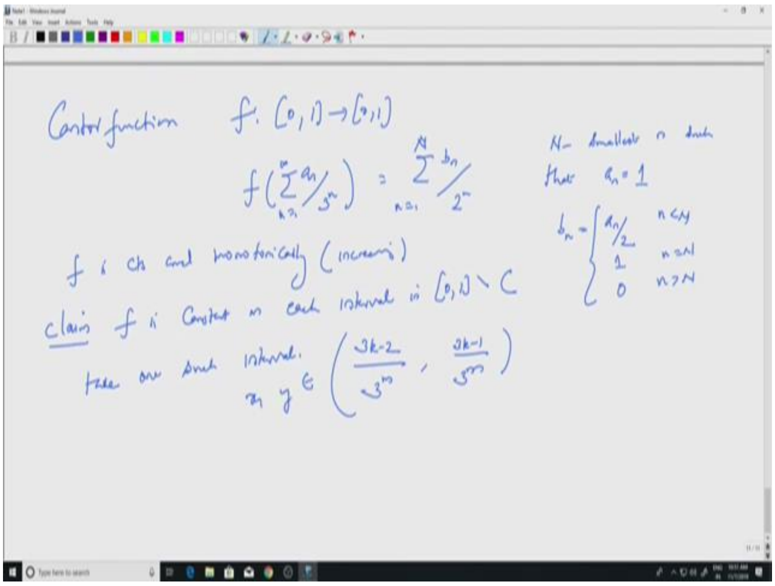
Measure Theory
Professor E. K. Narayanan
Department of Mathematics
Indian Institute of Science, Bangalore
Lecture 32 - Lebesgue Set Which is Not Borel

Okay, so we defined the Cantor function and we saw that it is continuous and monotonic. We need to complete the proof that it is a constant on the intervals which are thrown out during the construction of the Cantor set. So we just looked at one example where in the first step of the Cantor set, we throw out the interval one-third to two-third. We just showed that f is actually a constant in that interval. But it is true in general, so we will, let me justify that first.

(Refer Slide Time: 1:07)

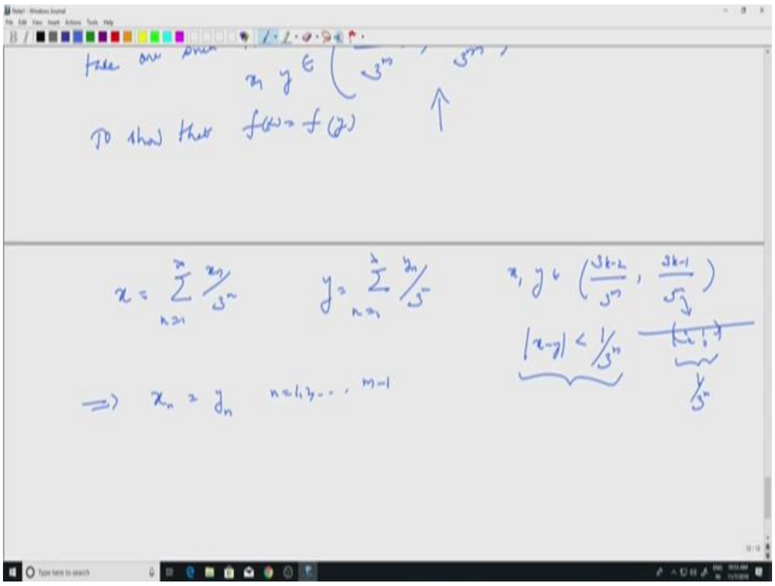


So, more generally, so let me recall Cantor function first, and then Cantor function so this was f from 0, 1 to 0, 1. How was it defined, remember the left hand side has ternary expansion right hand side has binary expansion. So, you look at f of summation a n by 3 to the n, n equal to 1 to infinity and that is defined to be summation b n by 2 to the n, n equal to 1 to capital N, where capital N was the smallest n, such that a n equal to 1. If you do not hit 1, you keep dividing by 2. So, otherwise b n is simply a n by 2.

And so b n is defined to be a n by 2 for n less than N, 1 for n equal to N, so, once you hit 1 you put 1 and then 0, that is my definition. So, this is the Cantor function, we just saw that f is continuous and monotonic. So monotonically increasing, it is not strictly increasing, monotonically increasing. Okay. So our claim right now is that f is constant on, f is constant on each interval in 0, 1 minus C, the Cantor set.

So, we take 1 such interval, so take 1 such interval, okay. So how does it look like? Well, it will look like 3 k minus 2 by 3 to the m for some m and 3 k minus 1 by 3 to the m. So it will have length 1 by 3 to the m, there are 2 to the n minus 1 intervals at the level n or at the step n of length 1 by 3 to the n, right, that is what we know. Okay, so let us take two points, x and y here in this. We want to show that f of x equal to f of y.

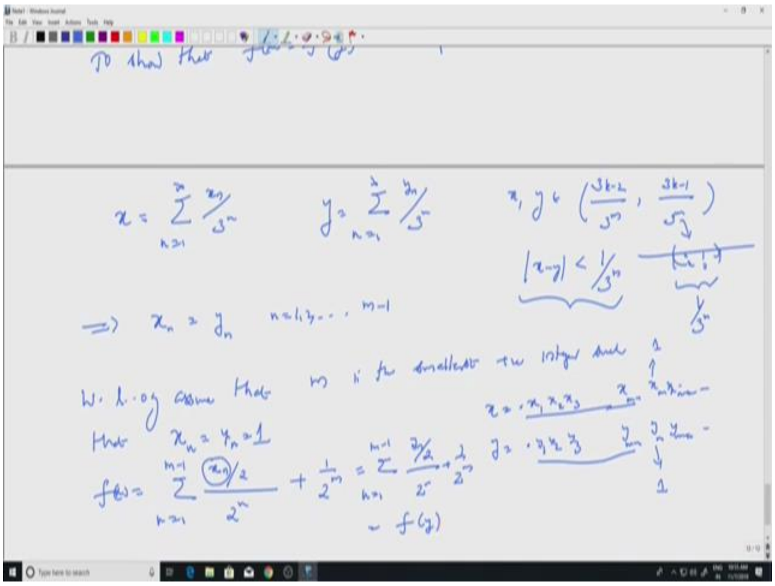
(Refer Slide Time: 3:32)

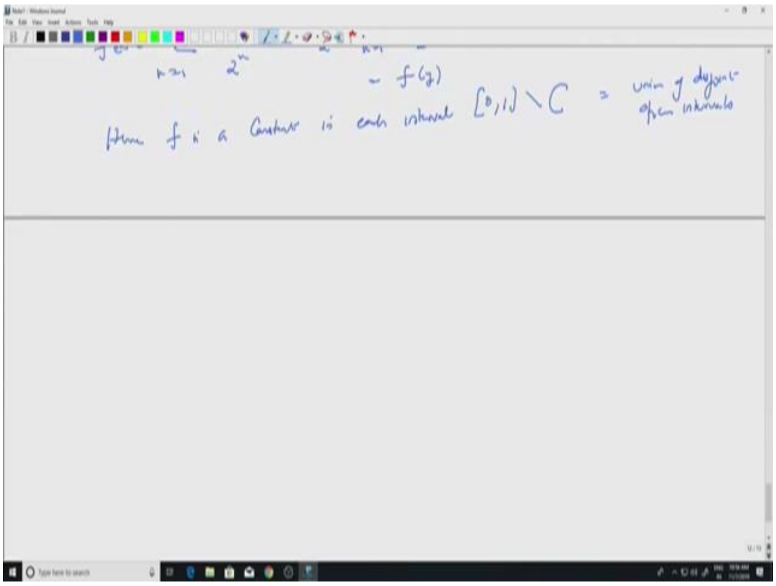


So to show that f of x equal to f of y, that is what you mean by f is a constant there. So, let us look at the expansion of x and y. So x can be written as, so x has a ternary expansion. So I write x as x n by 3 to the n, n equal to 1 to infinity, y equal to summation n equal to 1 to infinity, y n by 3 to the n. Remember xn's and yn's are either 0, 1 or 2. But they are in this interval. So, x and y are in the interval 3 k minus 2 by 3 to the m, 3 k minus 1 by 3 to the m, and so mod x minus y is less than 1 by 3 to the m.

Because they are in the interval of length, this interval has length 1 by 3 to the m, if you look at this interval. So, x and y are here, then of course, the distance between them is less than 1 by 3 to the m. Okay. Well, but we discussed this earlier, this means that the first m places x and y will have the same numbers. So, this tells me that, so this implies x n equal to y n for n equal to 1, 2, 3 etc, up to m or m minus 1.

(Refer Slide Time: 5:12)





So how will you write f of x and f of y? So, f of x is given by summation, well, okay so we can assume something here, otherwise it is well, it is the same proof but just to be clear, so without loss of generality assume that, assume that m is the smallest positive integer, such that x n equal to y n equal to 1. So remember, our definition is that you first hit 1 and then put 0's. So I am assuming that the first m minus 1, so I am writing x as point x 1, x 2, x 3, etc etc x m minus 1, x m, x m plus 1, etc. So this I am assuming to be 1.

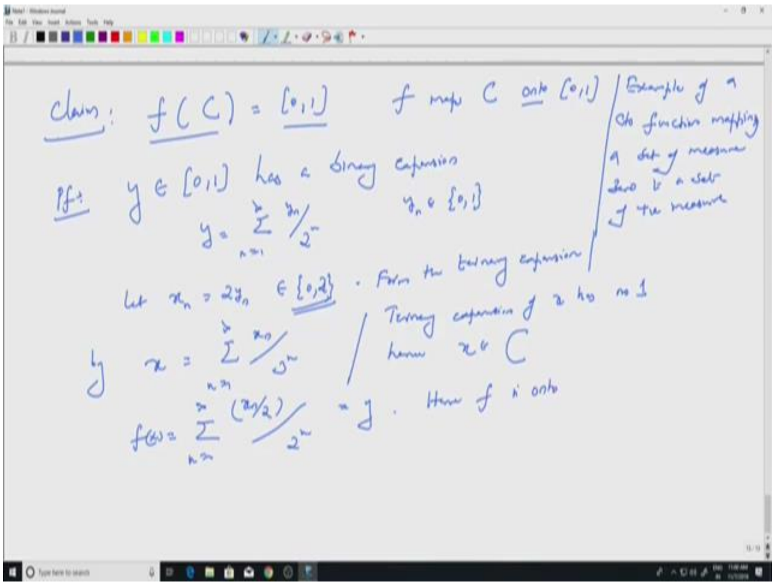
Similarly for y, right, point y 1, y 2, y 3, y m minus 1, y m, y m plus 1 and so on. These first m minus 1 places are same. So I am assuming this to be 1, if not, you can, you know you will hit 1 early, but then these places are same, so at the same place for x and y, so you will get the image to be same. So that is easy to see. In this particular case, when x m and y m are 1, well, what is f of x?

F of x is nothing but, well, you look at the first few places, they are all different from 1, so you divide by 2. So this is simply n equal to 1 to m minus 1 x n divided by 2 divided by 2 to the n, remember the image has binary expansion. Plus x m is 1, so in the image it is 1, but it is a binary expansion so it is 1 into 1 by 2 to the m. Then 0’s, okay. And this is, of course, equal to, because the xn’s are equal to yn’s.

So I can write this as n equal to 1 to m minus 1 y n by 2 divided by 2 to the n plus 1 by 2 to the m, which is precisely f of y, because x and y have the same property and so the images are same.

So you take any interval of this type, f of x is a constant there. So, hence f is a constant in 0, 1, well in each interval. So if I change the interval, the constant changes, but it is a constant in each interval. So constant in each interval outside the Cantor set, so 0, 1 minus C. So, remember this is a union of intervals. So union of disjoint open intervals, we will use this. These are the intervals which are thrown out in the construction of the Cantor set, they add up, their lengths add up to 1, that is why the Cantor set has measure 0.

(Refer Slide Time: 8:41)



Next claim, f of C, so that is the range of C under the map f, this is 0, 1. So this f maps Cantor set on to the closed interval 0, 1. So this might look weird, but f is a continuous function and it is mapping a…So, this is an example of a, so example of a continuous function mapping a set of measure 0 to a set of positive measure. So it is very important to realize that continuous functions need not map set of measure 0’s to set of measure 0. Okay, that happens if f is Lipschitz.

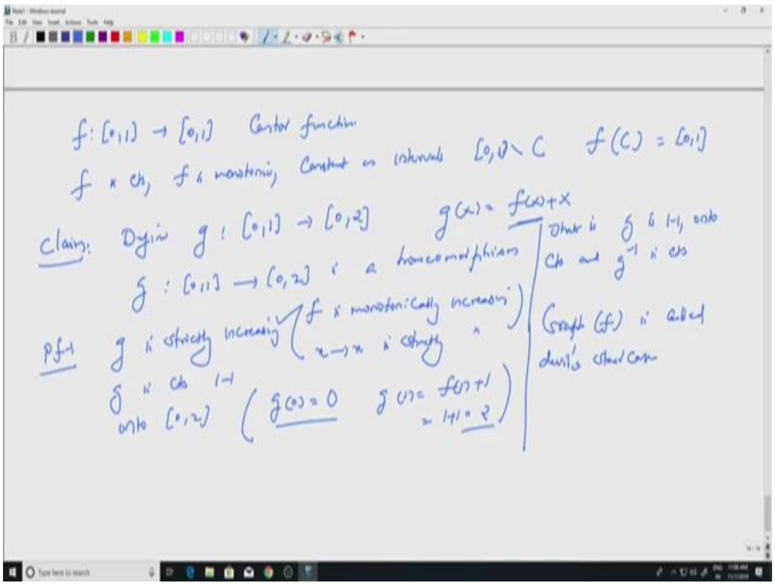
So, you have seen this for linear maps, which is, because linear maps are Lipschitz. Anyway, we will see some of those properties later when we look at differentiability properties of functions. Okay, so now the claim is that the range of f is under, I mean the image of the Cantor set is the whole interval 0, 1. So, let us see, again this is not very difficult, if I take some number inside 0, 1, well, it has a binary expansion.

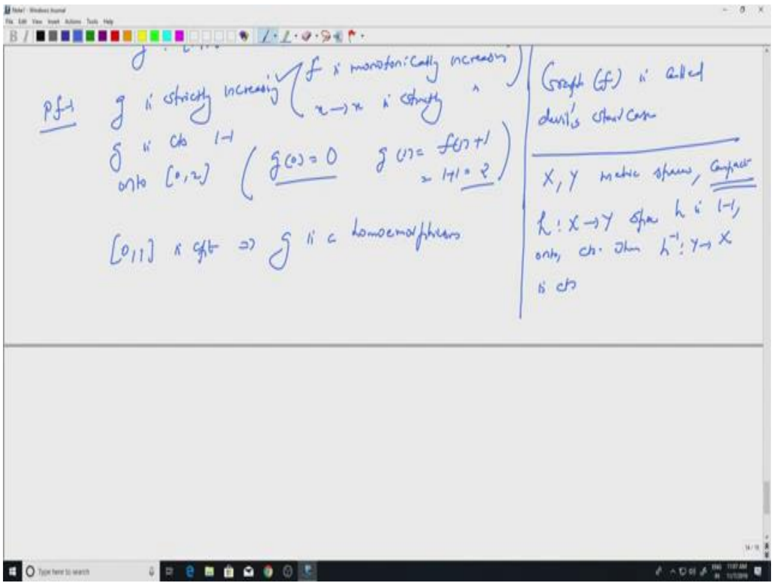
So, I need to find a point inside 0, 1 inside the Cantor set whose image is y. So, I have a binary expansion, so I write y as summation n equal to 1 to infinity y n by 2 to the n. But what are y n’s? y n’s are simply inside either 0 or 1, they are 0 or 1 because of the binary expansion. So, then it is clear what you should do to get the pre-image. So, let x n equal to 2 times y n. So they will be inside 0 or 2, this set. And then you form the ternary expansion using that.

So, form the ternary expansion by… So, you define x to be summation x n divided by 3 to m, x equal to summation x n by 3 to the m, well, what can you say about xn’s? xn’s are inside 0, is either 0 or 2. That means the ternary expansion of x, so ternary expansion of x has no 1, the digit 1 does not appear in the ternary expansion. Hence, this is a point in the Cantor set. Remember the Cantor set is also described as all those points whose ternary expansion does not have 1 in it.

So, this is a point in the Cantor set and f of x of course is equal to summation n equal to infinity x n divided by 2 because x n is never 1, so x n divided by 2 by 2 to the n, the binary expansion, which is precisely our y. So hence f is 1. So, we have proved several claims, of course, we are not finished, we still need some more properties of f, but let me recall the claims we did.

(Refer Slide Time: 12:49)





So f was defined from 0, 1 to 0, 1 Cantor function, we proved that f is continuous, f is monotonic, constant on intervals outside the Cantor set, so 0, 1 minus C. And we just proved that the image of the Cantor set f of C under the Cantor function is actually equal to 0, 1. Okay. So now next claim. So these are the claims we have proved, next claim, define g, g is another function from 0, 1 to 0, 1. Well, 0, 2 in fact, let us be clear here, taking 0, 2.

Well, definition is simple, g of x equal to f of x plus x, you simply add x to it. Well, the advantage of adding x is the function x is a strictly increasing function, continuous. So if I add x to f of x, it becomes strictly increasing. So claim is that g is, g from 0, 1 to 0, 2 is a homeomorphism, homeomorphism. Well, what does that mean? That is g is 1-1, 1-2, continuous. So, if it is 1-1 and 1-2 I have an inverse and g inverse is continuous. So, that is what you mean by homeomorphism.

So, what is the proof? Well, firstly, g is strictly increasing, because f is monotonic, so it is increasing, at times it can be a constant. So it is like, it can be like this. And then it can be constant like this in turn. It is actually like a staircase, so whatever I have drawn is not correct, f is called the Devil's staircase actually, graph of f is called Devil's staircase. Well, if you draw the picture, you will see that.

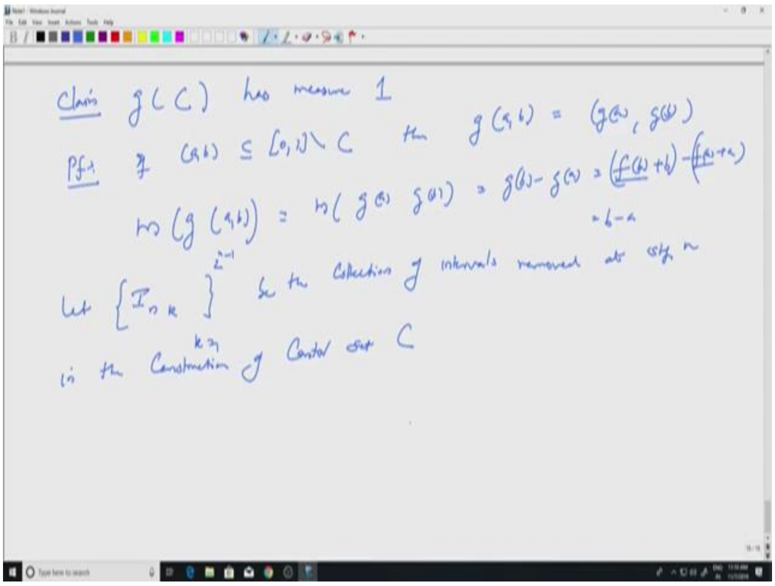
So it is like a staircase because it is a constant in various intervals. So it is 0, 1. So in 1 by 3 to 2 by 3, it is a constant. But in between, there are some points where it is increasing. And then it becomes a constant and so on. So it is difficult to draw because Cantor set has no interior. So there is no interval where I can draw something which is increasing, okay. Alright, so but because it is monotonic and x is increasing, so this follows because f monotonically increasing.

And the function x going to X is strictly increasing. I am adding them. So I will get g to be strictly increasing, and it is continuous g is continuous, so it is 1, it is strictly increasing, so it is 1-1, and of course it is onto to 0, 2, because you are adding x, f of x is going all the way from 0 to 1 and you are adding x. So it will, I mean, well, it has to be justified, but this is, if you look at g of 0 that is 1 and g of 1 equal to f of 1 plus 1 which is 1 plus 1 equal to 2. So, f of 1 is 1, so that you can check. So, g takes the value 1 and sorry, g 0 is 0 not 1. And it takes the value g 1 equal to 2.

So, it has to take the values between 0 and 2 and it is increasing so it cannot go outside 2. Now it is strictly increasing continuous 1-1 onto function and 0, 1 is compact. So that implies, so this is a trivial exercise.

0, 1 is compact implies g is a homeomorphism. So, this is generally, I can put this as in a very general setup. So, let us say I have matrix spaces x and y matrix spaces which are compact. Y need not be compact but it does not really matter. So, let us take a function h from x to y. Suppose h is 1-1, onto and continuous then h inverse, so h inverse will go from y to x is continuous. So compactness is crucial here because it will map close sets to close sets, that is all is needed to prove that inverse takes open sets to open sets. So we use this, so g is a homeomorphism from 0, 1 to 0, 2.

(Refer Slide Time: 18:42)



And next claim is that g of C, so Cantor set, remember f of c is the whole interval 0, 1 adding x is like translating but this requires some work, g C has measure 1, Lebesgue, so everything is Lebesgue measure of course. Of course this may…This also says that g of C has to be Lebesgue measurable, that is not very difficult to see. But anyway when we write down the proof, it will be clear.

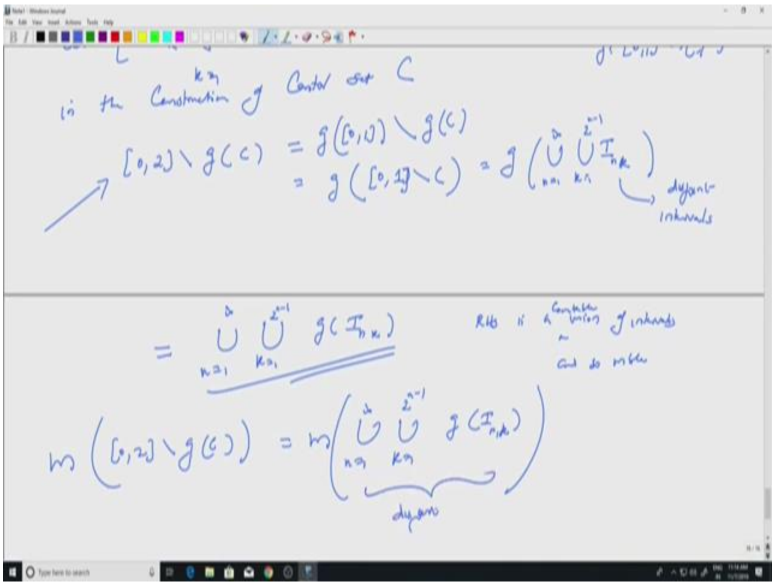
So, you take an interval a b, which is inside closed interval 0, 1 minus C outside the Cantor set. Then g of this interval, g is a strictly increasing monotonic 1-1, onto continuous function. So, this is of course, the interval g b minus g a, g b to g a, that is a trivial, sorry, the other way g a g b because g is increasing, so, g a to g b. So because it takes, so g of a, b the interval is a measurable set because it is an interval.

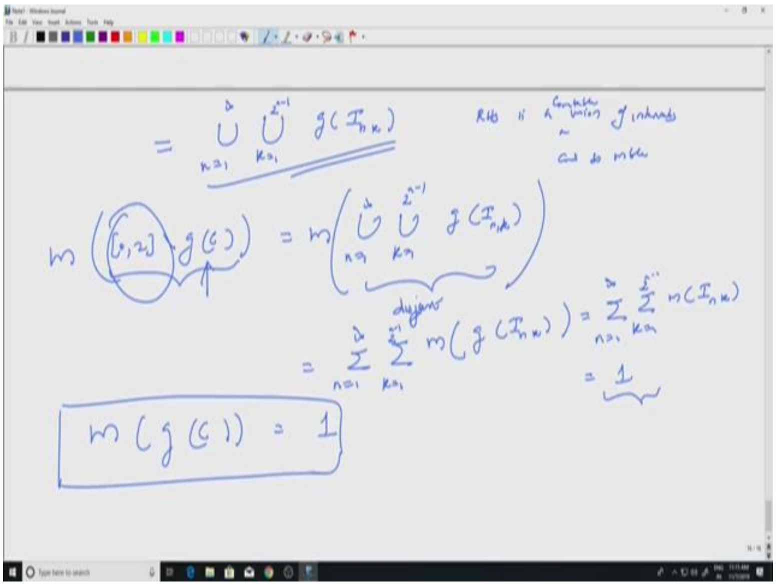
So, measure g of a, b so, g of a, b is the range of a, b under g. So, this is nothing but measure of the interval ga gb which is the length of the interval gb minus ga. But g is f x plus x, so f b plus b minus f a plus a equal to b minus a because, f a and f b are same, because you are outside the Cantor set and f is a constant there. So, f b and f a will be same constant in an interval, so that gets cancelled and you get to b minus a.

Which means that g preserves the measure in some sense. a, b, the interval a, b is taken to another interval whose length is b minus a. So, we can use that. So, let me introduce some notation, let I n k, k equal to 1, 2 to the n minus 1. So, remember there are 2 to the n minus 1 intervals you throw out at the nth level, be the collection of intervals removed at step n in the construction of Cantor set C. Right, at the first level, you remove 1 interval, second level, you remove 2 intervals, third level you remove 4 intervals and so on. So at the nth level, you are removing 2 to the n minus 1 intervals. So k going from 1 to n minus 1.

So our aim is to find the measure of g C. So we will look at the compliment of that and try to find out what that is.

(Refer Slide Time: 22:17)





So you look at, so remember g goes from 0, 1 to 0, 2, it is a homeomorphism. So it is 1-1, onto, continuous map with this inverse being continuous. So you look at 0, 2 minus g of C. Okay, I want to find the measure of g C. g C is in the rage, I want to find the measure of g C.

Well, what is this? Because g is 1-1, so I will write one more step this is g of 0, 1 minus this, there is set theoretic difference g of C but g is 1-1 and onto and things like that. So this is simply g of 0, 1 minus the Cantor set and you look at the image of that. So, this you have to be careful, but here g is 1-1, onto and so on, there is no problem. Otherwise, you have to be careful with the set theoretic operations.

But 0, 1 minus the Cantor set is a union of intervals. So, I can write this as g of, well, we denoted it by I n k, k going from 1 to 2 to the n minus 1, and then of course n goes from 1 to infinity, first step to final step. But these are all disjoint intervals and g is monotonic. So the image of this will be the union of the image of each part, so that is something you have to check. So this is equal to union of n equal to 1 to infinity, k equal to 1 to 2 to the n minus 1, g of I n k.

So, remember this is because they are all disjoint and g is 1-1. So, disjoint sets will go to the corresponding images which are also disjoint. So, all these are true. In general this, if g is not 1-1 etc., you will have to be very careful when you deal with set theoretic operations. But this tells me that the right hand side is a union of intervals, countable union of intervals, and so measurable.

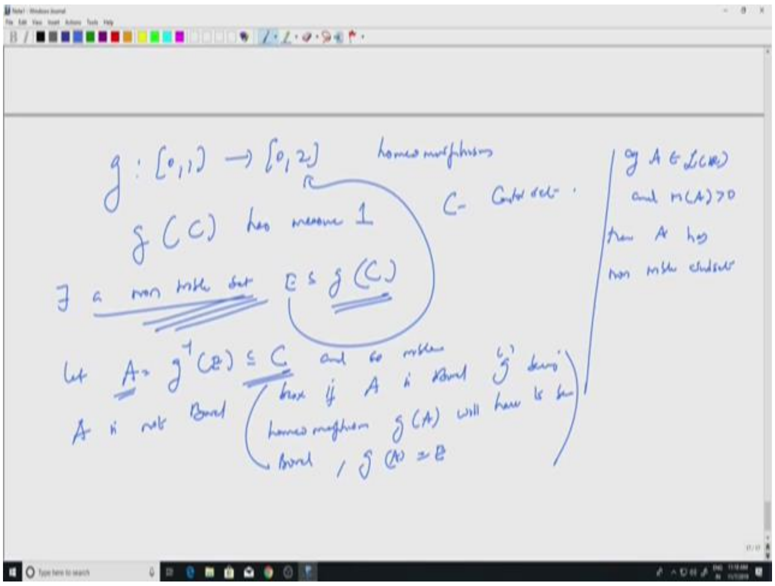
Of course g C, you can also argue that C is compact so g C is compact, so, its complement is open, which is what we have actually written down and so it is measurable. So, I can look at the measure of this. So, measure of 0, 2 minus g C, the Cantor set is equal to, well, measure of this guy, measure of union n equal to infinity, union k equal to 1 to 2 to the n minus 1. Remember g is 1-1, increasing and so on. So, if you take disjoint intervals, it will be mapped to disjoint intervals. So, this is a disjoint union.

So, this will sum up, countable additivity of whatever you want, n equal to 1 to infinity, k equal to 1 to 2 to the n minus 1. Measure of g of intervals I n k, where are these intervals? Outside the Cantor set. So, let us go to the first step. If I take any interval outside the Cantor set, measure of the image of g of that interval is same as b minus a, the length of the original length of.

So, this is nothing but length of the original interval, n equal to 1 to infinity, k equal to 1 to 2 to the n minus 1, measure of the interval I n k. But that is the measure of the intervals which you throw from Cantor set. So, that is 1, that is why Cantor set has measure 0. So, now if you look at this and this, you see that measure of the set g C, well, this is the compliment of g C that has measure 1. 0, 2 has measure 2 so remaining is just 1, so this is 1.

So that is what we wanted to prove. Okay, so now, we will justify the existence of the non-Borel set so, this will be quick.

(Refer Slide Time: 26:52)



So g from 0, 1 to 0, 2 homeomorphism, g of C has measure 1, that is what we have done with, with the Cantor ternary function we have constructed g such that g of C has measure 1 where C is the Cantor set. So, now we prove that since g of C has measure 1, there exists a non-measurable set E contained in g C. So, we recall that if A is a Lebesgue set and measure of A is positive, then A has a non-measurable subset. What we proved was, if measure of A is 0, all subsets are measurable.

So if measure of A is positive, there is some set which is not measurable. So there exists a non-measurable set E. So where is E sitting, E is on the range. Now you take the inverse image. So let A equal to g inverse of E, well, where is this? This is sitting inside the Cantor set, because I started with g C and I am looking at the inverse image.

So it will be inside the Cantor set and so measurable because Cantor has measure 0, so A will have outer measure 0, so A is measurable. But A is not Borel, A is not Borel because, this is because if A is Borel, g being a homeomorphism, g of A will have to be Borel. Because Borel set, homomorphisms will take Borel sets to Borel sets, because use good sets principles for that.

But g of A is the set E which is non-measurable. So the pullback of the non-measurable set under g is not a Borel set but it is Lebesgue set because it is a subset of the Cantor set. So, we can stop here. We, using the Cantor set, Cantor ternary function, we constructed this function g which was a homeomorphism from 0, 1 to 0, 2 and then we pulled back a non-measurable set inside g of C.

g of C had measure positive, so that is a very crucial part which you should understand. You have continuous functions which will take sets of measure 0 to sets of positive measure, that can happen. So, we exactly construct it as a function, using that we construct Cantor set which is not Borel, but Lebesgue. From the next lecture onwards, we will get back to our abstract settings and we will study LP spaces. Okay.