywhere. So, there may be a set which has measure 0 where this inequality is not true but, that we can (())(3:01) which, so, this also gives me the norm.

So, let us start with here L p norm of the function f in L p if a take the L p norm, L p norm of f is defined to be. So, we use this simple which is integral over x mod f to the p d mu to the 1 by p. So, remember when we talk about the norm there is always a 1 by p at the top which is what makes it into a norm.

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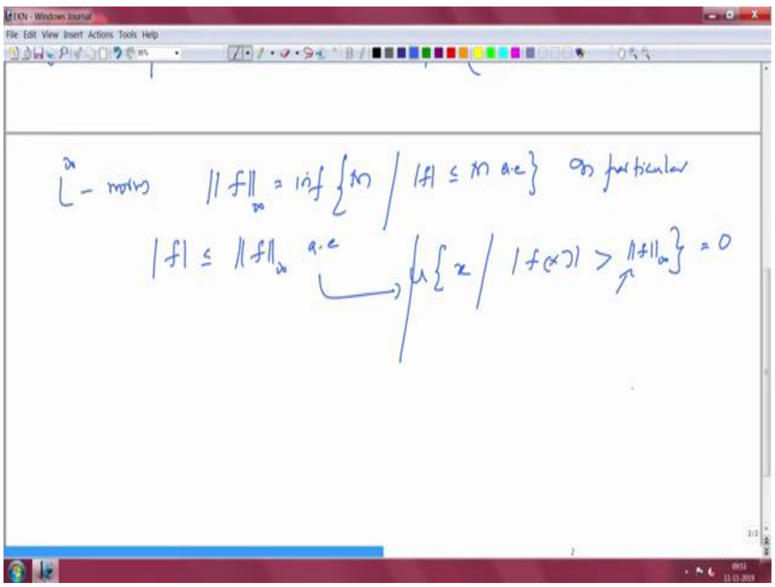

So, when p is infinity, L infinity norm well how was this then? So, you look at, you look at all those m. So, you look at the infimum of things which are bigger than m. So, infimum of all those m such that, mod f is less than or equal to m almost everywhere. So, this would be the smallest number which satisfy. So, in particular, in particular which will be, this will be used every now and then, mod f is less than or equal to L infinity norm of f almost everywhere.

This is the constant the L infinity norm and as a function f will be bounded by L infinity norm of f almost everywhere. So, this also tells me that if I look at the set x such that, mod f of x is strictly greater than L infinity norm of f, this will have measure 0. So, mu of this will be 0 because, you cannot have a positive, set of positive measure where, mod f is greater than this quantity. So, this is the space we are looking at.

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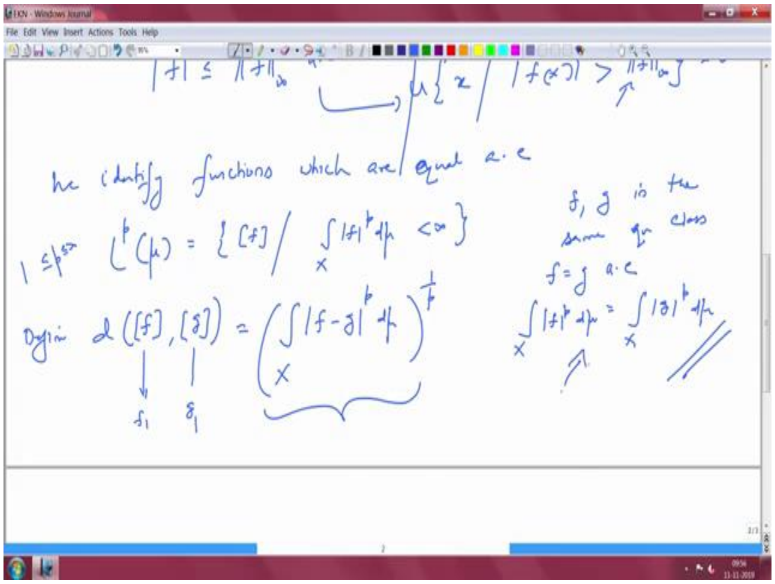

Also recall that we identify, we identify functions which are equal almost everywhere, functions which are equal almost everywhere. In another words, these spaces, spaces are collection of L p mu for 1 less than or equal to p, less than or equal to infinity is the collection of equivalence classes of f, d mu is finite. The norm does not change if I take f and g in the same equivalence class, in the same equivalence class then f equal to g almost everywhere and so, the norm does not change. So, integral over mod x mod f to the p d mu is same as the integral over x mod g to the p d mu.

So, the norm will not change whatever representative of f we take from the equivalence class. So, in this space we define. So, define the distance function d. So, I take two points, well strictly speaking I should take equivalence classes. So, let me do that, I will drop the equivalence class after some time when once you are used to it, it does not really make any serious difference. So, if I take two points in the space which two equivalence classes their difference is defined, the distance is defined to be integral of mod f minus g to the p d mu to the 1 by p.

So, again it is easy to see that this is well defined because, of this whatever instead of f and g if, I take f1 from this class and g1 from this class I can use f1 and the g1 in the definition of the distance and I will still get the same quantity because of this. So, that defines a distance function and d is a matrix.

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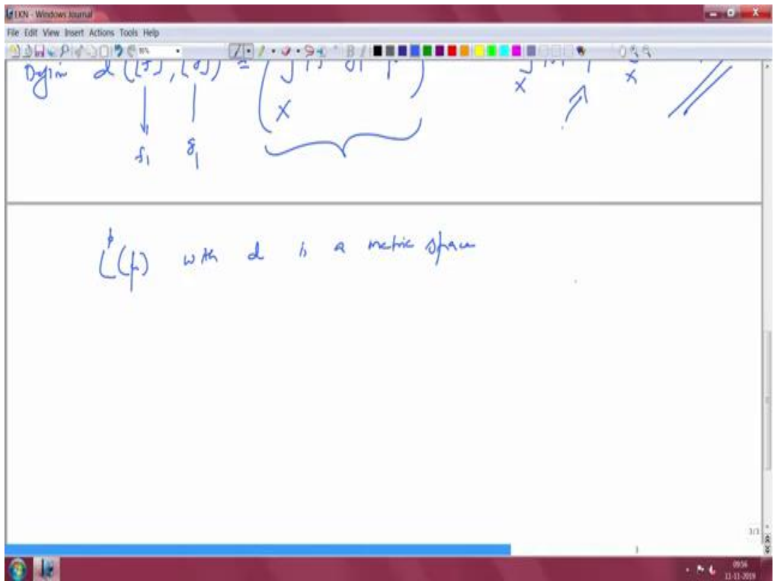

So, L p mu with d is a matrix space. So, this we had checked last time. So, that is where the equivalence classes come in, d f g equal to 0 implies f equal to g almost everywhere and that is what forced as the equivalence class property. So, now we want to prove that. So, this is the main theorem.

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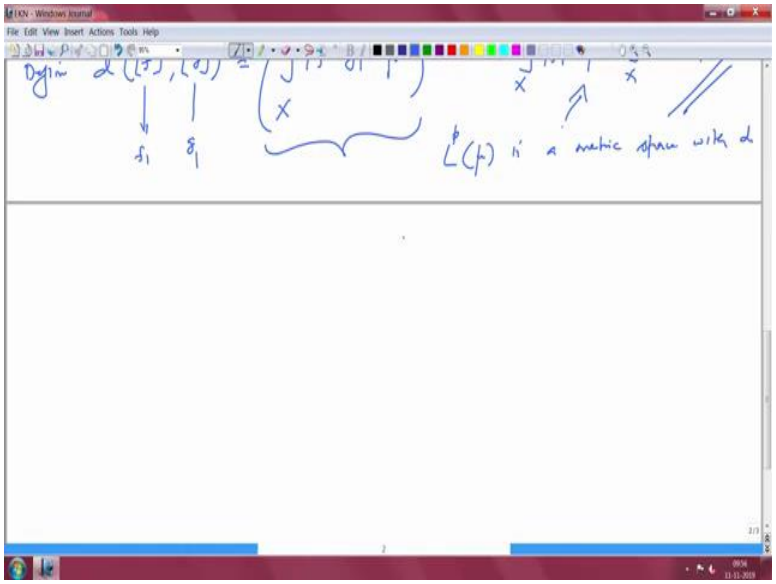

So, let me. So, L p is a matrix space, is a matrix space with the distance function d.

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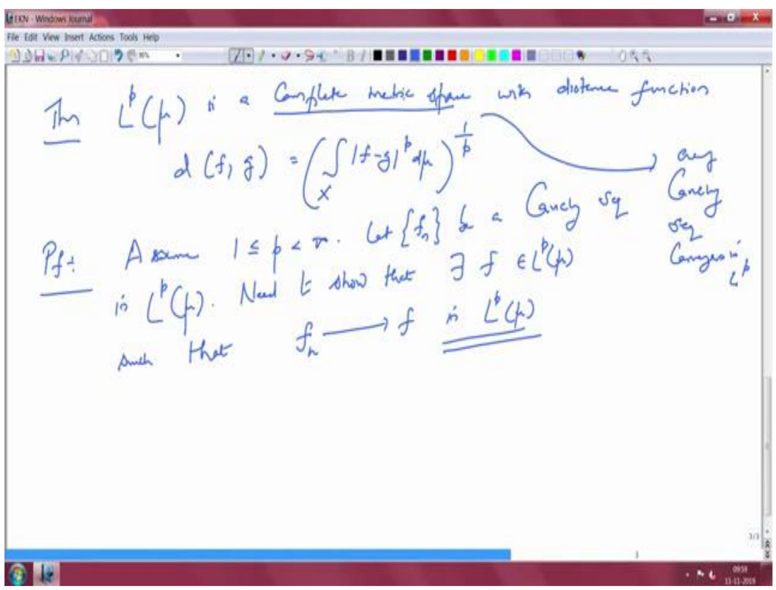

So, let me write the main theorem, one of the most important properties of L p spaces is that L p mu is a complete matrix space, complete matrix space with of course the distance function, with the distance function d f g. So, I am dropping the equivalence class now d f g to be integral over x mod f minus g to the p d mu to the 1 by p with respect to this matrix L p mu is a complete matrix space.

So, let us do the prove and then we will look at more properties of L p and some examples as well, proof. So, first assume. So, in many of this you will see that L infinity case is easier to do. So, assume that 1 less than or equal to p strictly less than infinity, well what do you mean by a complete matrix space? I need to show that if I take every Cauchy sequence converge, every Cauchy sequence converges, converge in L p off course it converge to something in L p. So, that is what we want to prove.

So, let us take a Cauchy sequence. So, let f n be a Cauchy sequence. So, strictly speaking one should take the equivalence class but as I said it does not really matter let fn be a Cauchy sequence, Cauchy sequence in L p of mu what do I want to show? So, I need to show, need to show that there exists f in L p such that, the sequence f n which is the sequence in L p converges to f in L p. So, this is important it has to converge in the given matrix space, not in some other space.

So, I need to find out the limit of fn which is the limit of fn in that matrix space. So, f n is Cauchy, what does that mean?

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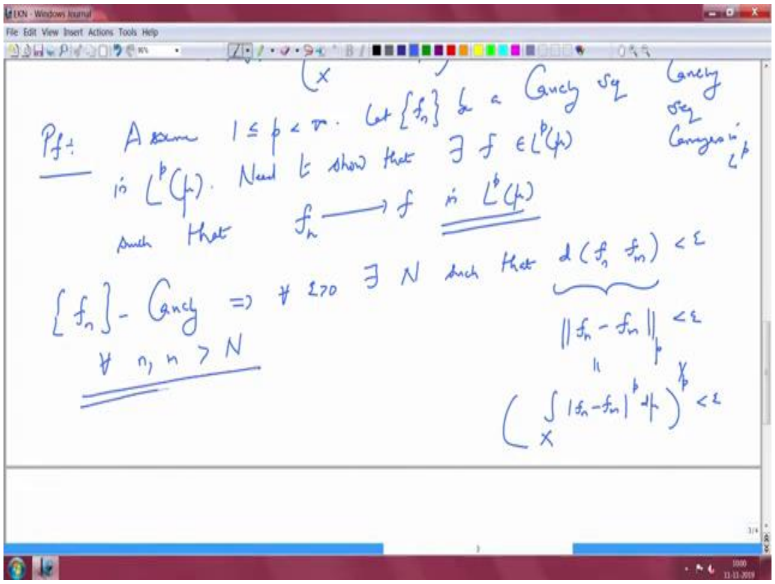

So, f n Cauchy. So, let us write down the definition of that f n Cauchy implies for every epsilon positive there exists sum capital N such that, such that the distance between f n, small f n, f small n and f small m will be less than epsilon for every n and m strictly greater than capital N after some stage you should be able to reduce the distance but remember the distance is define in terms of the L p norm, what does this mean? This means the L p norm of f n minus f m this is less than epsilon.

So, remember this is also given by the integral of mod f n minus f m to the p d mu to the 1 by p this is what is less than not epsilon is for every n and m greater than capital N. So, what we do is we choose epsilon to be 1 by 2 to the i.

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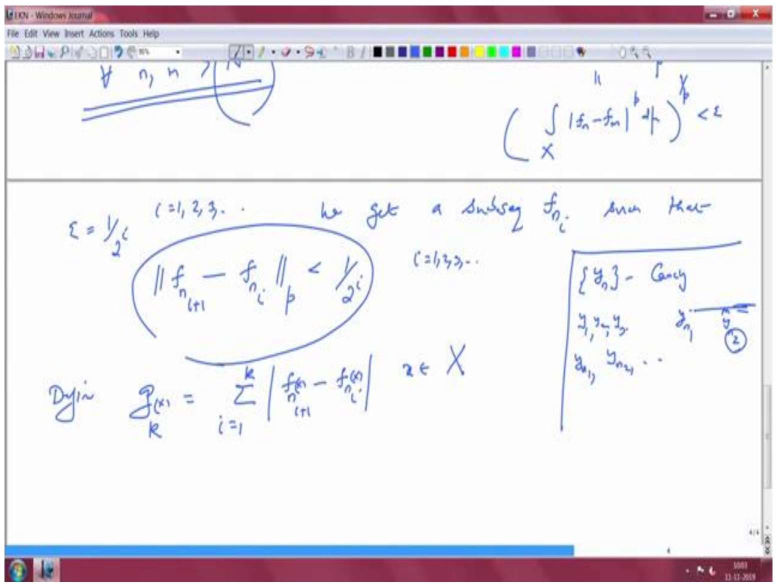

So, choose epsilon to be 1 by 2 to the i. i equal to may be 1, 2, 3, and so on. So, epsilon depending upon the value of epsilon we should be able to get capital N. So, that beyond capital N we will have this inequality. So, that immediately tells us that we get a subsequence, we get a subsequence f n i. So, i, equal to 1, 2, 3, etc., such that, such that norm of f n i plus 1 minus f n i this is the distance between f n i plus 1 and f n i.

So, this will come from the given subsequence fn less than 1 by 2 to the i, you will see why we are doing this i equal to 1, 2, 3, etc. Well, what exactly did we do? So, let us take some matrix space suppose, I have some sequence y n which is Cauchy, then I can choose epsilons to be 1 by 2 to the i. and I will get sum number. So, I have y1, y2 this is the sequence y3, etc. So, at some point let us say yn1 form here onwards distance will be less than or equal to 1 by 2. So, I have yn1 then I take epsilon to be 1 by 2 to the 2. So, I will have to go further and I will get y n 2.

So, that is the n 2th place after that it would be any two points will have distance less than 1 by 2. So, that I call y n 2 and so on. So, this is how the subsequence is chosen. So, this is the subsequence of the original sequence with the extra property that the distance is very small, what is the advantage of doing this? These are functions and these are we are looking at the L p norms of these functions.

So, because they are functions we define new function. So, define new function let us say g of k g sub k this is the function defined by summation i, equal to 1 to k f n i plus 1 minus f n i modulus well what does this mean? This means that the g k at x is you look at f n i plus 1 at x minus f n i. So, x n x. So, this how the function is defined and we take another function.

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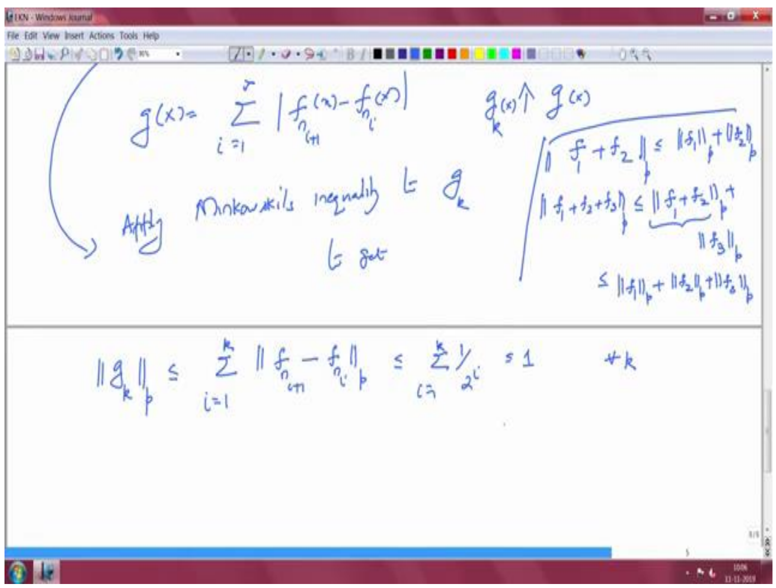

So, let us take the infinite sum. So, g of x to be the sum i, equal to 1 to infinity. So, that is just k going to infinity mod f n i plus 1 x minus f n i x. So, g what is g? So, g is simply the limit of g k. So, g k, well it is not just not simply a limit because you are adding positive numbers. So, g k increases to g for every x well it may converge infinity but these are positive numbers. So, there is no problem in talking about the limit.

Now, let us apply Minkowski's inequality here. So, apply Minkowski' inequality to g k. So, remember what is Minkowski's inequality if, I have two functions let us say f 1 and f 2 I add them and look at the L p norm that will be less than or equal to L p norm of f1 plus L p norm of f 2. Of course this you can extended to finitely many if I take 3 functions f1 plus f 2 plus f 3 then by the previous step we have this is less than or equal to f 1 plus f 2 and then you take the L p norm plus f 3 is L p norm, which is of course less than or equal to because, here we can apply Minkowski's again.

So, we will get L p norm of f plus L p norm of f2 plus L p norm of f3. So, I can do this finitely many times. So, I will get to get if, I look at the L p norm of g k.

So, L p norm of g k is less than or equal to L p norm of i equal to 1. So, let us go back to the definition of g k. So, we are looking at i equal to 1 to k the L p norm of f n i plus 1 minus f n i that is the function there. So, I take the L p norm. So, this is simply Minkowski's inequality but, this is less than or equal to summation i equal to 1 to k 1 by 2 to the i that is how we have chosen the subsequence which is less than to 1.

So, this is true for every k. So, whatever finite sum you take that is bound the L p norm is bounded by 1. So, this allows us to estimate the norm of g as well. So, what is g?

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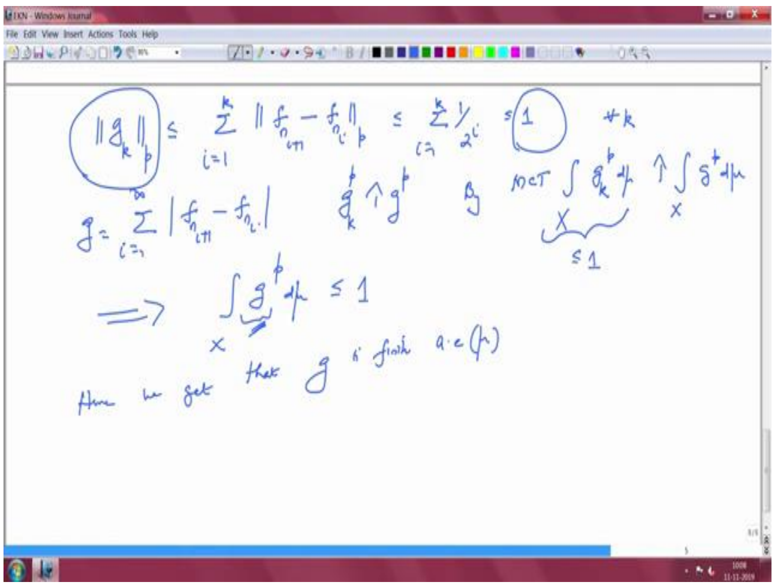

So, remember is the infinite sum, g is i equal to 1 to infinity mod f n i plus 1 minus f n i and we know that g k increases to g. So, g k to the p will also increase to the g k to the p, p is greater than or equal to 1. So, there is no problem here and so by Monotone Convergence Theorem. So, these are all positive functions.

So, there is no we do have take the modulus by Monotone Convergence theorem g k to the p d mu will increase to g to the p d mu that is the Monotone Convergence theorem because g k increases to g but what about this? These are less than or equal to 1 because that is the pth power of L p norm of g k that is less than or equal to 1. So, in particular limit also will be less than or equal to 1.

So, this implies integral over x g to the p d mu g remember is a positive function is less than or equal to 1. So, that is same as saying. So, that implies in particular. So, g will be in L p. So, in particular g is finite everywhere. So, hence we get. So, hence we get that g is finite almost everywhere, g is finite almost everywhere mu because pth power of g is integrable. So, it cannot be infinity on a set of positive measure anything which is integrable has to be finite almost everywhere.

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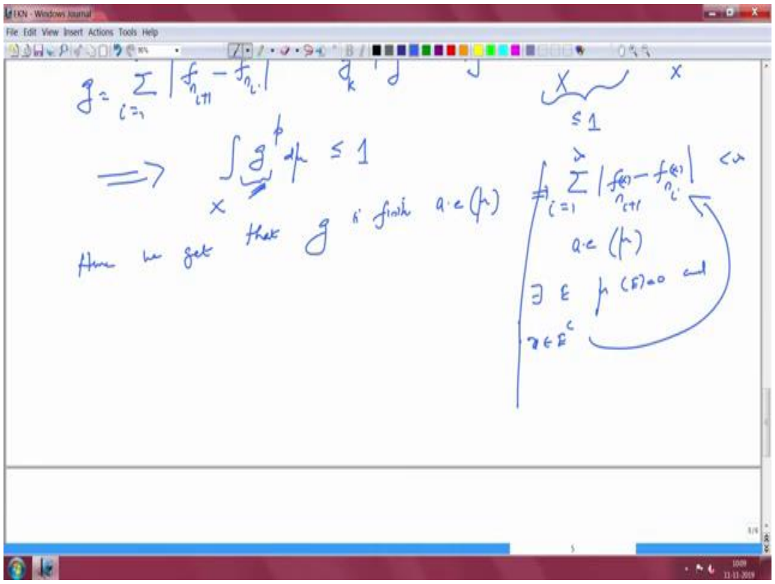

Well, what does that mean? This means that. So, this tells me that if, I look at g which is the infinite sum i equal to 1 to infinity mod f n i plus 1 minus f n i. So, this I put an x here is finite that means the series converges almost everywhere mu. So, that means there exists some set which has measure 0. So, let us say there exists E such that, mu of E equal to 0 and x in E compliment implies this converges. So, that is what it means. Well, what is the consequence of that? So, you can define. So, let us take the modulus out of g k. So, let us write it down again.

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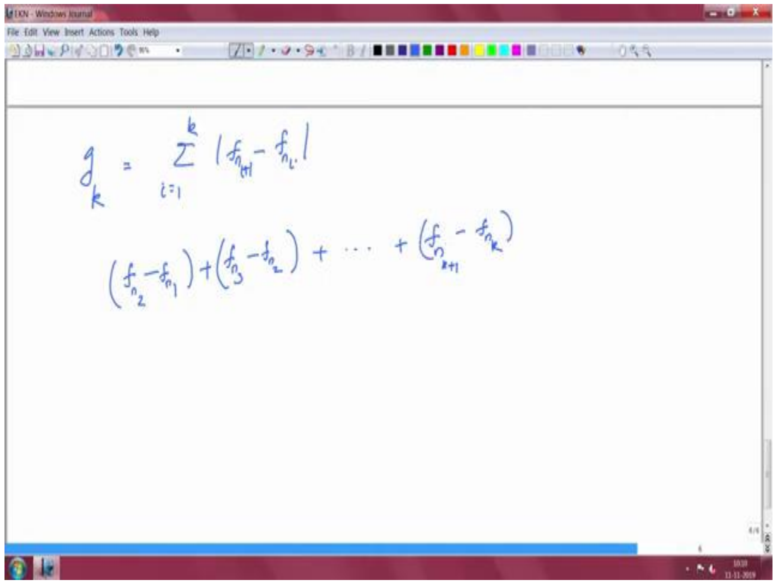

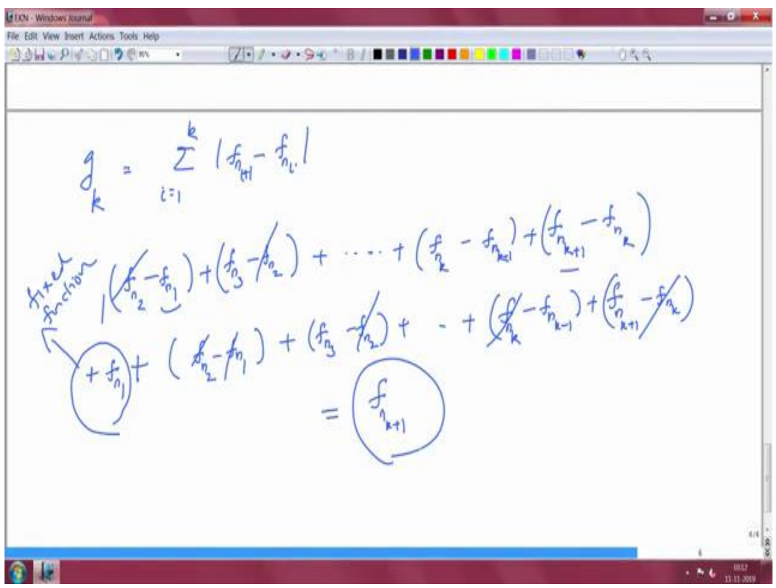

So, g sub k is simply summation i equal to 1 to k mod f n i plus 1 minus f n i. So, if I take the modulus out of this then I am adding. So, i equal to 1 will give me f n 2 minus f n 1 plus f n 3 minus f n 2. So, this is what is known as Telescoping sum dot, dot, dot plus f n k plus 1 minus f n k. So, maybe I should write one more term.

So, let us dot, dot, dot plus f n i. So, f n k minus 1 minus f n k plus f n k plus 1 minus f n k. So, I made a slight mistake. So, let us get this one correct this is f n k and this is f n k minus 1, f n k plus 1 and f n k. So, this is how it looks like if, I take the modulus outside and you see that this cancels with this will cancel with this etc, etc.

So, except this everything else will get canceled and the last term but if I put the modulus I know this converges. So, in particular without the modulus it has to converge because of absolute conversions implies conditional convergence. So, what we get is if, I look at f n 1 minus or minus f n plus 1. So, f n 2 minus f n 1 plus f n 3 minus f n 2 plus etc, etc., plus f n k minus f n k minus 1 plus the last term in g k n k plus 1 minus f n k if, I look at this well what will happen I know because so I have only added this which is a fixed, fixed function.

So, this does not affect the conversions. So, it is a fixed function which I have added. Well, this is nothing but, well this cancels with. So, maybe I should put a plus here this cancels with this and this cancels with this, etc, etc telescoping sum only the last term will remain this and this gets cancel and I will have f n k plus 1, f n k plus 1 but the left hand side as k goes to infinity, suppose to converge. So, that is same as saying the limit of f n k will exist.

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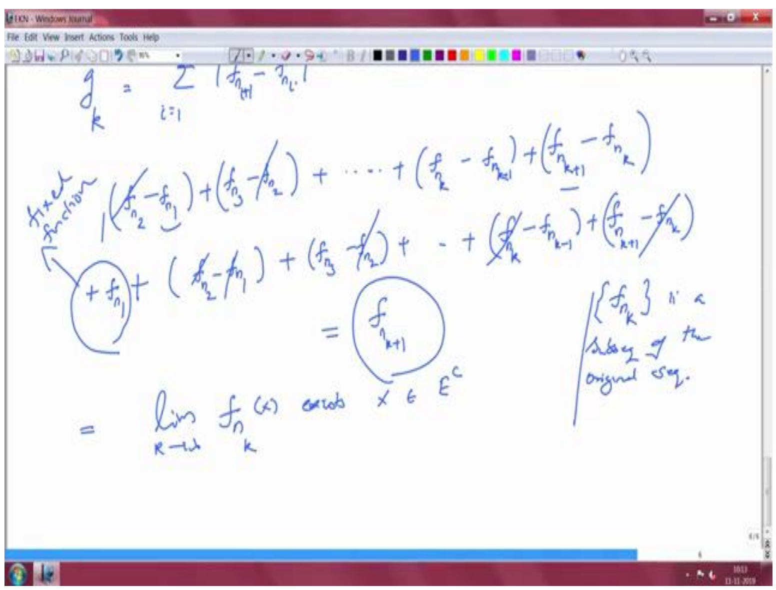

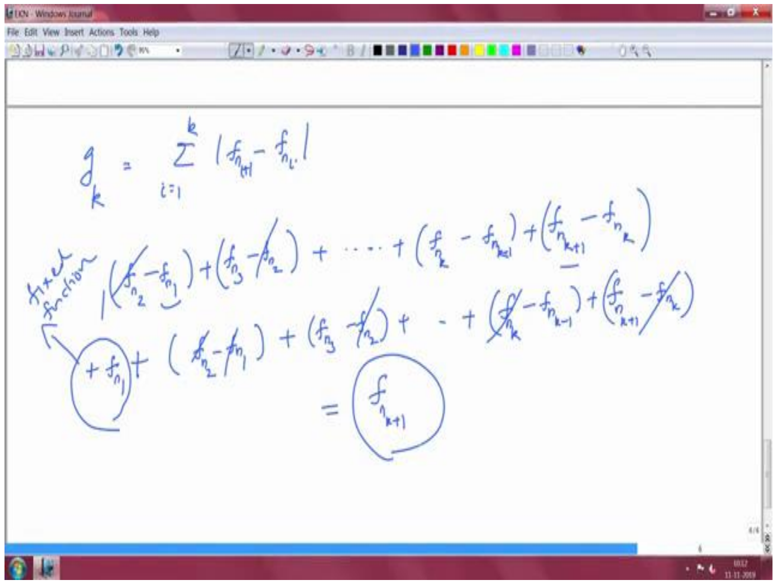

So, this implies that limit of k going to infinity f n k. So, this is the exists, exists for every x and E compliment remember, we had a set E where, measure was 0 and outside which this converged and wherever this converge is absolutely it will converge conditionally and for those points we have this limit existing but, f n k remember, f n k is a subsequence of the original sequence, is a subsequence of the original sequence, of the original sequence. So, what we have found is, we have found a function f as a limit.

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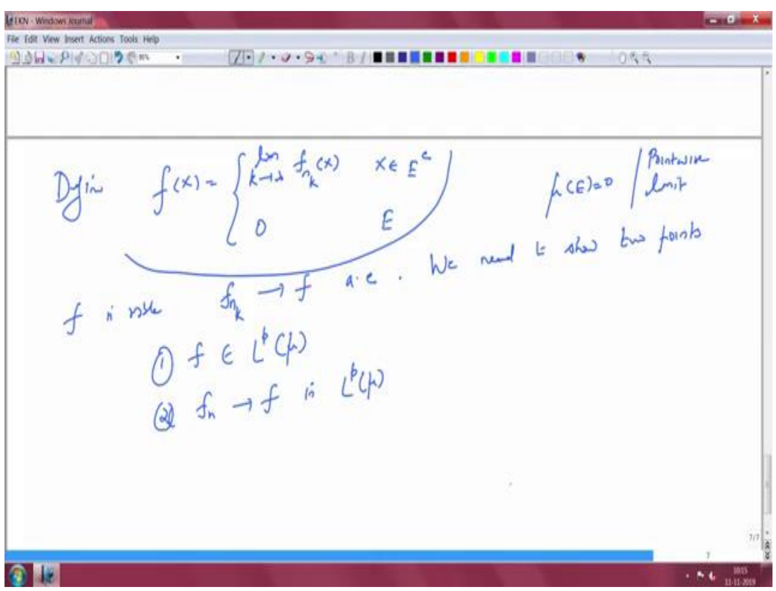

So, define, define f of x to be, so, limit k going to infinity f n k x. So, we know that this exists if x is outside that set of measure 0, on E which has measure 0 you can put 0 or 1 any number does not matter because, mu of E is 0, the measure of E is 0 but, let us put a constant because we do not know the sigma algebra is complete. So, if arbitrary define it may not be a measurable but with this definition f is measurable, f is measurable and f n k converge to f almost everywhere. So, we have found a limit but this does not prove anything.

So, we need to show two things, we need to show two points well what are the two points? One, f belongs to L p, two, f n converge to f, f n is the original sequence in L p these are different statements, here what we have is a point wise limit. So, this is point wise limit for each x we have some limit. So, point wise limit that does not prove any of these statements one and two but, it is not difficult now we have got a candidate we will just show that our sequence converges there. So, now apply Cauchy property again.

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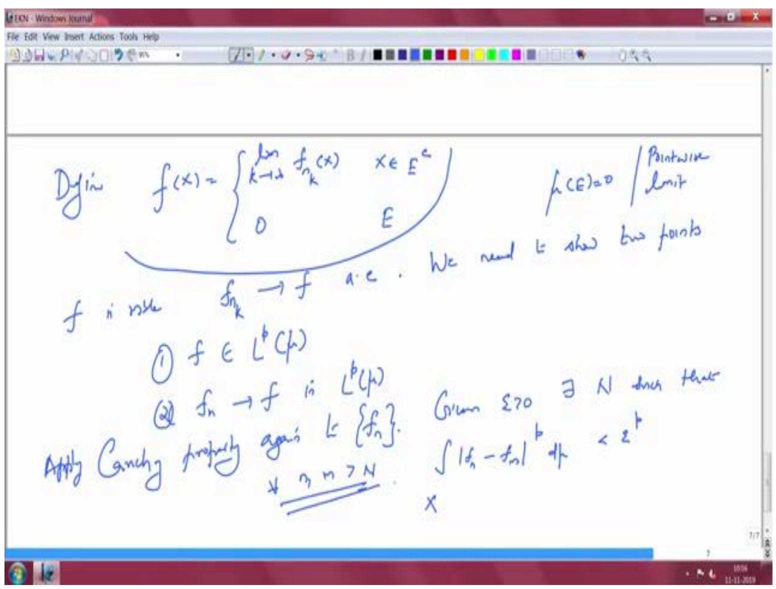

So, apply Cauchy property again, again to f n, f n is a Cauchy sequence we started with that. So, given epsilon I have sum n. So, given, so let us take sum epsilon positive there exists sum capital N such that, whenever these guy’s n and m or greater than capital N we have the L p norm. So, I will write this integral form mod f n minus f m to the p d mu to the 1 by p. So, that is less than or equal to the epsilon to the p. So, instead of 1 by p I have put epsilon to the p. So, this is true whenever this happens. So, whenever this happens this is true.

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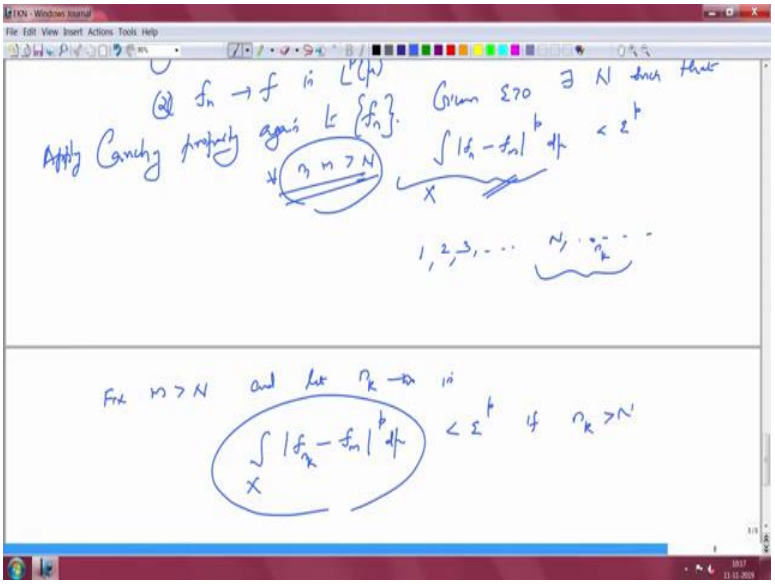

Now, the n k are subsequence of the given sequence we have constructed. So, you see 1, 2, 3, etc., we have capital N here and we have numbers here. So, whenever this happens we have this. So, in particular whenever n k is greater than n I know this happens. So, let us fix m. So, fix small m strictly greater than n and let n k go to infinity in integral over x mod f n k minus f m to the p d mu we can do that because if n k goes to infinity it is going to be bigger than capital N and so this will be this quantity will be smaller than epsilon.

So, this will be smaller than epsilon to the p if n k is greater than capital N. But then, what happens with limit? Let us apply Fatou's lemma there.

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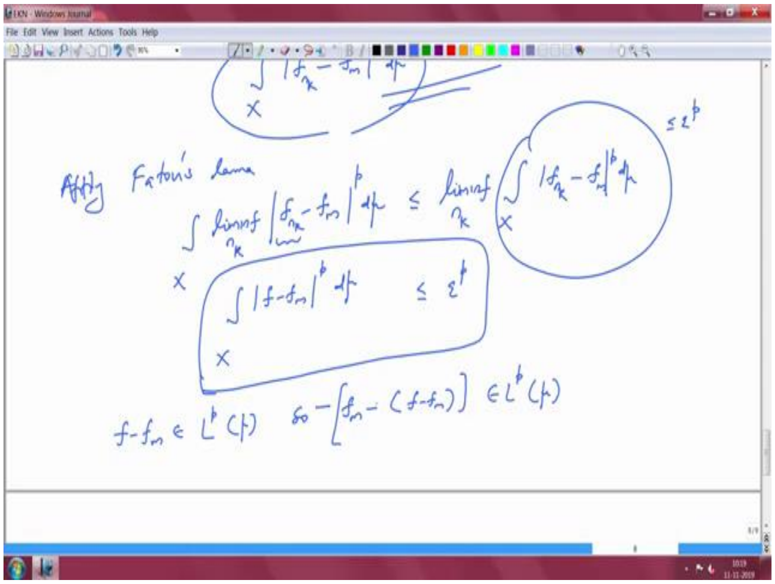

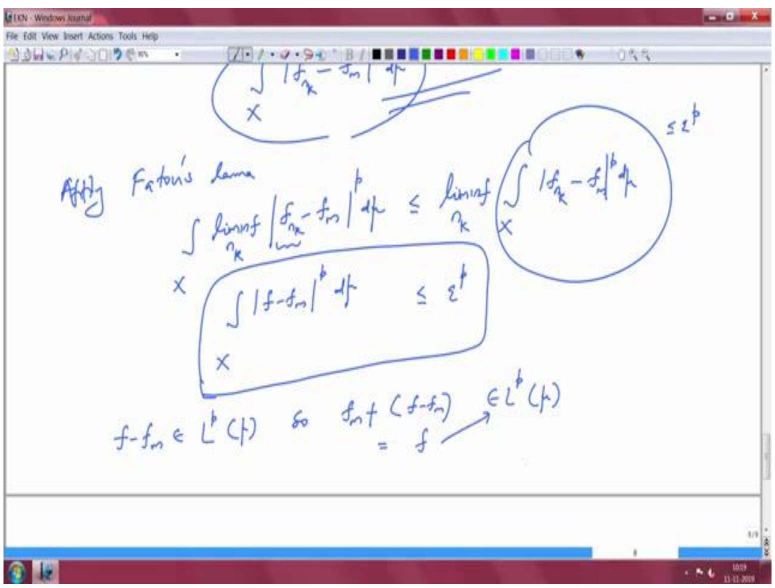

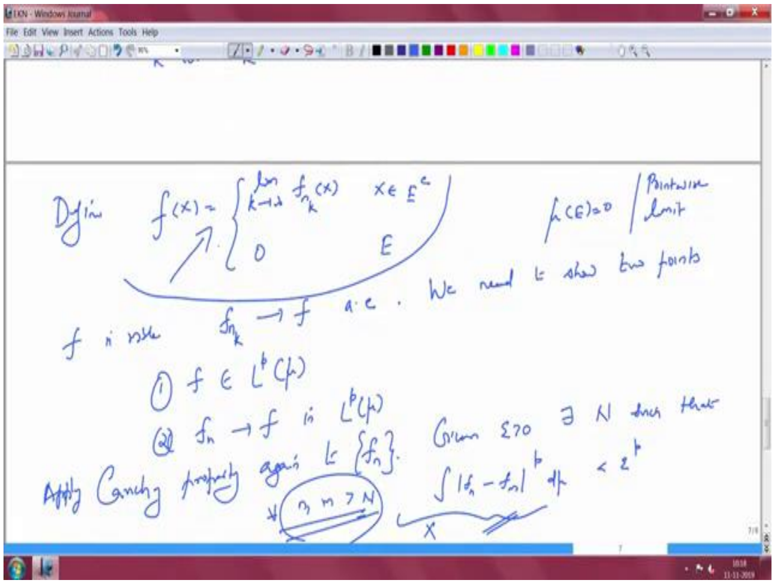

So, apply Fatou's lemma, apply Fatou's lemma, what does Fatou's lemma say? Integral liminf is less than or equal to liminf for integral. So, integral over x liminf, integral liminf with respect of n k is what I am taking f n k minus f m, f m is sort of fixed but, large enough d mu, m is large enough this is less than or equal to liminf of the integrals. So, liminf of the integrals, integrals of, so mod f n k minus f m to the p d mu but, I know because, of this, this is less than or equal to epsilon to the p.

So, the whole thing is less than or equal to epsilon to the p but, what happens to the left hand side? I know what happens to f n k, f n k will converge to f, f we have define here. So, f n k converges to f almost everywhere. So, the set E has measure 0. So, we will forget that. So, the left hand side becomes integral over x mod f minus f m to the p d mu. So, this proves both the things we want in one go because this tells me epsilon to the p is a small, small number but, what does it tell me?

It tells me that f minus f m it is a measurable function that is in L p. So, f m minus f minus f m that will also be in L p because, it is a difference of two L p function. So, minus of this also should be in L p well. So, I should, I made it unnecessarily complicated. So, f m, f m is in L p, f minus f m is in L p. So, if I simply add them that we will do, this is equal to f that will be in L p. So, that was one of the things we wanted to prove, we remember two things. So, one of them is done but, I need to show f n goes to f in L p. So, let us apply this again here, what did we get?

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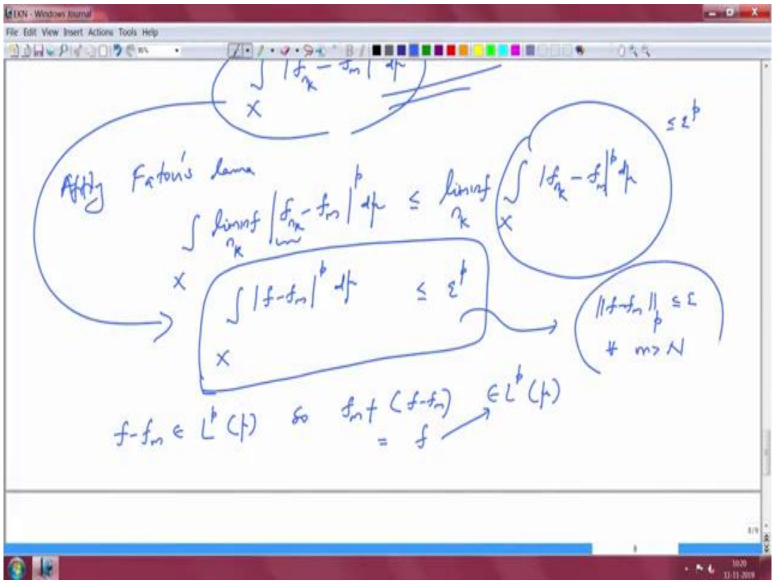

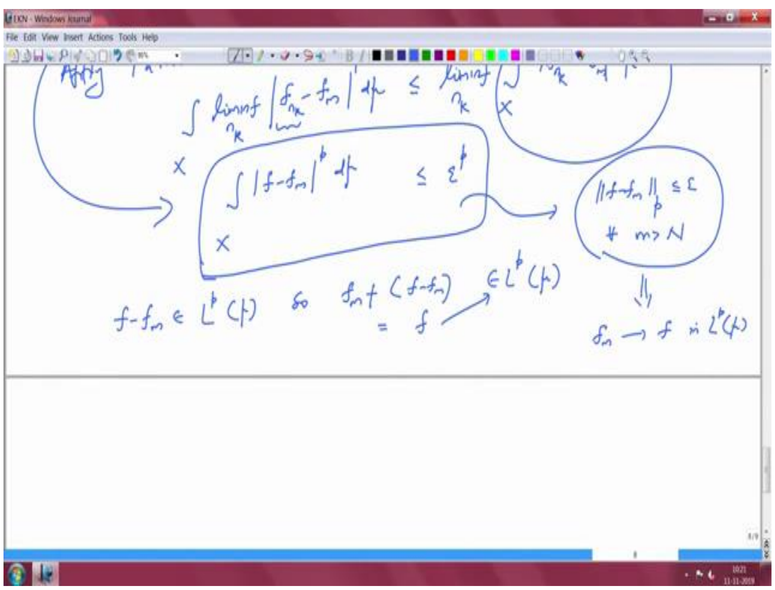

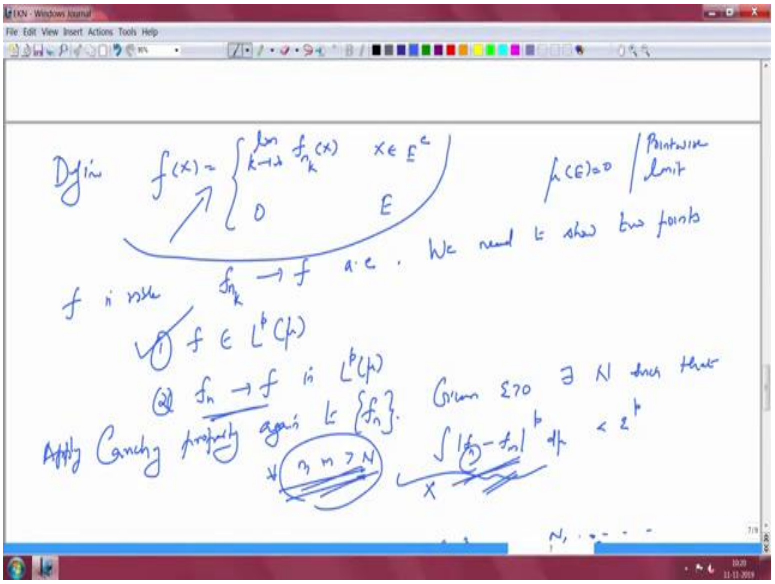

From here we got this, this tells me that L p norm of f minus f m is less than or equal to epsilon remember when we take L p norm there is a 1 by p that is why epsilon to the p became epsilon for every n, every m greater than capital N.

So, that is how we started with let us go back here again, for every n and m we had this and this is where we change the n to n k took the (())(31:31). So, for every small m greater than capital M, N we have this which is same as saying f m converge to f in L p. So, this implies f m converges to f in the L p matrix that is very important L p. So, that proves that f is, that proves that L p is complete.

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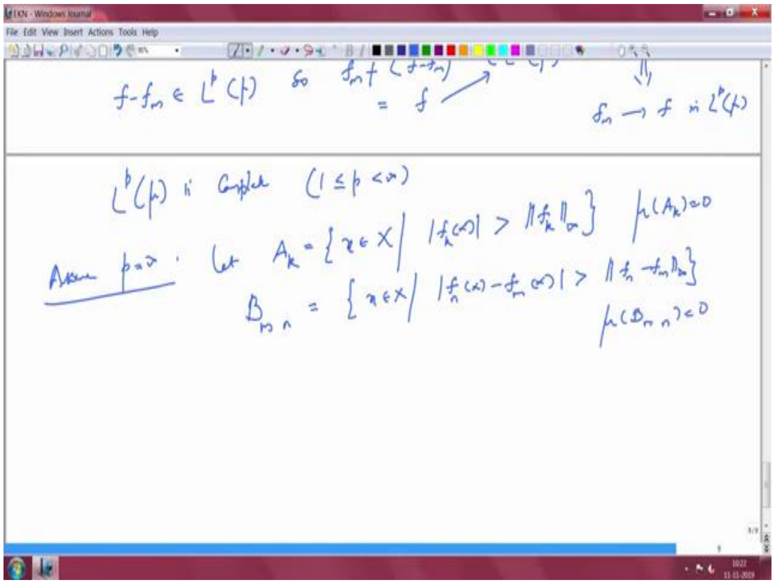

So, L p mu is complete but we assume that 1 is less than or equal to p strictly less than infinity. So, let us quickly get rid of p equal to infinity as well. So, assume, assume p equal to infinity. So, let A k to be the set x in x such that, mod f k x is strictly greater than the L infinity norm of f k. So, then of course mu of A k will be 0 and also B m n.

So, again it is not very difficult to see what is going to happen we are trying to get rid of the problematic sets but, all of them will have measure 0. So, f n x minus f m x is strictly greater than the L infinity norm of f n and f m and of course this will also have measure 0 because, of the definition of the L infinity norm.

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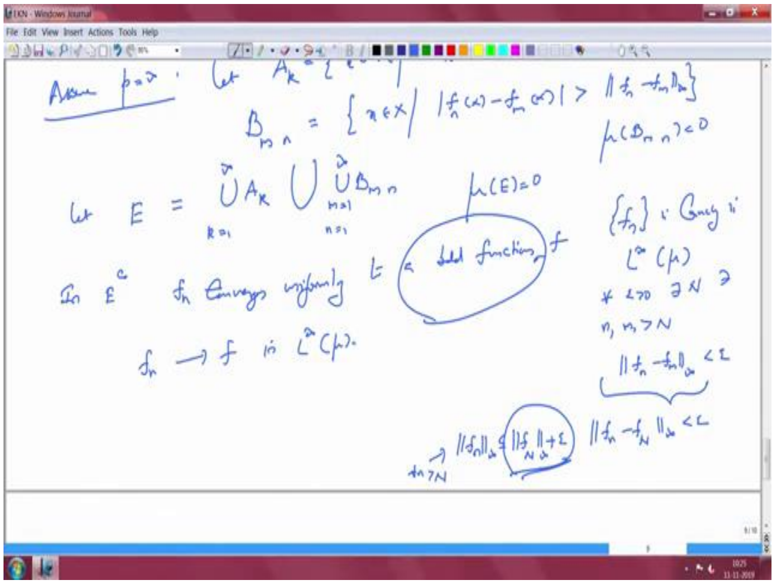

Now, outside. So, let. So, you take the union of these sets well in fact the union A k, union of B m n. So, I will put m equal to 1, n equal to 1 to infinity, k equal to 1 to infinity this is a countable union of sets of measure 0. So, Ee will also have measure 0, well what will happen outside E? So, in e complement.

So, recall that f n is Cauchy in L infinity, f n is Cauchy in L infinity we are looking at p equal to infinity case what does that mean? For every epsilon positive there exists sum capital N. Such that, whenever small n and small m are greater than capital N the norm which is the L infinity norm now f n minus f m L infinity norm is less than epsilon.

So, that is like uniformly Cauchy. So, if you have for each x if you have f n x to be Cauchy it will converge because complex plain is complete that we will do. So, apply that on E complement you will prove that So, this is something that you can check f n converges, converges uniformly to a bounded function, to a bounded function, boundedness is very easy because all the f n's are bounded by some fixed quantity because, of this. So, if I look at this I know that norm of f n minus f N L infinity norm is less than epsilon. So, because, of that L infinity norm of f n will be less than or equal to L infinity norm of capital f N plus epsilon.

So, you take sum epsilon you will have this, this is true for every n, for every n greater than capital N. So, all of them are bounded. So, if I take the limit that will also be bounded by this fixed constant. So, I will get bounded function. So, outside E, I know f n converges to f. So, f n converges to f bounded function let us called that f in L infinity. So, we will stop here we just proved that the spaces we were looking at L p of mu is complete as long as p is between including 1 and infinity we will look at finer properties of L p spaces.

For example, we know there are simple functions in L p spaces, there are also continuous functions with compact support and so on. So, we will look at such finer properties in the next lecture.