Measure Theory
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Lecture 34 - LP spaces

So, last session, we defined LP spaces. Now we are going to look at the so called LP norm on this basis. Our aim is to prove that it is actually a norm, of course, there will be a certain things to be modified, we will see that, especially with the property that if the norm of a vector is 0 the vector is 0. So, that is not strictly speaking true in LP spaces, we have to mod out the space with an equal insulation but, that is not really going to trouble us with any kind of computation.

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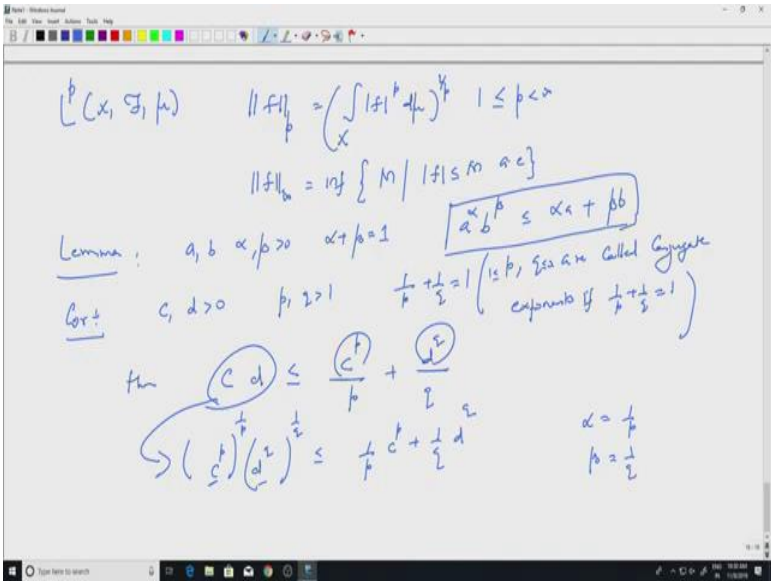

So, let us continue with the proof. So, recall that we are trying to prove LP norm is a norm actually. So, if I take f then the LP norm is defined to be, so this is integral over x mod f to the p d mu to the 1 by p but, this was true for 1. So, we are looking at only p between 1 and infinity, is strictly less than infinity and you have L infinity norm to be the infimum of all those functions, all those m such that, mod f is less than or equal to m almost everywhere with respect to mu of course.

So, you look at all such m and then take the infimum. So, we proved the following lemma. So, let me recall that. So, lemma we proved was if I take a, b, alpha, beta all positive with the alpha plus beta equal to 1, which is a converse combination then we have a to the alpha, b to the beta, is less than or equal to alpha a plus beta b. So, this was simply a consequence of minus log b in convex.

So, this has an immediate corollary. Well, what is the corollary? So, I take two numbers c, d positive p and q greater than 1, 1 by p plus 1 by q equal to 1. So, the p and q are called conjugate exponents. So p and q are called conjugate exponents if 1 by p plus, conjugate exponents if 1 by p plus 1 by q equal to 1. So, these are between 1 and infinity, infinity is allowed. So if p is infinity 1 by p is 0, so q will be 1.

So, 1 and infinity are conjugate exponents. So, if you take p and q like this and positive numbers, c and d then we have c times d is less than or equal to c to the p by p plus d to the q by q. So, that is exactly this inequality here. So we just have look at this and then decide. So, what do you do? You take c to the 1 by p, d to the 1 by q, well, c to the 1 by p to the p and d to the 1 by q to the q so, this will be less than or equal to maybe I should, should write the other way because, alpha plus beta should be 1.

So, my alpha is 1 by p, so, you take alpha to be 1 by p and beta to be p 1 by q. So, a and b are these quantities so, these quantities are a and b. So, c to the p to the 1 by p, d to the q to the 1 by q will be less than or equal to, so this is my a, this is my b. So, I have alpha times a which is 1 by p into c to the p plus 1 by q into d to the q. So, that is all this, so this is LHS is just this. So, this is a trivial application of the inequality, so, we use that.

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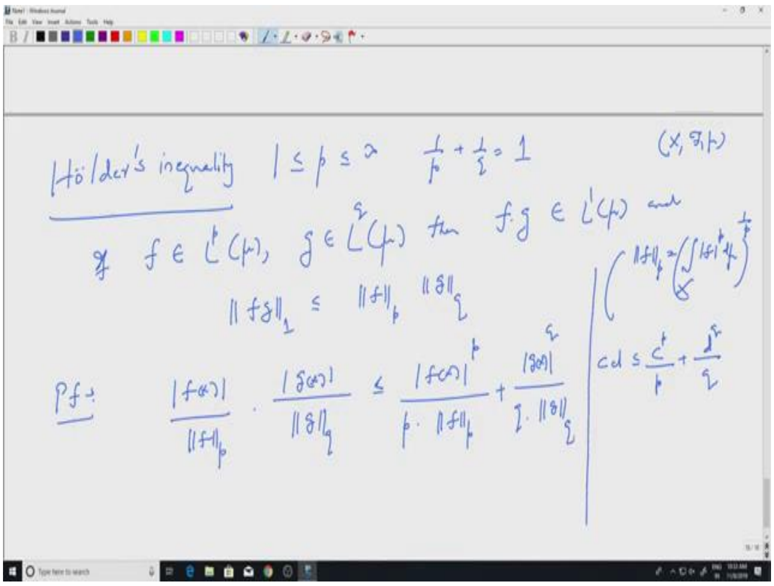

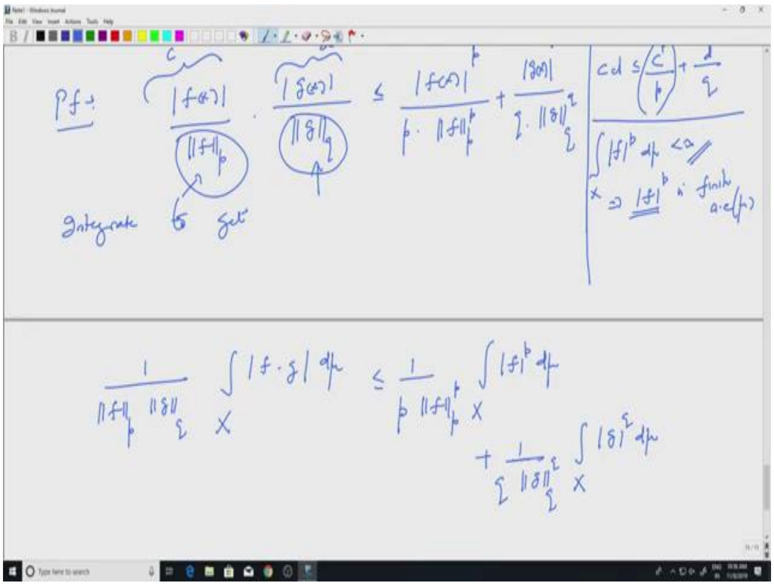

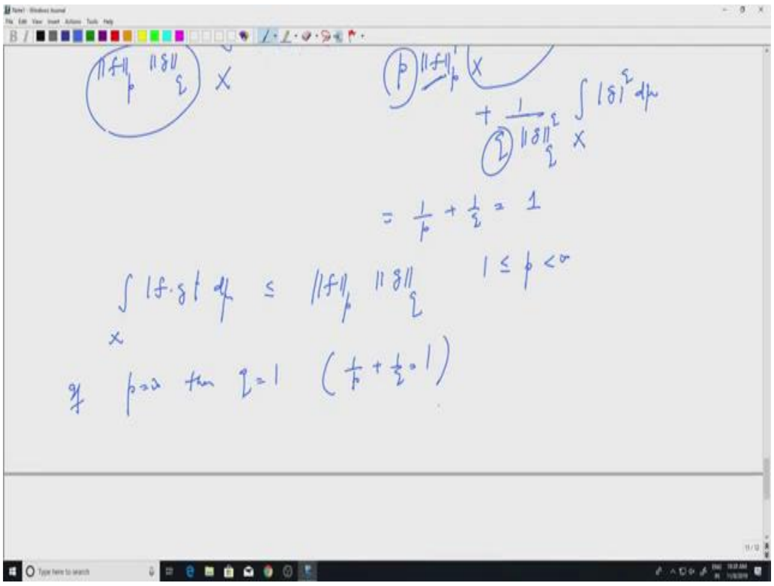

So, we will use this in the following inequality, it is called Holder's inequality. So, you have seen some cases of this, which is the Cauchy–Schwarz inequality. Holder's inequality is a generalization of Cauchy–Schwarz. So, you take 1 p less than or equal to infinity and q to be the conjugate exponent. So, 1 by p plus 1 by q equal to 1, of course, we have the space x, f, mu always. So, we have a major space with the positive major and we are looking at the LP space on that.

So, Holder's inequality says, if f belongs to LP of mu g belongs to L q of mu then f times g. So, this is another measurable function, this will belong to L1 mu and the L1 norm of f g is less than or equal to LP norm of f into L q norm of g. So, recall that LP norm was defined to be so, LP norm is simply integral over x mod f to the p d mu to the 1 by p. The norm is always on a with respect to the major.

So, that is understood so, I am not going to write the major in these exponents. So, we will just stick to 1 and p. Whenever there are two majors etc., we will be very clear about what it is. So, proof, so, we use the earlier inequality. So, let us look at this inequality. So, we use this. So, c d is less than or equal to, so, I can, let write it here c times d is less than to c to the p by p plus d to the q by q, 1 by p plus 1 by q equal to 1. So, we use that.

So, if we use that, we will get mod f of x by LP norm of f times mod g of x divided by L q norm of g, this is less than or equal to, I will explain after writing down, mod f x to the p by p times norm f to the p plus mod g x to the q divided by q times norm g q. So, let us do this. So, let us look at this carefully just to see if it is ok. So, this is my c, this is my d. So, c d is less than or equal to c to the p by p. So, what is c to the p by p? Well, c to the p will be, so, I should put a p here and I should put a q here ok, well, I can assume everything to be norms to be 1 but let us not worry too much about it.

So, c to the p will be mod f to the p by L p norm of f to the p see, remember f and g are in LP and L q. So, these are all finite quantities. So, this we can divide by them, there is no problem, these are numbers. Now, f and g are measurable functions. So, for some x mod of x can be infinite but, recall our one of the earlier results where we did if f is in LP so, that means mod f to the p d mu is finite, which would imply mod f to the p is finite almost everywhere, almost everywhere with respect to mu. If it is infinity on a set of positive major when I integrate, I will get infinity but, finiteness implies that this cannot be infinity on a set of positive major.

So, except on a set of major 0 f of x, g of x everything should be finite. So, I can choose those x, so this is fine. Now I apply. So, integrate, so integrate to get, so if I integrate, what do I get? Well, this is a constant, this is a constant, I have f and g as variables with variable x. So, these constants come out. So, I will have norm of p, norm g q. So, these are constants, they come out integral over x, mod f into g, mod of x gx.

So, that is mod f g d mu, that is the left hand side. If I integrate with respect to mu, this is the left hand side I will get, this is less than or equal to 1 by p. So, in the first component I have 1 by p and L p norm of f that comes out and integral over x mod f to the p d mu. So, that is the first term plus 1 by q times norm g q to the q integral over x mod g to the q d mu. So, all I have done is integrate because, the functions are bounded like this mod f, mod g is less than or equal to something the integrals will satisfy that, the integral monotonicity of the integral.

So, this is true. Well, so what is the right hand side? Let us look at this once again, this quantity is simply the denominator because, the LP norm is the 1 p th root of whatever is in the bracket and similarly further. So, this is simply 1 by p plus 1 by q which is 1. So, now you can take this to the left hand side. So, I will get integral over x mod f g d mu is less than or equal to LP norm of f into L q norm of g.

So, this is true for 1 less than to p strictly less than infinity because, we have taken, we have divided by p and q and so on. So, infinity is not included there. So, the proof works for only that but, if one of them is infinity this is trivial. If, p equal to infinity then q will have to be equal to 1 because I have 1 by p plus 1 by q equal to 1. So, then Holder' inequality is very trivial. So, let us see what is the Holder' inequality.

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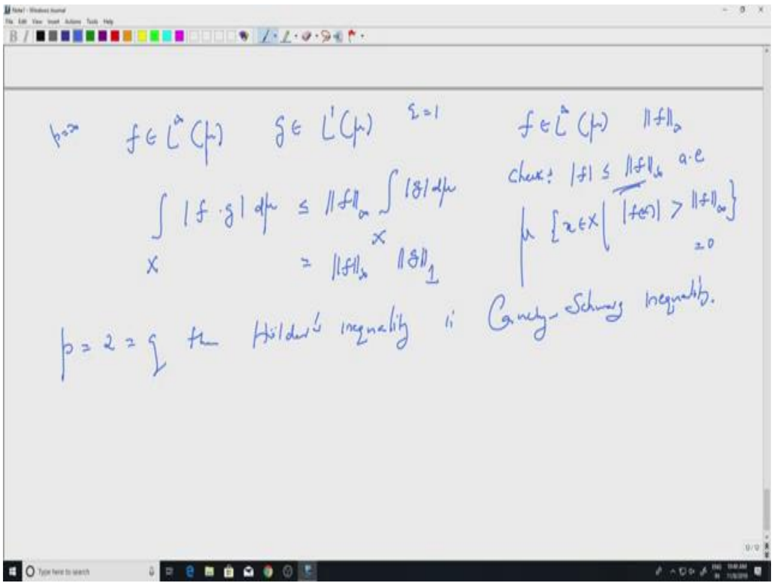

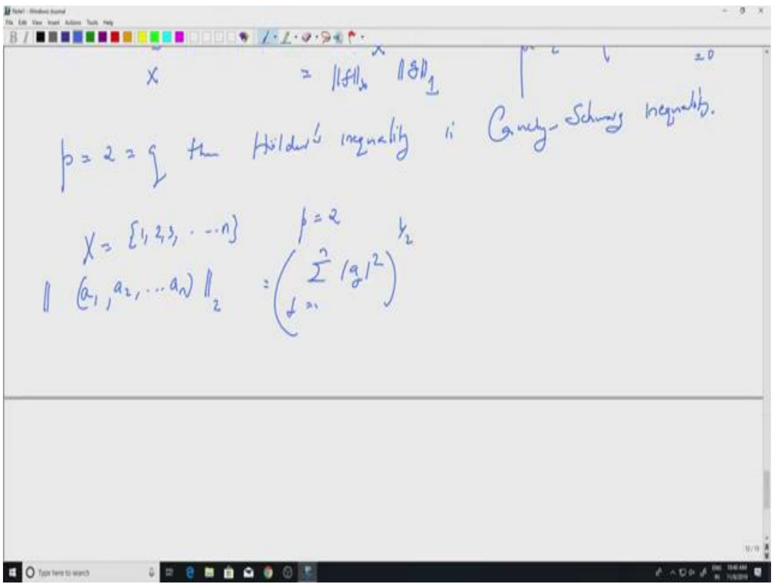

So, I look at f in LP which is L infinity of mu then g is in L q which is L1 because, q is 1, so p is infinity and q is 1 and product is in L1 that is what we want to prove. So, you look at product f and g d mu. Now, you see f is in L infinity, what does that mean? The L infinity norm of f is finite. So, check. So, this is something which we should verify, mod f is less than or equal to L infinity norm of f almost everywhere.

So, in other words, the mod f is a function. So, in other words, if I look at all those points x, x in x such that mod f of x is strictly greater than L infinity norm of f, this has major 0 because, L infinity norm is the infimum of m such that mod f is less than or equal to m almost everywhere. So, that is why this is true. So, f is less than L infinity norm of f almost everywhere. So, I can take that outside and I will have integral over x mod g d mu which is the L1 norm.

So, this is L infinity norm of into L1 norm of g. So, if you multiple two functions which are at conjugate exponents then the product is in L1. So, let us this is something which you, as I said it is a generalization of Cauchy–Schwarz inequality. So, when p is equal to 2, q is also equal to 2 then, Holder's inequality is, Holder' inequality is called Schwarz inequality, Cauchy–Schwarz inequality, is the Cauchy–Schwarz inequality. Why is that? So, let us look at the case you know which is the finite dimensional vector spaces and so on.

So, if I take x equal to 1, 2, 3, etc., n then p equal to 2 norm. So, if I take a point here so, that is a vector. So, a 1, a 2 and a function on x is a, is an n vector, what is the L2 norm of this? This is usual Euclidian norm, summation mod a j square j equal to 1 to n to the half, this is my Euclidian norm. So, what does Holder's inequality say?

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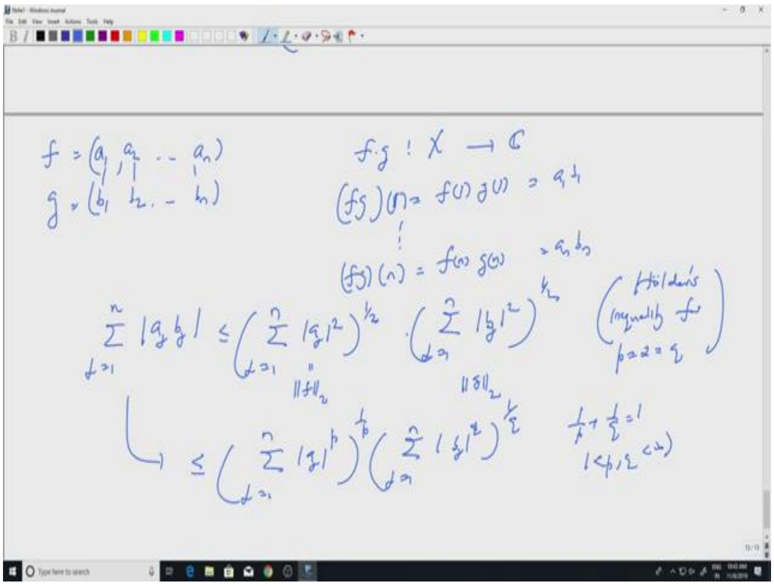

I take two functions. So a 1, a 2, etcetera, a n and another function b 1, b 2, etc., b n, you multiply them. So, this is my f, this is my g. So, what is f g? Well, f g is a function on x. So, this is a function on x taking values in the complex state, f g of 1 will be f 1 into g 1 which is a 1 into b 1 and so on. So, f g of n will be f n into g n equal to a n into b n. So, you simply multiply pointwise, that is your f g and you take the L1 norm. So, summation mod a j b j, j equal to 1 to n so, this would be the L1 norm of f times g, this is less than or equal to j equal to 1 to n mod a j square to the half, that is the L2 norm of f.

So, this is L2 norm of f times summation j equal to 1 to n mod b j square to the half, this is L2 norm of g, that is the Holder's inequality and that is the Cauchy–Schwarz inequality you know but, in general we have so, this is the Cauchy–Schwarz. So, Holder's inequality, Holder's inequality for p equal to 2 equal to q. When p equal to 2, q equal to 2 because, q is the conjugate exponent. In general, we have this to be less than or equal to summation j equal to 1 to n mod a j to the p, to the 1 by p and summation j equal to 1 to n mod b j to the q 1 by q. 1 by b plus 1 by q equal to 1, 1 less than p, q strictly less than infinity. If one of them is infinity, you take the maximum out.

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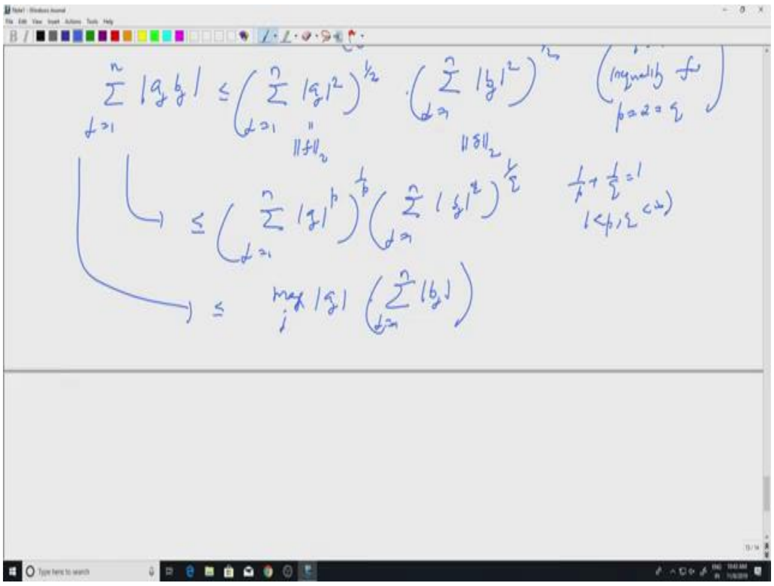

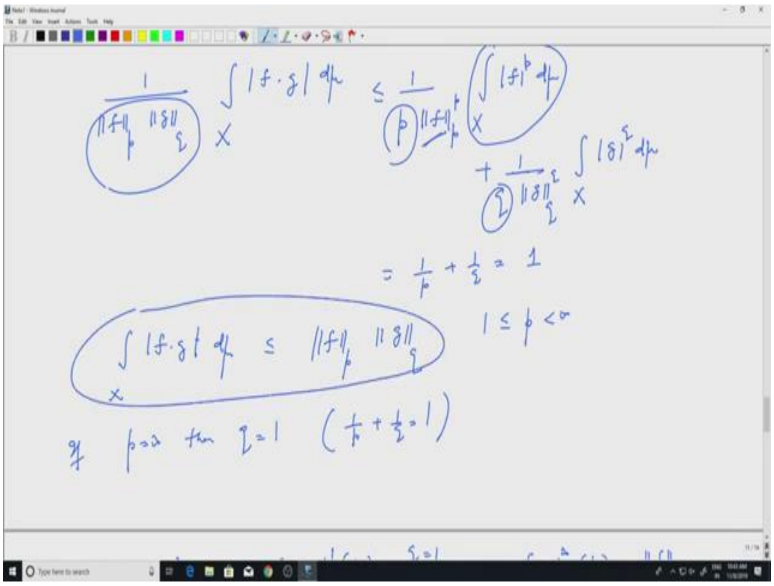

So, if one of them is so, maybe I will denote that as well. So, this is less than or equal to, I can take the infinity norm of let us say a j. So, that is the L infinity norm of f times whatever is remaining is the L1 norm of. So, these are all trivial in the case of finite vectors like this and this is true in general is what Holder's inequality tells you. So, this is the Holder's inequality. So, Holder's inequality implies what is known as, we still have not proved that LP norm is actually is a norm. So, we will do that in the next two proofs.

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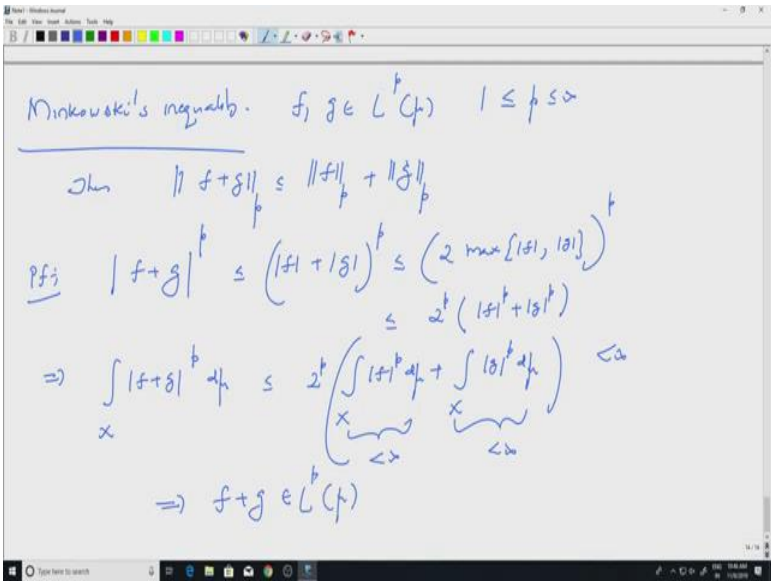

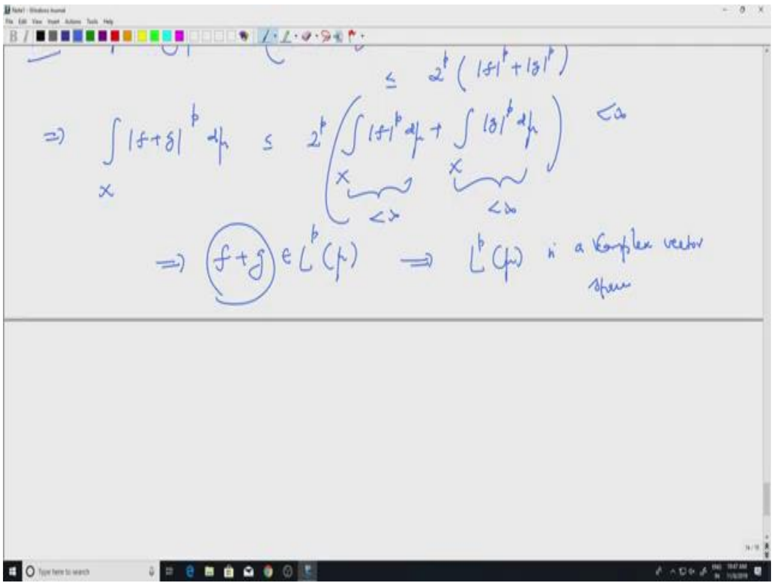

So, the next one is Minkowski's inequality, Minkowski's inequality which will tell us that it satisfies the triangle inequality. So, I take f g in LP, p is fixed between 1 and infinity, it can be infinity but, infinity proofs are generally much easier. Then, the LP norm of f plus g is less than or equal to LP norm f plus L q norm of LP norm of g. So, there are many things here, one is that it satisfies the triangle inequality, next but, you have to say that if f and g are in LP then the sum was in LP.

So, LP mu becomes a vector space. So, let us start with that. So, the first part is very easy. So, you start, look at mod of f plus. So, what is f plus g norm? So, you are looking at mod f plus g to the p integral over x d mu, this is what you want and you want to say that is finite and you have those inequality. So, we look at f plus g. So, mod f plus g, well, this is of course less than equal to mod f plus mod g. At any point x, this is what will happen but, I need to take the power p.

So, this is of course less than to the power p here which is less than or equal to, I can put, I can replace each of them by the maximum. So, you look at the maximum of mod f and mod g which means you are looking at each point, if I take a point x, mod of f x plus g x to the p is less than or equal to maximum of, each them you replace by the maximum and then take, multiply by 2. So, f is replaced by the maximum f and g, g is replaced by maximum of f and g and so, you have the power p, which of course is less than or equal to 2 to the p and the maximum is of course less than to mod f plus mod g.

So, this is fine and this implies that if I integrate with respect to mu now, this is less than to 2 to the p times integral over x mod f to the p d mu plus integral over x mod g to the p d mu and so, this is finite because, I know this is finite because g is in LP, I know this is finite because f is in LP. And so this implies that f plus g is in LP. So, if f and g are in LP then f plus g is in LP and you can multiply f with a constant, that constant comes out as modulus from the norm.

So, this all this discussion simply implies that LP mu is a vector space, this is a complex vector space and the norm is defined on that. So, I know that f plus g is there and if I multiply by alpha, I still land there, that is a trivial thing.

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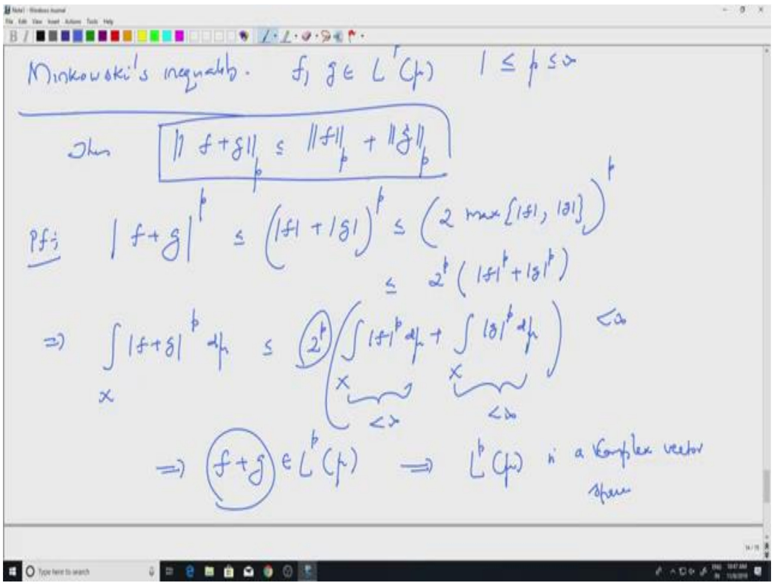

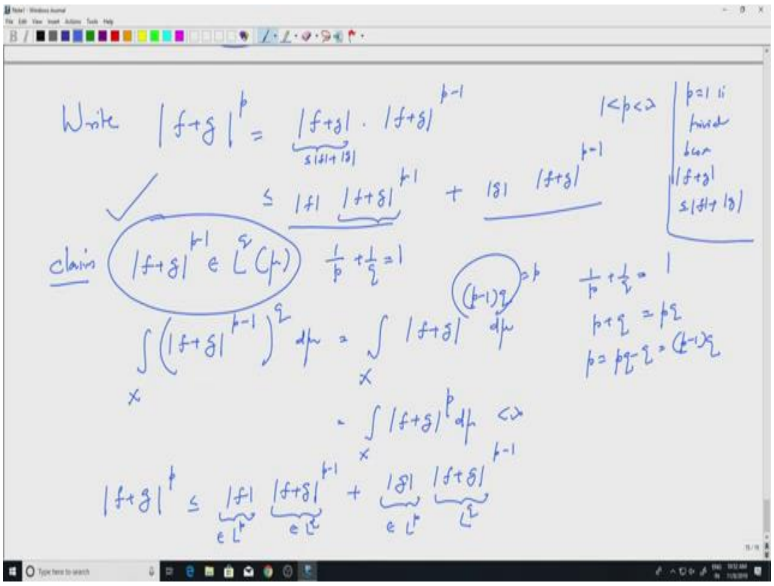

So, now our aim is to prove Minkowski's inequality. So, we want to look at the LP norm of f plus g, I want to say it is less than to LP norm of f plus LP norm of g. So, this does not give us exactly what we want because of the 2 to the p here. So, we have to get rid of that, for that we use Holder’s inequality.

So, write mod f plus g to the p. So, this is equal to, I can write this as mod f plus g times mod f plus g to the p minus 1. So, remember p is between 1 and infinity, p equal to 1 case is trivial. So, I will leave this to you, p equal to 1 is trivial because, that is mod f plus g is less than to mod f plus mod g and you simply integrate, trivial because, mod f plus g is less than to mod f plus mod g. So, then you integrate and you will get L1 norm of f plus g is less than to L1 norm of f plus L1 norm of g.

So, now this is less than to so, here I know this is less than to mod f plus mod g. So, I get two terms, one is mod f into mod f plus g to the p minus 1 plus mod g into mod f plus g to the p minus 1. So, all I have done is to distribute mod of g to the p minus 1, two terms. So, now the point to note is that, so this function I claim that is in L q. So, claim mod f plus g to the p minus 1 is in L q, I want to apply Holder’s inequality, for Holder’s inequality I need two functions, one in LP, the other in L q where 1 by p plus 1 by q equal to 1, I claim this.

So, let us see why is that true. If I look at the L q norm of this function so, I will be taking the q th power of this and then 1 by q. So I will forget the 1 by q for the time being, I want to know this is finite if this integral is finite, well, this integral is simply I have first mod f plus g to the p minus 1 into q. So, what is p minus 1 into q? Let us compute that. Well, we can compute that because, I know that 1 by p plus 1 by q equal to 1, correct. p and q are conjugate exponents.

So, 1 by p plus 1 by q equal to 1, what does this mean? This means p plus q is equal to p q. I want to compute p minus 1 into q. So, p equal to p q, I take the q to the right hand side. So, I will get minus q which is p minus 1 into q. So, p minus 1 into q is simply p. So, this is simply p. So, this is equal to integral over x mod f plus g to the p which I know is finite because, f and g are in L p. So, f plus g is in L p, we just proved that it is the vector space.

So, the claim is true. So, now you look at these two terms, I am multiplying two functions, apply Holder's inequality. So, let me I will write that step once more. So, mod of f plus g to the p is less than to mod f into mod f plus g to the p minus 1 plus mod g into mod f plus g to the minus 1 same, same qualities. So, this is in LP, this we just proved that, that is in L q, this is in LP and this we just proved that is in L q. So, we integrate and then apply Holder's inequality.

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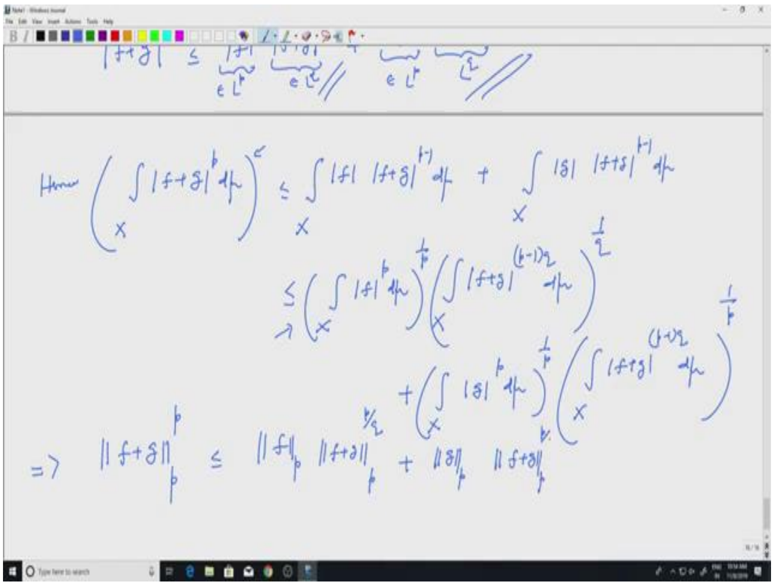

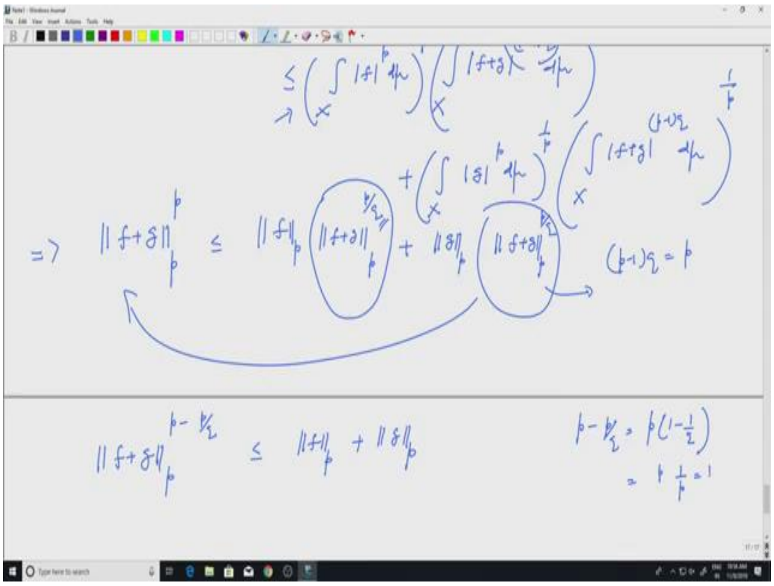

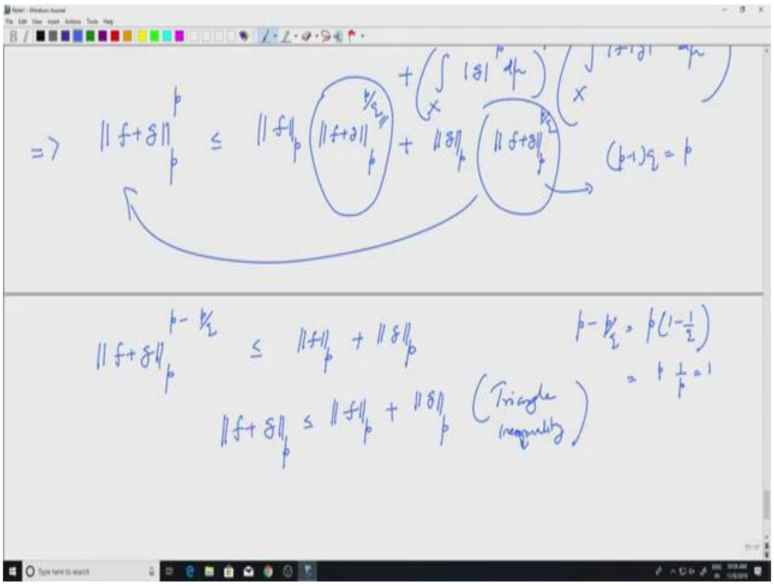

So, hence integral over x, remember we are trying to bound the LP norm of f plus g. So, this is less than or equal to, so, we will put the 1 by p later on. So, I have, well, I simply integrate first. So, I have mod x over x mod f times mod f plus g to the p minus 1 d mu plus integral over x mod g mod f plus g to the p minus 1 d mu. So, now we can apply Holder's inequality because they are in the correct spaces, conjugate exponents.

So, by Holder's inequality this is less than or equal to integral over x. So, the first one will give me mod f to the p d mu to the 1 by p into integral over x L q norm of the second function. So, mod f plus g to the p minus 1 to the p. So, into q d mu to the 1 by q. So, this is simply the Holder's inequality in the first term plus again the second term will be exactly similar with f and g replaced.

So, mod g to the p d mu to the 1 by p into integral over x mod f plus g to the p minus 1 into q, you have to take the L q norm d mu to the 1 by q. So, this tells us that so, if I look at this inequality, what is the left hand side? You look at f plus g and you look at LP norm and take the power p because, here there is no 1 by p. So, that is why you have the power p, this is less than or equal to LP norm of f that is this part and you have mod f plus g to the. So, I will, let me write it f plus g to the p norm to the p by q plus LP norm of g into LP norm of f plus g to the p by q.

So, now you can. So, this is a common factor here, you can bring this to this side, it is same quantities. So, maybe I did not explain this part. So, how do you get this? p minus 1 into q equal to p. So, here I will be writing p but, I have to write 1 by p and then cancel it with another p. So, that is why you get p by q power. So, that is a trivial calculation. So, these quantities you bring it to the left hand side. So, remember these are all finite, if it is 0 there is nothing to do.

So, if we will leave that part if f plus g is 0, we have nothing to do. So, we will assume that it is a positive quality and bring it to the left hand side, we will have the LP norm of f plus g, I have the p here and I am dividing by whatever is on the right hand side which is f plus g to the p by q. So, that would be p minus p by q this is the left hand side we will get, this is less than to LP norm of f plus LP norm of g but, what is p minus p by q?

But what is p minus p by q? So p minus p by q equal to p minus so, p you can take outside, it simplifies very fast p into 1 minus 1 by q. Remember, 1 by p plus 1 by q equal to 1. So, 1 minus 1 by q is 1 by p which is 1. So, that is all we want on the left hand side. So, that is LP norm of f plus g is less than or equal to L p norm of f plus L p norm of q. So, that is the triangle inequality for the norm triangle inequality.

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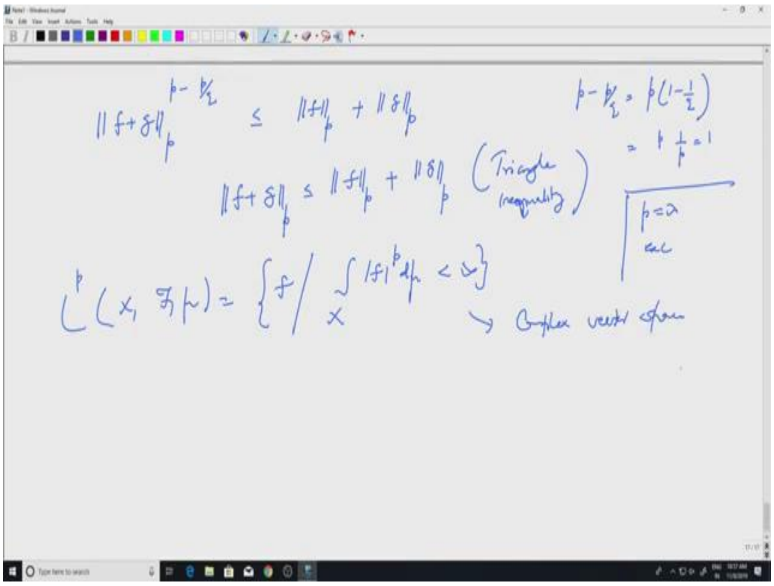

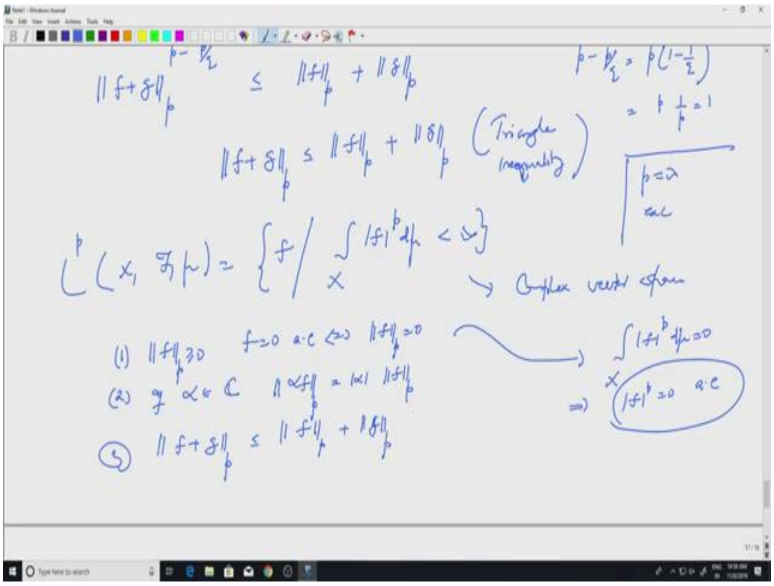

So, there is a small point which I should emphasis on. So, if I look at, so the same proof, so for p equal to infinity, I will leave it as exercise. So, this is a trivial exercise, so let us go back to LP, LP x f mu. So, I have all those functions f such that the LP norm is finite. So, p equal to infinity is included but, I will not keep writing it, it is finite. So, what do we know, we know it is a vector space so, this is a complex vector space and we have defined the norm. So, we want to say it is actually a norm.

So, let us see why. So, first thing is so, there are three properties, one, LP norm of f is positive and f equal to 0 if and only if LP norm of f is 0, this is what we want for the norm but, this is not entirely true because, if I look at what mod f p d mi is 0 then all that I will get is mod f to the p is 0 almost everywhere. So, remember that property that we have something positive function integrating to 0 will give me only integrant is 0 almost everywhere. So, it is not entirely equal to 0.

So, we have only this much there but, that is fine, we will see why. So, if alpha is a complex number and I look at alpha times f, this I know is mod alpha times LP norm of and we have the third property, the triangle inequality I take two elements in LP. I know that it is LP norm of the sum is less than to sum of the LP norm, that is the Minkowski's inequality. So, the only property which is missing from the norm is the first one. So, all that we do is we identity function which are almost everywhere.

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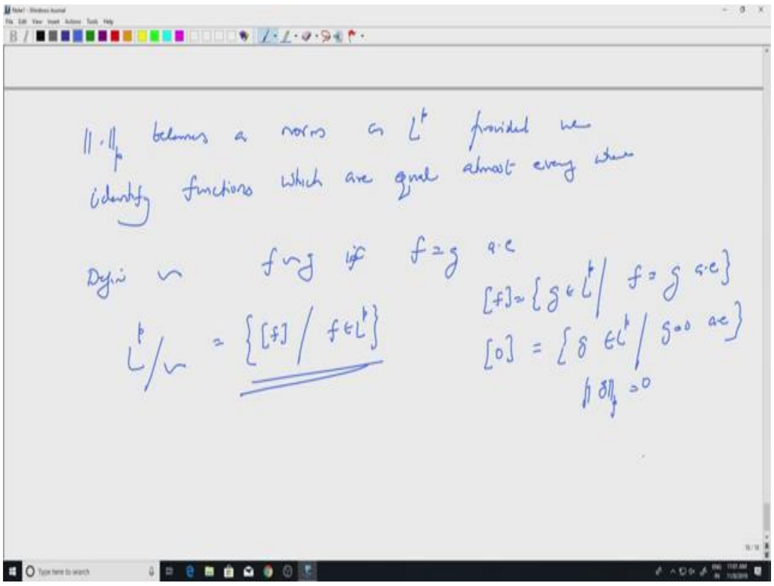

So, LP space. So, this becomes a norm, becomes a norm on LP provided we identify, identify functions which are almost everywhere, functions which are almost everywhere equal almost everywhere, functions which are equal almost everywhere. So, we want to identify 0 and functions which are 0 almost everywhere but, we can do this for all functions. So, what does it mean? So we define a new, define an equivalence relation f is equivalent to g if f equal to g almost everywhere and then you look at the space LP quotient it with that thing, which means that this will have equivalence classes. So, you decompose LP with respect to the equivalence relation.

So, this would be equivalence classes of f, f is an LP. So, that is the space actually we are looking at. On this space if I so, what is the equivalence class of f? So, this is all those g in LP which are equal to g or equal to f almost everywhere. So, equivalence class of the constant function 0 would be all those g in LP such that g equal to 0 almost everywhere but if g equal to 0 almost everywhere the LP norm of g is 0 because, when you integrate you do not distinguish between almost everywhere functions.

So, that is why g will be considered as 0. So, after this identification this becomes a genuine normed linear vector space but, we will not bother too much about this equivalence relation because for all practical purposes we can deal with it as usual function space. So, we will stop here. We have defined the function space called LP spaces and we have proved that the natural norm on that space is actually, actually a norm provided we identify functions which are equal almost everywhere.

So, the next step will be to study these spaces a bit more, in bit more detail, we will see that they are actually complete matrix spaces with respect to this norm. So, remember the norm gives you matrix. If I take f and g, the distance between f and g is the norm of f minus g. With respect to that matrix these spaces are going to be complete and if x a nice topological space like R N we will see that continuous functions with comeback support will be dense in this.

So, we will look at some examples in the next after proving the completeness, we will try to get some idea about these spaces by looking at some examples in the next lecture.