Measure Theory
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Lecture 33 - LP spaces

So, we go back to our abstract settings where we have a Space X, sometimes it will have a topology, locally compact (())(0:39) space, with some additional properties like sigma compact methods and so on like R n. And we will be studying function spaces on these spaces associated with the measure. So, given a positive measure we can define what are known as LP spaces. Our aim is to study those spaces. So, let us start, we will of course look at some examples to give you a good idea about it.

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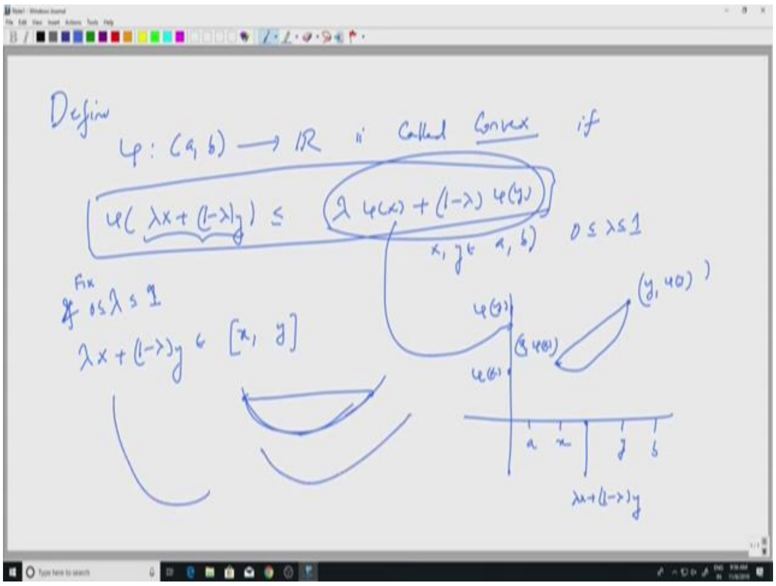

So, let us start, so before we go to LP spaces, we need some preliminaries. So we will define something. So define. So, I have a function phi from an interval a, b to R. This is called convex, if phi of lambda x plus 1 minus lambda y is less than or equal to 1 minus lambda into phi of x plus 1 minus lambda into phi of y. Well, this is true for all x and y in the domain of phi and lambda between 0 and 1. So, we have some inequality. So, let us see what that says.

Okay, what does this mean, what does this inequality mean? So, let us look at the graph of phi. So, let us say this is the interval a, b and I take some points there, x and y. So phi x would be some points on the y axis, so, it would be let us say this is x phi x and this is x y phi y. So, this would be x phi x and this is y phi y as points in R2. Well, if lambda is between 0 and 1, any point of the form lambda x plus 1 minus lambda y will belong to the interval x, y.

Okay, in fact, closed interval x, y because I am taking lambda equal to 0 and lambda equal to 1. So lambda equal to 0 will give me y, lambda equal to 1 will give me x, lambda between 0 and 1 will give me some points here. So the point, if I fix a lambda between 0 and 1, a point in the middle would be actually lambda x plus 1 minus lambda y. It is an arbitrary point inside x, y. So that also has some image. Now let us look at those inequality, what does it say?

Phi of the point in between x and y should be less than or equal to 1 minus lambda. Okay, so there is a slight mistake here. This is just lambda, convex combination. So, phi at lambda x plus 1 minus lambda y should be less than or equal to 1 minus lambda phi x into lambda y. The way I am writing is slightly confusing, so let me correct this. So, the lambda is with x, so lambda into phi x, 1 minus lambda is with y so I have 1 minus lambda here. So, I had written it the other way.

So, if I look at a point in between x and y, which is lambda x plus 1 minus lambda y, phi of that point should be less than or equal to lambda x plus 1 minus lambda y, lambda phi y. What does that mean? Phi x and phi y are two points and lambda and 1 minus lambda will make a point in between. So, if I join this line, the lambda phi x plus 1 minus lambda phi y will be somewhere here, because this is phi of x and this is phi of y and this point will be somewhere here.

And the inequality says that, if I look at the graph of phi that should be below the line. So, that is why it is called convex. So, convex graphs will be something like this. So, you take any two points on the graph, join the line, the graph in between should lie below the line. That is what convexity means. So, functions like this, functions like this and so on.

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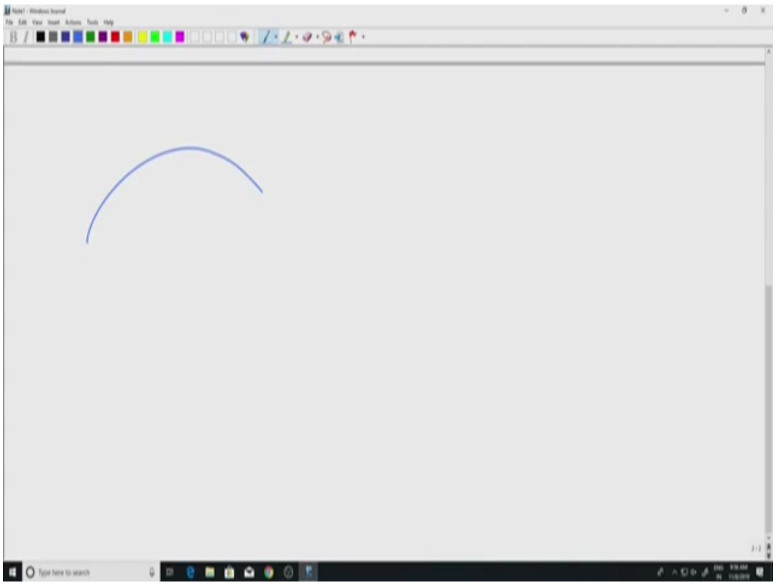

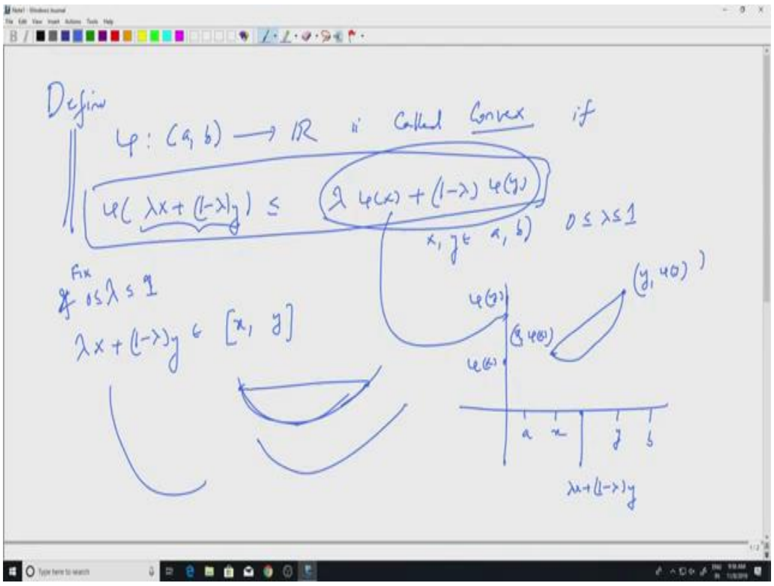

But functions like this will not be. So, this would be what is known as concave, which we will not talk about, we will stick to convexity. So take this as the definition of convexity and we use that. Now, it is possible to write it in different formations. So, let me give several exercises here to give you an idea about what is a convex function actually.

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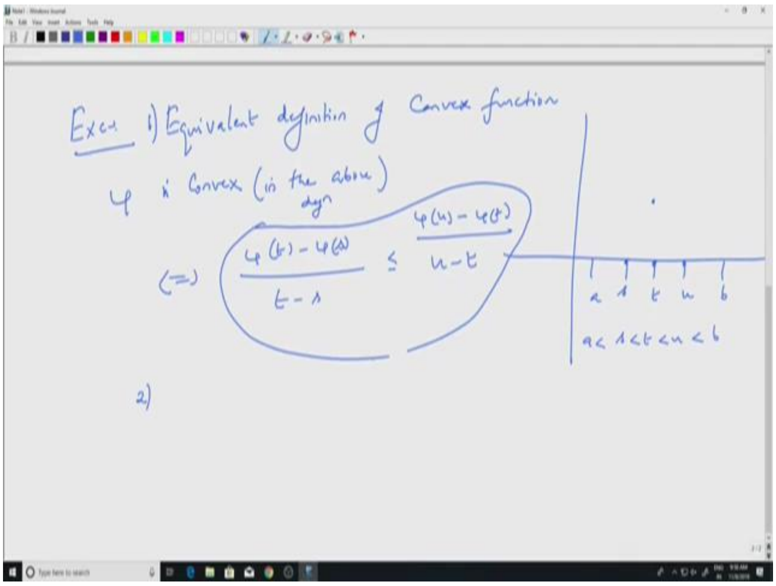

So exercise, so equivalent formation of, equivalent definition of convex functions. Well, what would be this? This is same, so you take points, so I have, so let me draw the picture again, I have the interval a, b. Take points in between so I can take s, let us take t and let us take u. So s is less than t less than u and of course everything lies in a, b, the interval a, b. The equivalent formation is phi is convex. So phi is convex in the above definition.

So in the above definition, if and only if phi satisfies the inequality, phi t minus phi of s, so I am taking t and I am thinking s and u. So I am looking at the image of phi t and image of phi s and you are looking at phi t minus phi s by t minus s. So, this is what you form for making the derivative. This is less than or equal to, if you go to the other side, phi u minus phi t by u minus t. So, this is same as saying phi is convex. So, the earlier inequality if you rewrite, you will get this.

So, let us look at second one. Suppose, so if you look at this quotient, so you know how derivatives are formed by taking this quotient and taking the limit.

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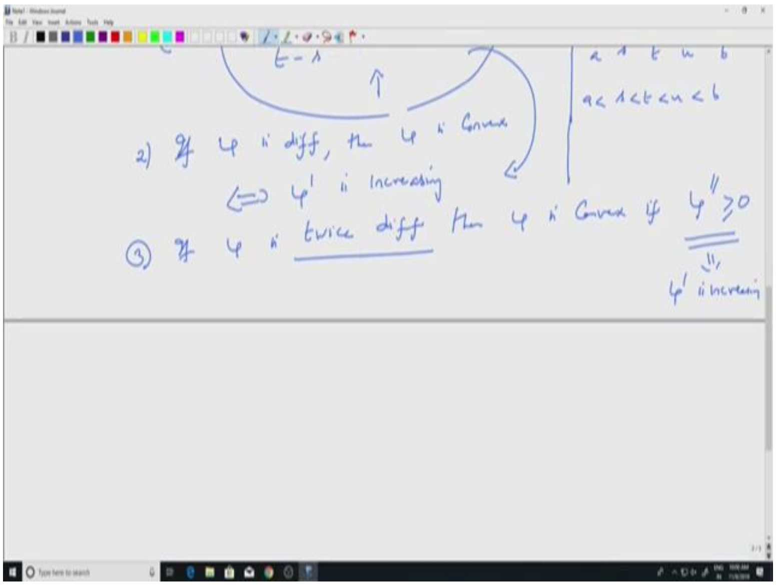

So, this will tell you that if phi is differentiable, if phi is differentiable…So we are not assuming anything on phi, phi is just a function which satisfies this inequality, that is the assumption. Suppose phi is differentiable, then because of this inequality, then phi is convex. So convex you can take either the equivalent definition or the definition earlier if and only if the derivative is increasing, phi prime is increasing. So, it is an increasing function. So, this inequality essentially says that.

If I take the limit of appropriate t going to s and t going to u and so on, you will see that that is the inequality you get. So, if I have a function whose derivative is increasing, then it is convex. Well, this immediately implies that if phi is twice differentiable, so 2 times differentiable, then phi is convex if phi double prime is positive, greater than or equal to 0, so that will do.

Well, that is obvious from the second exercise because if phi double prime is positive, this immediately implies that phi prime is increasing. A function is increasing if its derivative is positive. So, that tells me that phi prime is increasing and exercise 2 tells me that phi is convex. So, this is one way of checking something is convex or not. Most of the functions we deal with are smooth enough so you can differentiate them. In fact, somewhat the converse of this is also true but we will not get into that.

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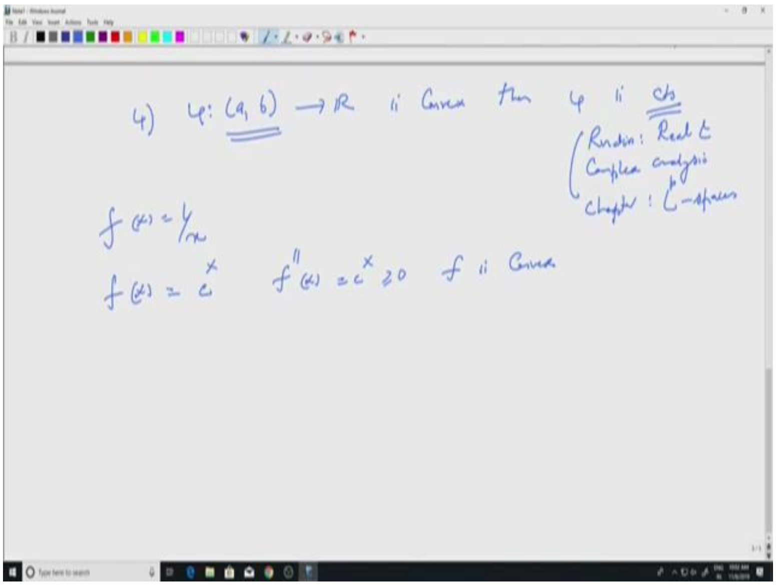

And more importantly, if phi is actually convex, if phi is convex, then phi is continuous. So, convex functions are continuous on open interval, so the open interval is important. You can, so if you look at the book by Rudin, Real and Complex Analysis, so this is the chapter on LP spaces, that is what we are looking at now, chapter on LP spaces, I think it is chapter 3 or 4.

So, phi is, if it is a convex function, then it is already continuous. So, it is like in fact one can prove that it is differentiable almost everywhere and things like that. So, but we do not need any of those. We simply look at the convex functions whenever it is necessary and use this test, you know, if I take some function in a domain, if I can differentiate I simply look at that. So, for example, you can look at the function f of x equal to 1 by x and see if it is convex or not.

So, you look at or f of x equal to e to the x for example, that is easy. If you draw the graph, you will see that it is convex, but you can also check that the second derivative is e to the x itself which is positive, so f is convex. So which functions are convex can be sometimes easily checked through taking derivatives which we will use these techniques sometimes.

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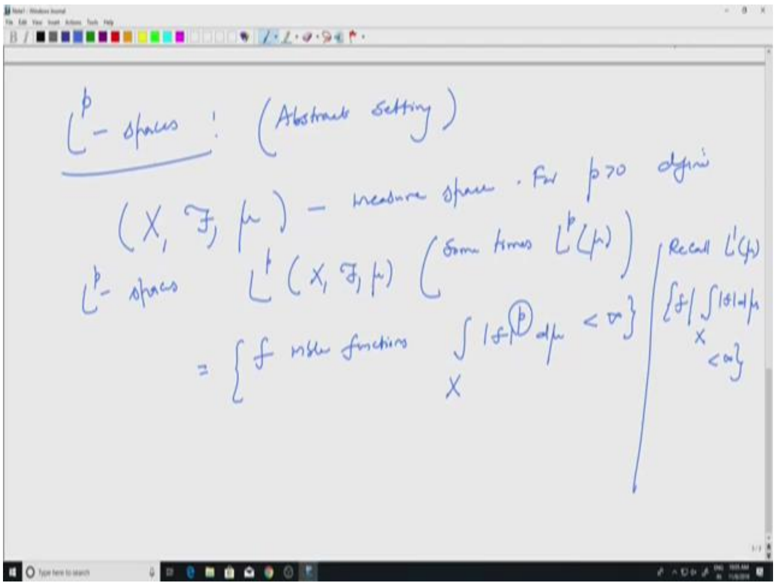

Okay, so, now I can start with, so let me rub off this part. We will start with LP spaces, okay. So, these are function spaces, so these are spaces consisting of various functions, which we will, we will study them, so LP spaces. So, the functions, convex functions are defined on intervals. Now we are going back to our abstract settings, so abstract setting and of course, all this will be applicable to R n. So, I have a space X, so this is our usual triplet X, I have a sigma algebra and I have measure mu, this is my measure space.

So, right now there is no assumption on capital X, except that it is a space and mu is a positive measure. So, for p greater than 0 define LP spaces, so strictly speaking I should say LP of X, f, mu, sometimes we will denote it by, sometimes simply LP of mu when the space and the Sigma algebra are all understood, we simply look at LP of mu. Well, what is this? This is the collection of measurable complex valued functions, measurable functions whose LP power is integrable.

So you look at integral over X, you look at mod f to the p, so f is measurable, so mod f to the p is measurable and I can integrate this against a measure mu, because everything is positive here. And you want this to be finite. So, recall that we defined L1, recall L1, L1 of mu was collection of functions, such that the mod f d mu was finite. And that of course agrees with our LP space when p is 1. When p is 1, we are simply looking at mod f.

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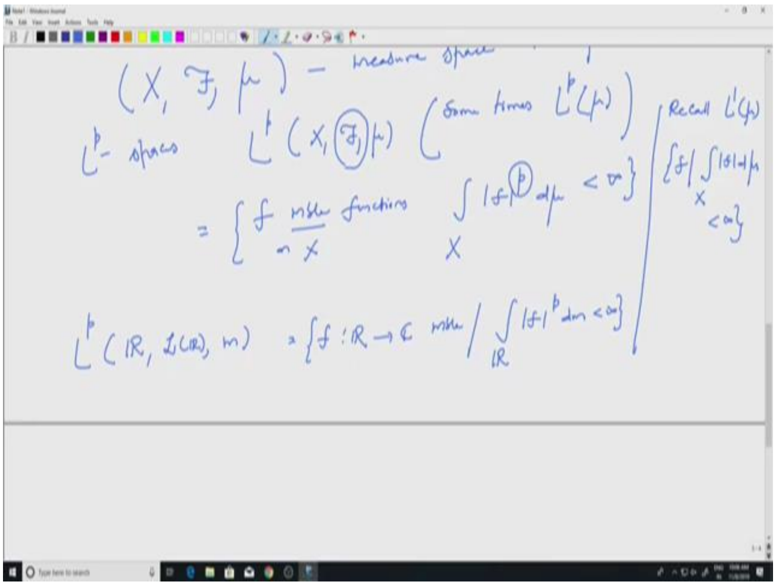

So, this space is defined as a collection of functions on X, so measurable functions on x. So measurability, remember, depends on the Sigma algebra you are looking at. So, you can look at LP of let us say the space R with the Lebesgue sigma algebra and the Lebesgue measure. Well, what would be this? You will be looking at all those measurable functions from R to C or sometimes it can take infinity values as well, measurable such that integral over R mod f to the p dm, that is our measure, is finite.

So this is a collection of function. Of course, we do not know how big the collection of functions or the LP spaces, it will all depend on how the space is, how the measure is and all that.

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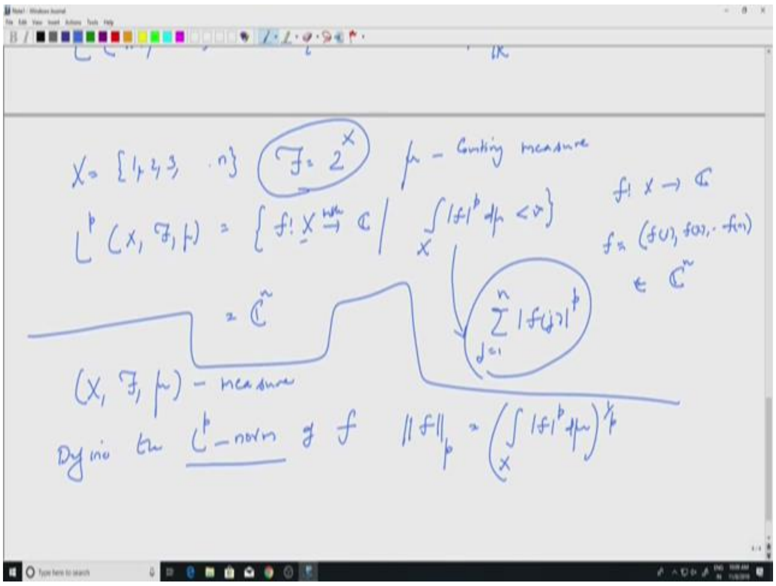

So let us look at some trivial examples before we study the spaces. So let us take x to be a finite space. So that is the easy case always, 1, 2, 3 etcetera. n, so it is a finite space. The Sigma algebra is of course the power set and mu is the counting measure. So, these are examples, which we looked at very early in the course and easy examples to understand. So, what do I mean by LP space? So I have LP of x f mu. So what do I do? I look at all those functions from X to C measurable.

Well, the Sigma algebra is the power set. So any function is measurable such that the integral of mod f to the p d mu is finite. Well, what does that mean? What is a function on x? So, f from x to C, then f is identified with the n tupple f 1, f 2 etcetera. F n. So, these are complex numbers. So, this space can be identified with, so these are all elements of C n or R n, if they are real valued. And what is this quantity?

Well, you have counting measure, so this is this becomes a summation we have seen that, so this is simply mod f j to the p, j equal to 1 to n. And this has to be finite, of course this is a finite number because everything is complex valued. So, this is simply C n with this condition. So, this condition give us what is known as a norm. So let us define that. So maybe I should have done this before the examples. So but let us look at this.

So, I have a Space X, I have a sigma algebra and I have a mu, measure space, this is general, we are not looking at the example now. Define the LP norm, so remember, we defined what a norm is, I will repeat that of f, LP norm of f to be, so f is a measurable function, to be….So it is denoted by this symbol LP norm f equal to integral over x, mod f to the P d mu. So if f is an LP, this is finite and you take the power 1 by p. We will see why the power 1 by p is necessary.

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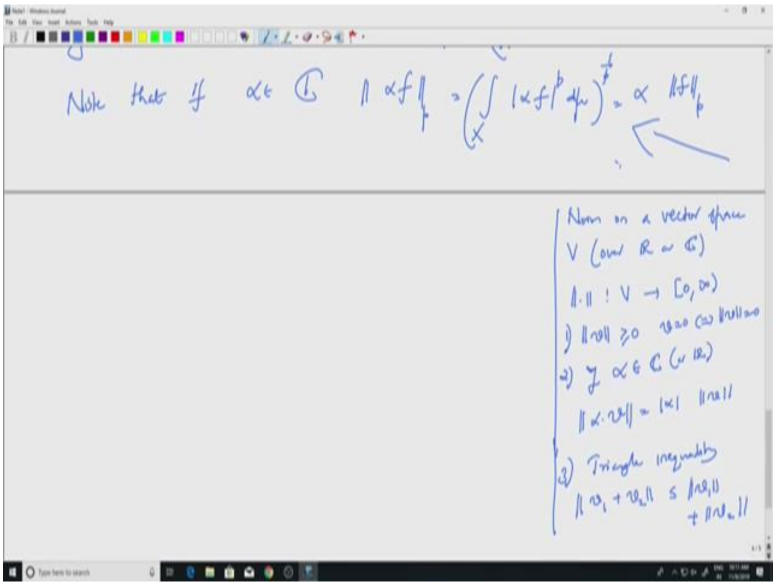

So, before I go to the examples, let us just notice one property. So, note that if alpha is a complex number, what is the norm of alpha f? So, alpha f, alpha is a complex number, you multiply f with that, you will have x modulus of alpha times f to the p d mu to the 1 by p. But alpha is a constant. So, that comes out of the integral as alpha to the p and there is a 1 by p. So, this is just alpha times LP norm of f. So, this is one of the properties of the norm.

So, let us, let me recall the properties of the norm. So norm on a vector space V. So the vector space can be over R or C, it can be more general but we will not bother about that. So what is a norm? So norm is a function from V to close 0, open infinity, so it is a positive function, which satisfies 3 properties. One, norm of v any vector is greater than or equal to 0, and V equal to 0 if and only if it is norm is 0.

Two, if alpha is in the field or R, if the vector space is real space, then alpha times v. So that makes sense because it is a vector space, the norm of that is mod alpha times norm v. So that is the property here we have. And three is the triangle inequality, norm of v 1 plus v 2 is less than or equal to norm of v 1 plus norm of v 2. So in the LP norm, these two are sort of easy to see, the first two properties, one of them we have already checked, we will see the third property.

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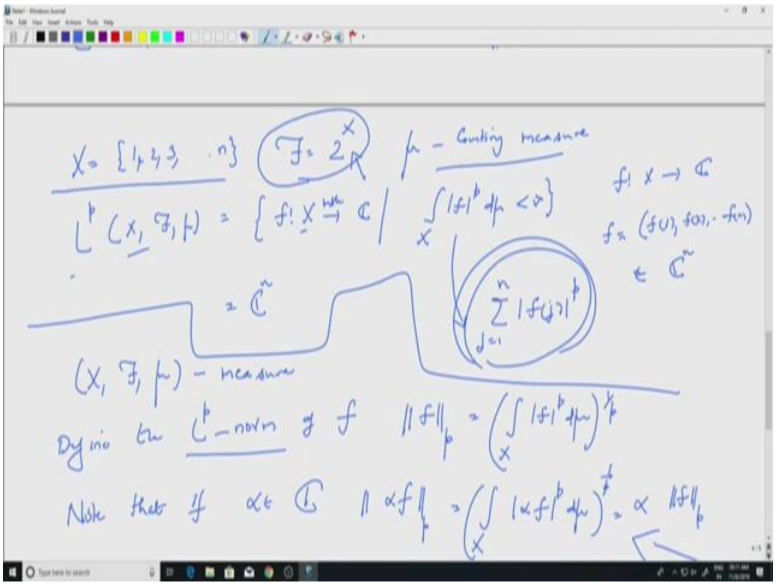

But let us go back to our example. I have x to be equal to 1, 2, 3, n, I have the Sigma algebra, which is the power set and I have the counting measure. The LP space is simply all functions on x, because this is always finite.

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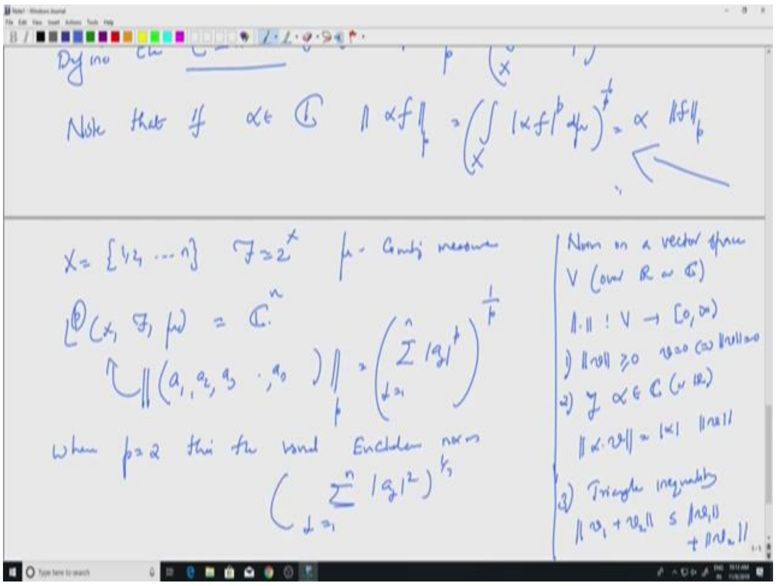

But let us see what is the norm. So if I take, so instead of writing, so let us write the sets again. So I have the finite set and the power set as the sigma algebra and mu counting measure. I can define, so if I look at LP of x f mu, well, this is same as the space C n, because functions are simply n tupples, but the difference is with the LP norm.

So, if I take the p, if I take the element here, so let us call it a 1, a 2, a 3 etcetera a n, I write it as a tupple, so that is the element here. What is the LP norm of this? Well, LP norm of this is summation mod a j to the p, j equal to 1 to n to the 1 by p. So, when p equal to 2, this is the usual Euclidian now, when p equal to 2, this is the usual Euclidean norm which you are familiar with, Euclidian norm. So summation j equal to 1 to n mod a j square to the half.

So the p norm generalizes the L 2 norm. Of course, we have to show that it is a norm, we have done the triangle inequality for example, for LP norm. So we will do this in general. So let us start with some lemma.

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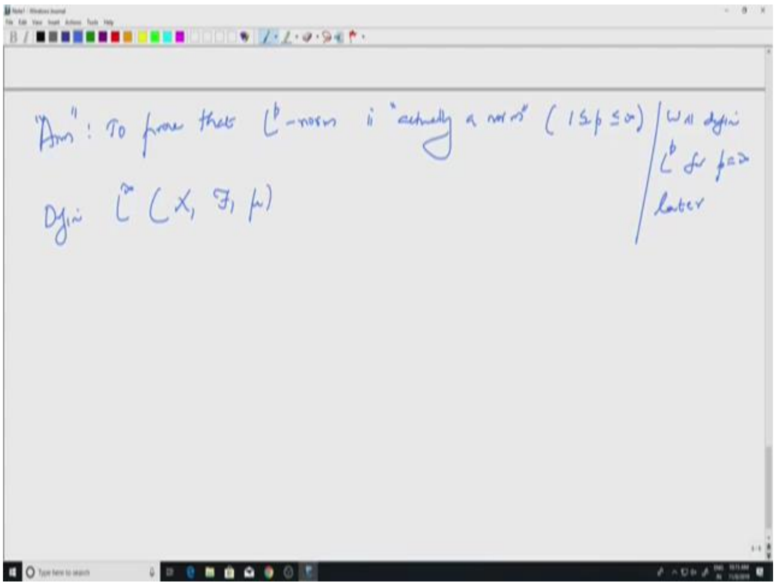

So aim is to prove that LP norm is actually a norm. So I will put this in apostrophe, so we will see that it was a norm under appropriate condition. But this will happen only for p between 1 and infinity. So p equal to infinity is included, I have to define. So we will define LP for p equal to infinity later. So p equal to infinity there is a slight problem, because we are taking the power of f with respect to p. So p equal to infinity does not make sense. So it has to be appropriately modified.

I will motivate that and then define. Okay, so let us, maybe let us do that first. So define L infinity of X, f, mu….Well, what should be L infinity?

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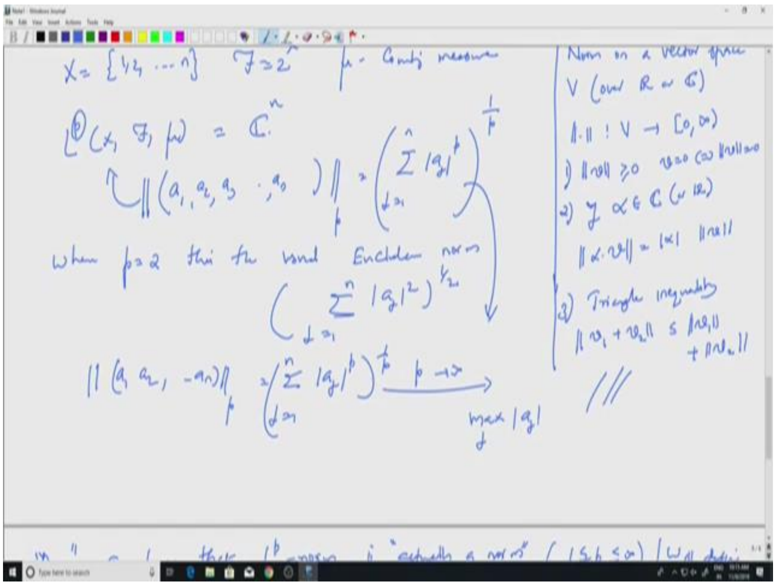

So, let us look at the trivial example we have of Euclidean spaces. So that is the LP norm. So a 1, a 2, if I look at it up to a n, then this is summation j equal to 1 to n. Mod a j to the p to the 1 by p. So, what will happen to this as p goes to infinity? So, if you compute the limit of p going to infinity, you will see that this goes to maximum of over j mod a j. So, that is the supremum of the values mod a j. So, that is what we want to define L infinity as.

But you see this is a counting measure, so there are no sets of measure 0. So if there are sets of measure 0 and the function takes the value infinity there, then the maximum will be infinity, we do not want that. So we will throw away sets of measure 0 and then take the maximum. So, how do we do this?

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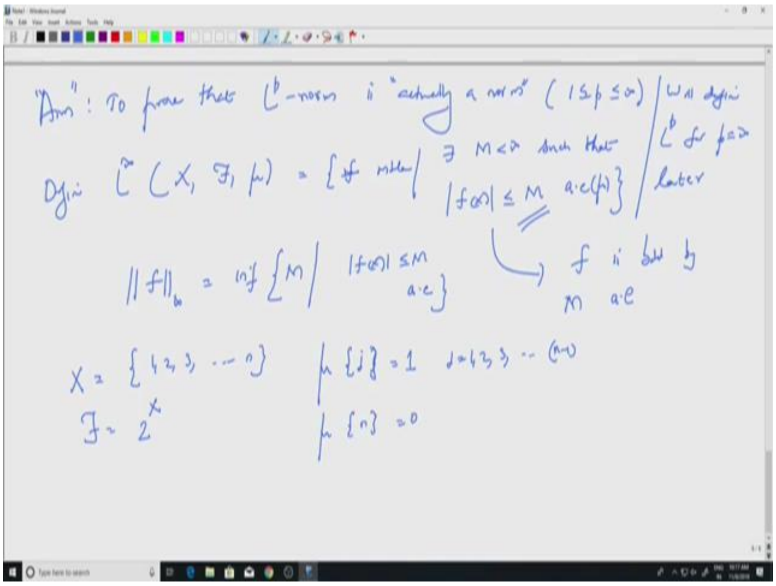

Well, define L infinity to be the collection of measurable functions, such that there exist some M less than infinity such that modulus of f of x is less than or equal to M almost everywhere. So f is bounded by, so this says that f is bounded by the constant function m almost everywhere. So, there may be a set of measure 0 where f is bigger than m, if it will be infinity there, it does not really matter. But what should be the L infinity norm? Well, this is the infinium of all such m's. So, you take the smallest value m which works.

So, m such that mod f of x is less than or equal to m almost everywhere. So, that is what we mean by L infinity norm. Let us look at some trivial example, it will be very clear. So if I take x to be again 1, 2, 3, etcetera, n and instead of counting measure, let us define mu of the Singleton j to be 1. So I will take the Sigma algebra to be the power set of course. Mu of j equal to 1 if j equal to 1, 2, 3, etc up to n minus 1. So mu of n is left out, so I put that to be 0. So there is a set of measure 0, that is all that is all I wanted.

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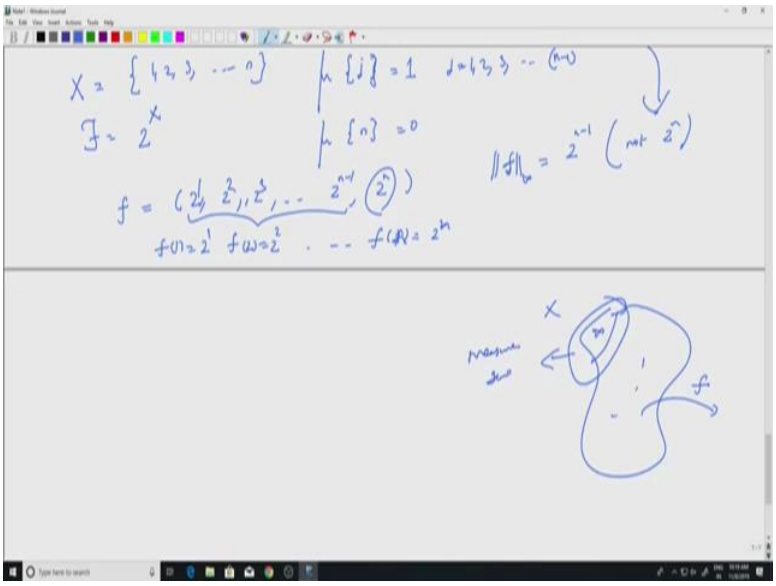

So in this case, what how will the L infinity norm look like? Well, the set of measure 0 will not matter. So, if I take f to be, so f is a tupple, so f I will take to be, let us say 1, 2, 4, 6, etcetera. Or 2 to the n let us say. So, this is, I will take 2 to the 1, 2 to the 2, 2 to the 3 etcetera up to the n minus 1, 2 to the n. So what does this mean? This means that f of 1 equal to 2 to the 1, f of 2 equal to 2 to the 2 etcetera. This is my function, f of 2 to the n equal to, f of n equal to 2 to the n.

So, what would be the L infinity norm of f in this case? Well, I have to look at the bound of f, of course, everything is bounded by 2 to the n. So, f is of course less than or equal to 2 to the

n. But the value of f on 2 to the n is a set of measure 0, I can forget that. So, I can simply look at this and take the supremum. So, this is simply 2 to the n minus 1, it is not 2 to the n. Because that set of measure 0 does not play a role when I look at this definition.

So, in general, you can have, so x could be some set like this and you may have a function f defined there and there may be a portion where f is infinity. But if this is contained in the set of measure 0, then this will not matter. Okay, we look at only f here and take the supremum. So, that is the L infinity norm. So, when you deal with L infinity norm, just see that it can, sets of measure 0 you can discard essentially.

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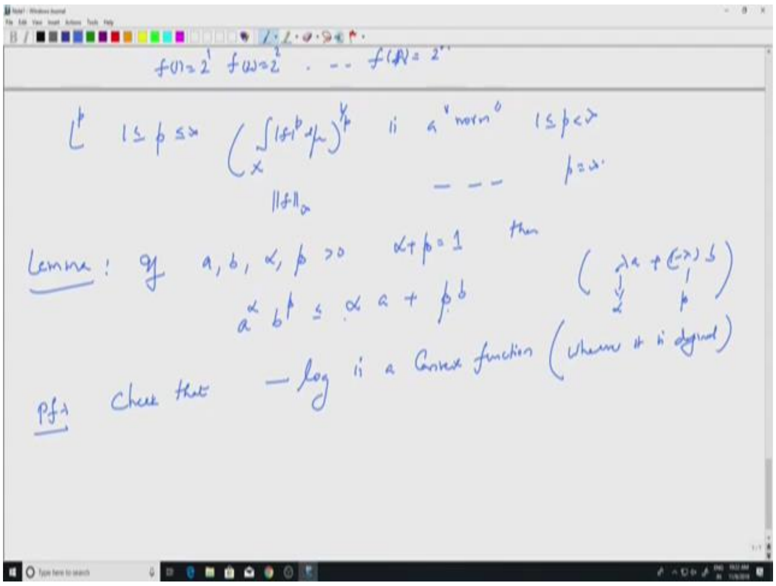

So, we have defined LP and L infinity norm. So, our aim is to prove that LP is a norm. So, LP for 1 less than equal to p less than equal to infinity, the norm is, so we will prove that LP norm is a norm. So, integral over x, mod f to the p d mu to the 1 by p is a norm. So, norm with some, I will explain that for less than equal to infinity and we have L infinity norm equal to p equal to infinity. This is what we want to prove. So, let us start with the lemma.

It is a technical lemma to prove that it satisfies certain inequalities, which is necessary to prove triangle inequality. So, this is an easy lemma. So, if a, b, alpha, beta, they are all positive numbers with alpha plus beta equal to 1. So, this is like the convex combination, then we have a to the alpha, b to the beta is less than or equal to alpha a plus beta b. So, there is a convex thing here, alpha plus beta is 1. So, this is like lambda a plus 1 minus lambda b, this is my alpha, this is my beta.

So, you will see that convex function has to come in, so let us prove this. Proof, well, check that minus of log is a convex function. So, you can differentiate wherever it is defined. So it is a convex function wherever it is defined. Well defined. So you can take the second derivative and see that it is…..the first derivative is minus 1 by x, second derivative is minus, 1 by x square which is positive. So, that is enough.

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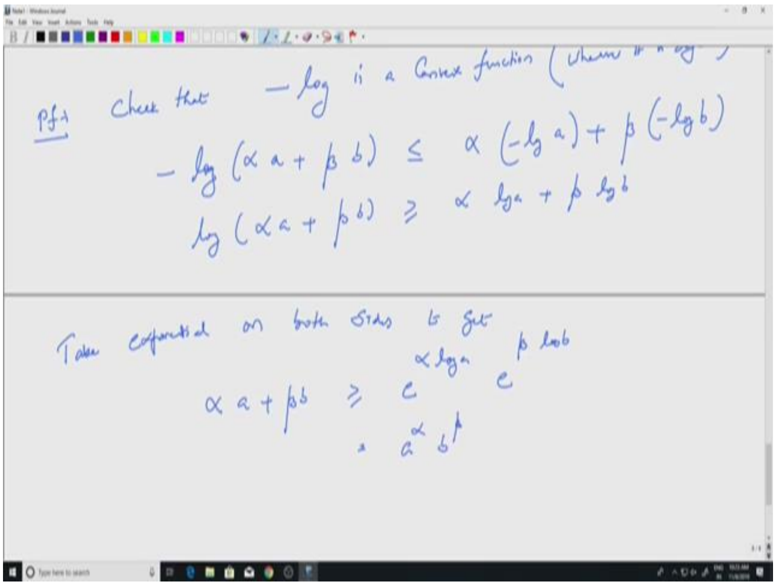

So, because it is convex, we get an inequality. What is the inequality? Minus log alpha a plus beta b is less than or equal to, so this is like phi of lambda x plus 1 minus lambda y, that will be less than or equal Lambda, here it is alpha, phi of a, the first, so you evaluate the function at the first element plus 1 minus lambda is beta here and I have log b. So, minus you can multiply, you will get the inequality the other way. So log of alpha a plus beta b is greater than or equal to alpha log a plus beta log b. And then you take powers.

So take exponential on both sides, take exponential on both sides, we will get, to get, well, what do you get if you take exponential on both sides? On the left hand side you get alpha times a plus beta times b, which is greater than or equal to e to the alpha log a times e to the beta log b. Which is same as a to the alpha and b to the beta.

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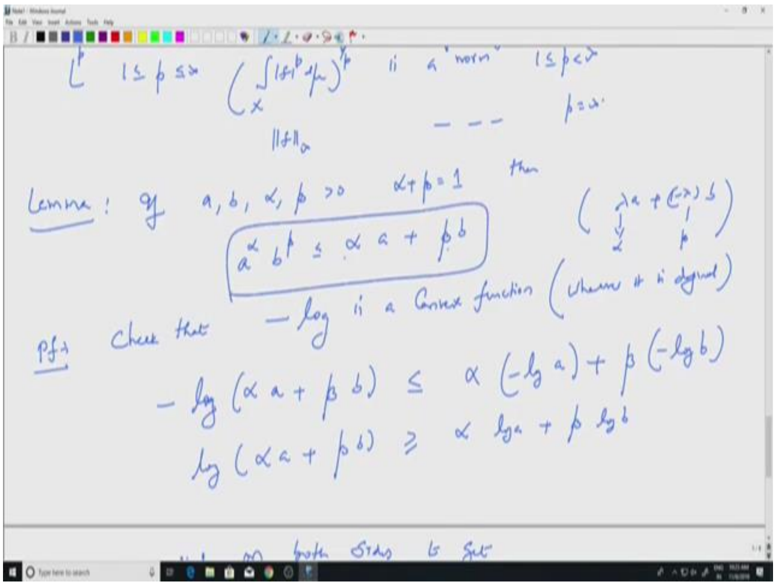

That is the inequality we wanted to prove, we wanted to prove this one. And that is what we have done here. Okay. So we will stop here. We just defined LP spaces, including p equal to infinity, L infinity space. So the L infinity norm is slightly tricky, you need to understand that. It simply says that the function is bounded almost everywhere. So there is a set of measure 0, on the compliment of that, function is actually bounded, that is enough for the function to be in L infinity.

Our aim is to show that these quantities, the LP norm which we defined is actually a genuine norm, modulus at n things, which I will explain. So, we have started with the technical lemma. In the next session, we will prove that these are actually norms. Okay.