Measure Theory
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Lecture 31 - Cantor Function

So, our aim in this lecture is to construct an example of a Lebesgue measurable set which is not a Borel set in the real line. Also, there are cardinality arguments which will prove this easily. You can start with intervals of with n points rational numbers, that will give me a countable collection of open intervals in the real line. They generate the Borel sigma algebra. So, one can show that the cardinality of the Borel sigma algebra is the cardinality of the real line, first uncountable Cardinal.

Now, when you look at Lebesgue sigma algebra the Cantor set, which has measure 0, which is also uncountable has 2 to the C many subsets and all those subsets are Lebesgue sets, because they all have outer measure 0. So, the Lebesgue sigma algebra has at least 2 to the C many sets and the Borel sigma algebra has only C many sets where C is the cardinality of the real line. And so, there are many sets which are Lebesgue measurable but not Borel measurable.

So, our aim today would be to sort of construct this using what is known as the Cantor function. Instead of using the cardinality argument, we will justify using the Cantor function that there is a non-measurable, non-Borel set which is Lebesgue measurable. Okay, let us start.

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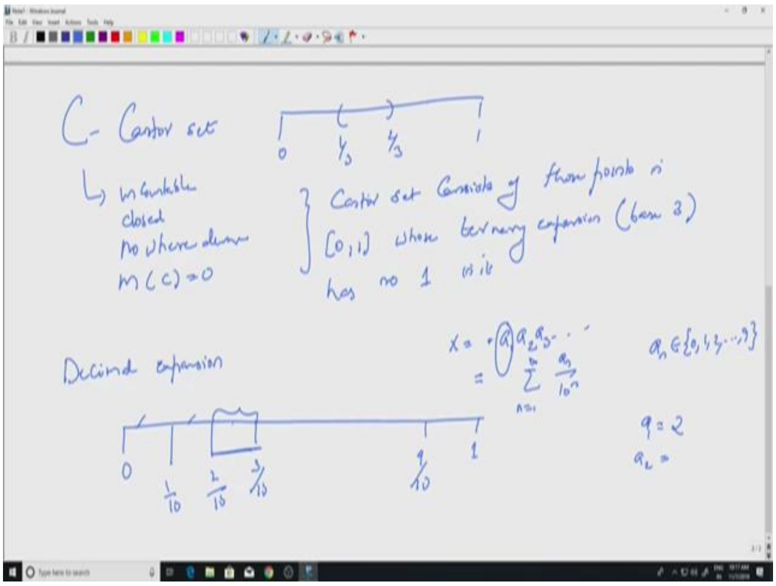

So, C will denote the Cantor set, Cantor set. So recall that we constructed this inside 0, 1. And you throw out the middle third and then you continue. So this was a set which was uncountable. Cantor set is uncountable, closed, nowhere dense, nowhere dense set and has measure 0. This is what we know about Cantor set. Also, there is a different description of Cantor set, Cantor set consists of, Cantor set consists of those points in closed interval 0, 1 whose ternary expansion….

So this is the base 3 expansion, so base 3 instead of decimal 10, ternary expansion has no 1 in it. Okay, so let me explain what that means. So let us look at decimal first. So what is decimal expansion? And then we will go to other expansion. So decimal expansion is something which you know. So if I look at 0, 1, and a point, let us say x here, so x can be written as point, A1, A2, A3, etcetera, where are An's? An's belong to the set 0, it can be 0, it can be 1, it can be 2, etcetera. It can be all the way up to 9.

Well, what does this mean? This is a series A n by 10 to n, n equal to 1 to infinity. Well, how do you plot this point in 0, 1? Well, first you decompose 0, 1 into 10 equal intervals, 3 by 10, etcetera, I have 9 by 10. So, you get 10 equal intervals and the first digit A1, so if A1 equal to 5 or let us say A1 equal to 2, well, it will fall here. So, let me…So, if A1 is 2, it will be between 2 by 10 and 3 by 10, so it will fall here.

That means it is in the third interval. So, this is the first interval, second interval, third interval. Now, depending on what is A2, we divide this interval, so you divide the 2 by 10 to 3 by 10 interval.

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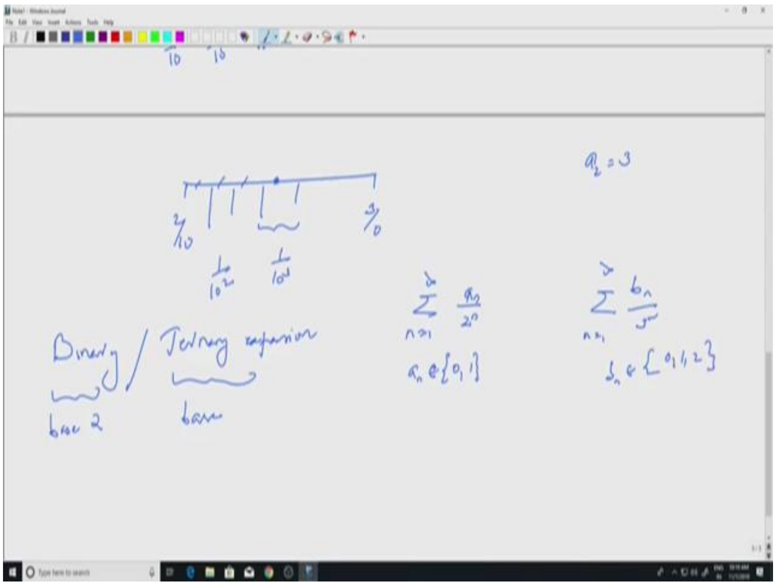

So, now we have 2 by 10 here and 3 by 10 interval into 10 equal parts. So, this would be of length 1 by 100, right, 1 by 10 Square and then you look at what is A2. If A2 is let us say 3, this will be in the 4th interval, so it will fall here, so somewhere here. And then you divide this into equal 10 parts, so that would be of length 1 by 10 cube and so on. So, these numbers here tells you where exactly your point is, and that series will converge and that is your number.

Now, whatever base you take, so ternary expansion would be to ternary or binary. So, we will use both binary and ternary expansion. Well, what does that mean? So, this would be base 2 and this is base 3 and you expand a point in terms of 1 by 2 to the n's. So, that would be your binary expansion or you expand in terms of 1 by 3 to the n's. So, that would be your ternary expansion. So, you can do exactly like what we did for the decimal. So here the An's would be either 0 or 1, because you are in the binary situation, here the Bn's will be in, you are allowed 3 numbers, so you have 0, 1, 2.

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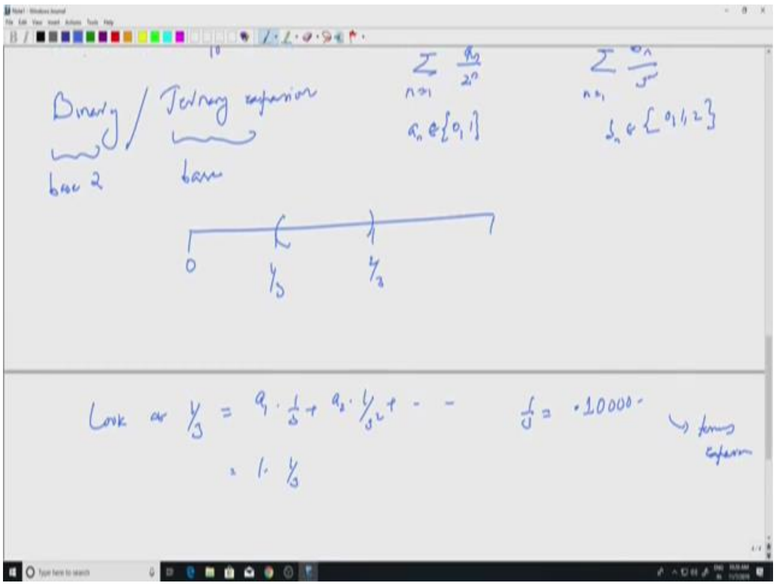

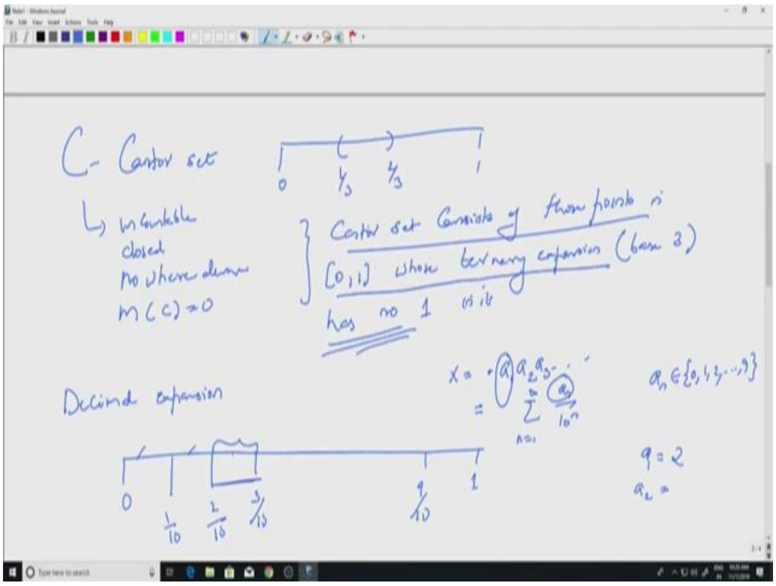

And if you look at the way we did decimal expansion, the Cantor set is nothing but, so you look at 0, 1. You divide this into 3 equal parts, and then you threw away the middle one third, which means that numbers which fall in the second interval is thrown away, which means that whenever you have 1, that is thrown away. So that is the statement here, consists of those points in 0, 1, whose ternary expansion has no 1. Okay. So the 1 does not appear. Well, one has to be a bit careful because the endpoints can cause problems.

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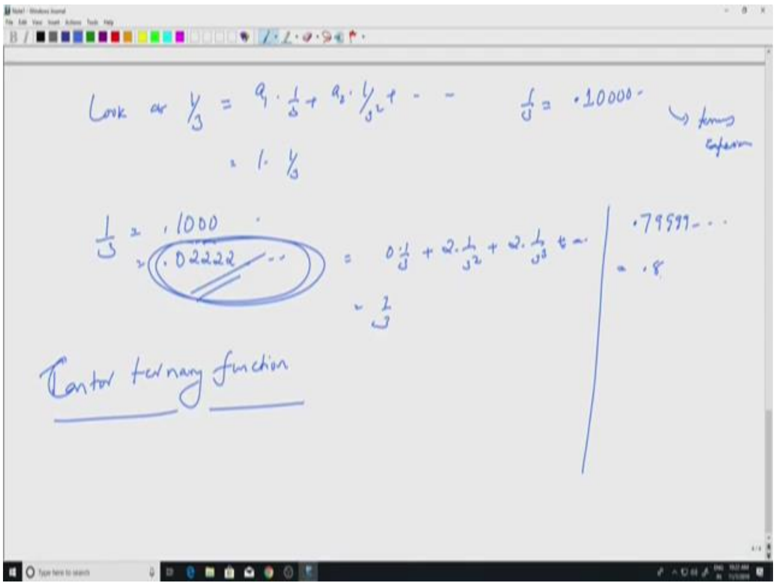

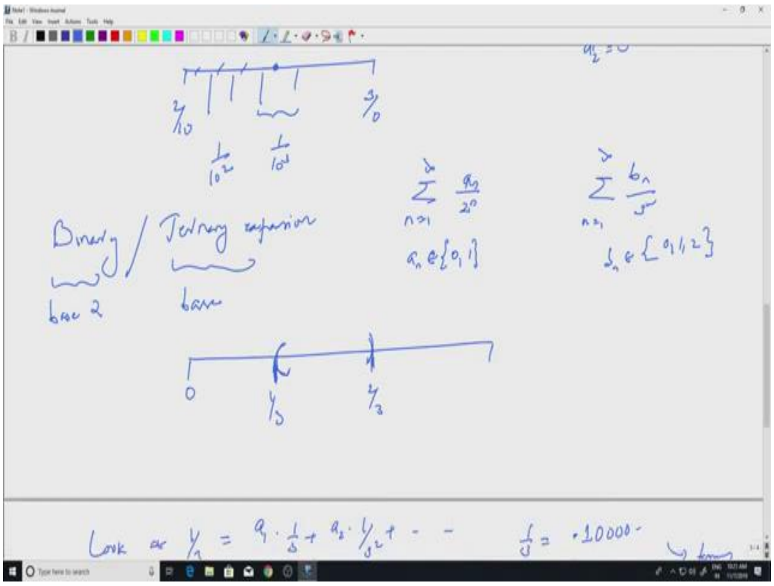

So let us let us see the endpoint 1, 3. So look at 1 by 3, what is the ternary expansion of 1 by 3? Well, I want to write this as A1 into 1 by 3 plus A2 into 1 by 3 square plus etcetera and this is obvious, right, it is 1 into 1 by 3. So, the ternary expansion of 1 by 3 is equal to 0.1000000, this is the ternary expansion. But it has 1 in the ternary expansion and 1 by 3 is in the Cantor set because I am only throwing out the open intervals.

So the endpoints here, they are all in the Cantor set. Well, the point is 1 by 3 has another expansion. So, let us see 1 by 3, it can be written as 0.100000. Well, which can also be written as 0.0222222 never ending 2s, right. Well, why is that? This is, so let us look at this portion. This is nothing but according to the ternary expansion, this is 0 into 1 by 3 plus 2 into 1 by 3 square, plus 2 into 1 by 3 cube, etc, etc. So you add that, you will see that this is actually equal to 1 by 3.

There is nothing surprising here. This is something which you have seen for decimal expansion. Recall that whenever you have some number like this, let us say 0.799999. This you round off as 0.8, right. That is precisely what I am doing. If I have a terminating expansion, I can put my 9s at the end to make it never ending. Similarly, you can do that here. So in this expansion there are no 1s, there is only 2. So that is why 1 by 3 is allowed in the Cantor set. So with that preliminary let us look at the ternary function. So this is the Cantor ternary function, we will use this to construct our set.

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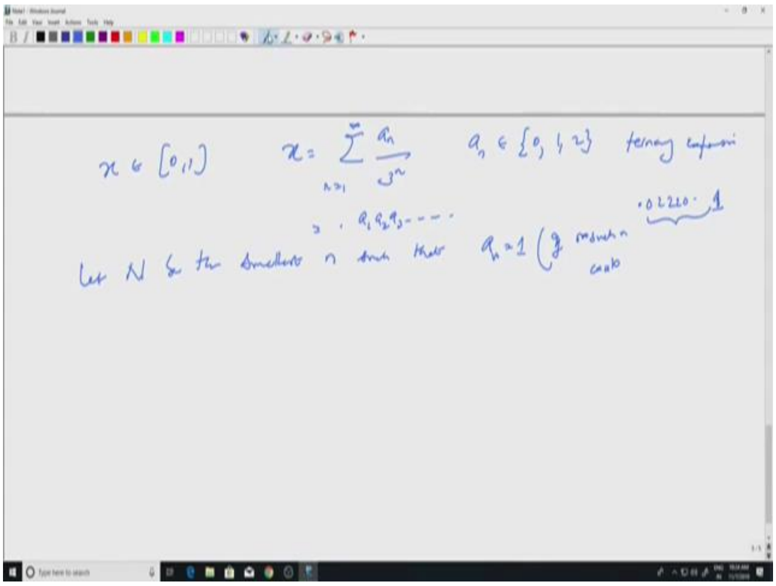

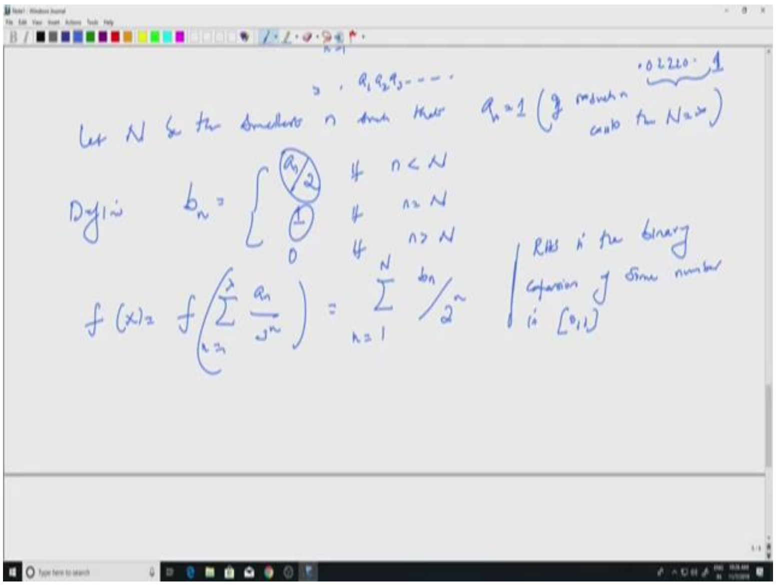

So let us define this. So the definition is sort of easy to understand, define f of x, so f is going to be from 0, 1 to 0, 1. So f will be from 0, 1 to 0, 1. This is my Cantor ternary function. Well, how do I do that? I take a point. I take a point x in 0, 1. I need to tell you, what is f of x. Okay, so x has a ternary expansion. So, x is written as a n 3 to the n, n equal to 1 to infinity, a n's are either 0, 1 or 2. So, this is the ternary expansion in base 3. So, this is written as point a 1, a 2, a 3, etc.

So out of this a 1 could be 0, 1 or 2, a 2 could be 0, 1 or 2 and so on. So, you go until you hit 1 first. So, let capital N be the smallest n, small n such that an equal to 1. So, which means so you have something like 02220, etc, etc and at the nth position you have 1, okay. Before that you do not have 1. Okay. So of course, it can be start from 1. So then you will have capital N to be 1. Well, you may not have 1 in that case, n is infinity. If no such n exists, then you call it then capital N is infinity. So this is not a problem.

So, you define b n to be a n by 2, so you divide by 2 if n is smaller than capital N, so remember capital N is the first place where you hit 1. So, all the numbers before that is either 0 or 2. So, you divide by 2 you will get either 0 or 1. So, a n by 2s are either 0 or 1. And you define it to be 1 if n equal to 1 so, that means at that position you are changing and 0 if n is greater than N.

So, that is your b n and define f of x, so remember x is given by the ternary expansion a n by 3 to the n, n equal to 1 to infinity, you write this as binary expansion now, so b n by 2 to the n, n equal to 1 to capital N. So, this is the RHS is the binary expansion in base 2, binary expansion of some number in 0, 1. So, that is the image. So, you think of right hand side as a binary expansion, you will get a number in 0, 1 and that is your image.

Well, why is it a binary expansion? Because b n is either a n by 2 which is either 0 or 1 or 1 or 0. So, that is bn's are always 0 or 1. So that gives me a binary expansion of a number in 0, 1. So, you have to check that it is well defined. Well, let me explain what that means.

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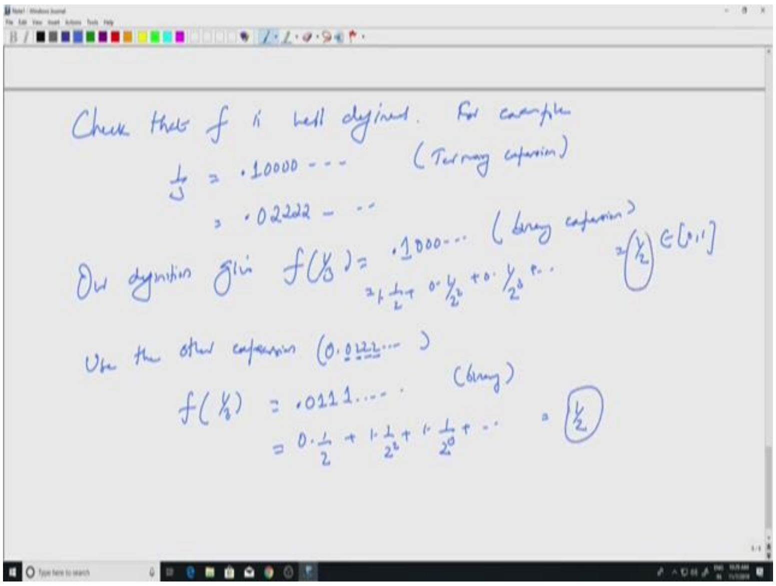

So, check that, I will leave it to you, check that f is well defined. What does that mean? Because there are points with 2 expansions. So remember 1 by 3, we had 2 expansions. And if you use this formula we may get 2 expression, so we do not know if they are same. So let us check. So for example, 1 by 3 can be written as 0.10000 in ternary expansion, so remember this is ternary expansion, not decimal, ternary, which is also equal to 0.022222. So, what are the images if I look at?

So, our definition gives, so our definition gives 2 numbers, well, f of 1 by 3 equal to, well, what do you do? You look at where the first 1 comes. So, that is in the first place. So, you put 1 and then you put 000, after that it is 000. So, it is the same number, but it is in the binary form, so this is binary expansion. In other words, we will be looking at 1 by…1 into 1 by 2 plus 0 into 1 by 2 square plus 0 into 1 by 2 cube etcetera, which is equal to 1 by 2.

So, this is my number inside 0, 1 that is the image of 1 by 3. That is using what expansion? So let us use the other expansion. So, use the other expansion, use the other expansion. So what is that? That is 0.02222, this number. So then if I look at f of 1 by 3, well, what do I do? I divide by 2 if I do not see 1, if I see 1, I put 1 and then I put 0s. So I have point, first element here is 0, so I divide by 2, I get 0. The next element is 2 it is not 1, so I divide by 2, I get 1.

Here, it is 2, so I will get 1, here it is 2, I will get 1 and it is all 2, so I get 111. I do not get 0s anyway anymore. But this is binary, remember that the image is supposed to be calculated in the binary expansion, which is 0 into 1 by 2 plus 1 into 1 by 2 square plus 1 into 1 by 2 cube plus etc, etc. and this, you can calculate to be equal to 1 by 2. So, this value and this value are same.

So, whether we use different expansions or not, we will still get the same thing. Of course, we have to check that this is true for all such terms. So that I will leave it to you.

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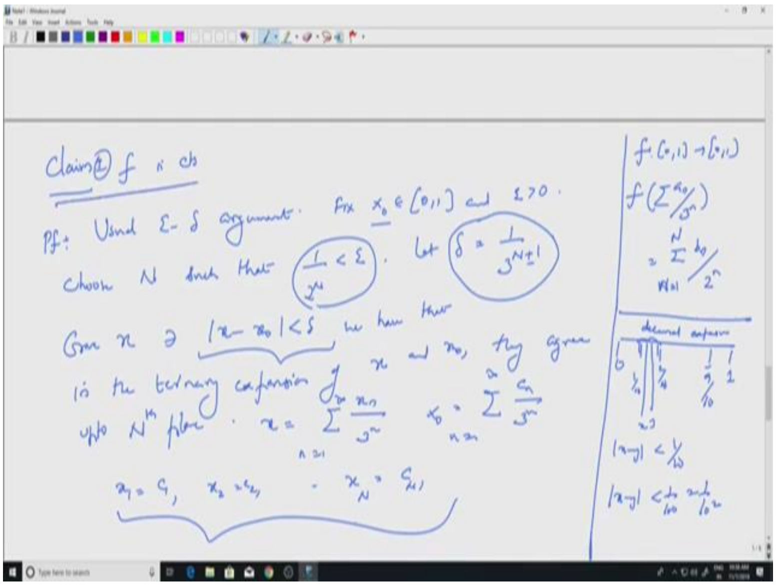

So, let us start with various claims. So, let us recall the Cantor ternary function here f of 0, 1 to 0, 1, f of summation a n by 3 to the n equal to summation b n by 2 to the n, n going, small n going from 1 to capital N, sometimes it can be infinity sometimes it can be a finite. So, claim. So, lots of claims about this function. F is continuous, that is the first thing. So it is well defined, it is continuous. So, let us take the, let us look at the proof. So proof is by usual epsilon delta argument.

So usual epsilon delta argument. So, given epsilon we have to find a Delta so that something happens. So, let us fix a point, fix x naught in 0, 1 and epsilon positive. I need to find a Delta so that whenever x is at a distance less than delta from x naught images are at a distance less than epsilon. So, let Delta, so what do we do is we, so remember that the domain has ternary expansion and the range has binary expansion.

So, choose N capital N such that 1 by 2 to the n is less than epsilon. So, that is possible because if I choose the capital N going to infinity, the left hand side goes to 0. So, this is possible. After choosing the capital N, we can choose delta. So delta you take to be 1 by 3 to the n plus 1. So, the 3 and 2 because one side we have ternary and other side we have binary. So, given x such that mod of x minus x naught less than delta. So, x naught is this fixed point and we want to look at a (())(18:33) around x naught and see how the images are, that is how you prove continuity.

So, what does it mean to say that x minus x naught is less than delta? This is same as saying…So, let us take a short digression here. If I have decimal expansion, let us take 0 and

1 and decimal expansion. Decimal expansion. So, you divide this into 10 equal parts, say n by 10. So if I take 2 numbers here, so let us say x and y, then mod of x minus y is less than 1 by 10, because that is the length of the interval.

But if let us say mod of x minus y is less than 1 by 100, that is 1 by 10 Square, what does that mean? That means if x is point a1, a2, a3, etc, y will have to be equal to point a1, a2 and then something we do not know. At the first 2 places, they will have to agree because otherwise the difference will be only in the first place, if they agree or if they do not, then the difference will be greater than 1 by 10.

If they agree on the first place, the difference will be less than 1 by 10, if they agree on the first 2 places then it will be less than 1 by 100 and so on. Because, I mean write down the expansion you have a1 by 10, a2 by 100 plus something, similarly the…So, when I take the difference only this part will matter, let us use that. So, when I say mod of x minus x naught is less than delta, remember delta is 1 by 3 to the n plus 1. So, because of the n plus 1, on the first n places x and x naught should agree.

Given x such that this, we have that in the ternary expansion of x and x naught, they agree they agree up to nth place, up to nth place. Okay, what does that mean? That is if x equal to summation x n by 3 to the n, n going from 1 to infinity and y equal to, sorry, x naught equal to summation let us say c n by 3 to the n, n going to 1 to infinity, then the first n places should agree.

So, x 1 will be equal to c 1, x 2 equal to c 2, etc, x n equal to c n and then we do not know. Okay, so, we do not know whether the equality continues or not but the first n places will have to agree.

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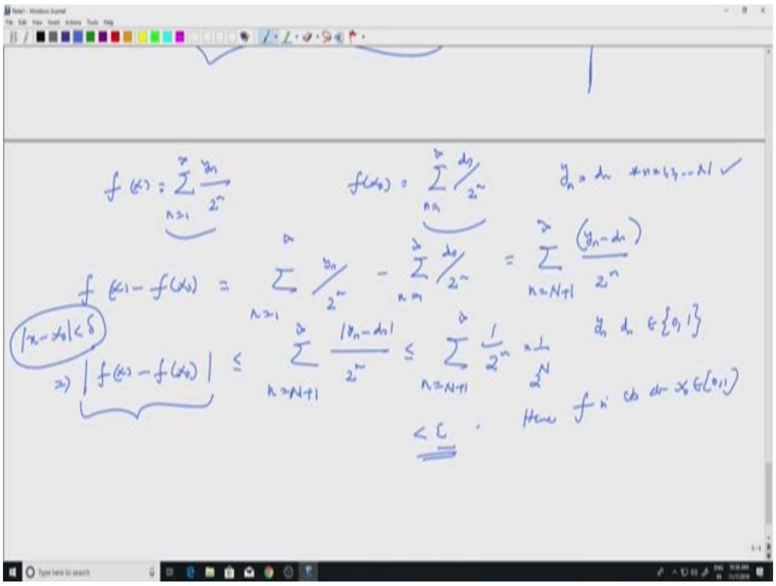

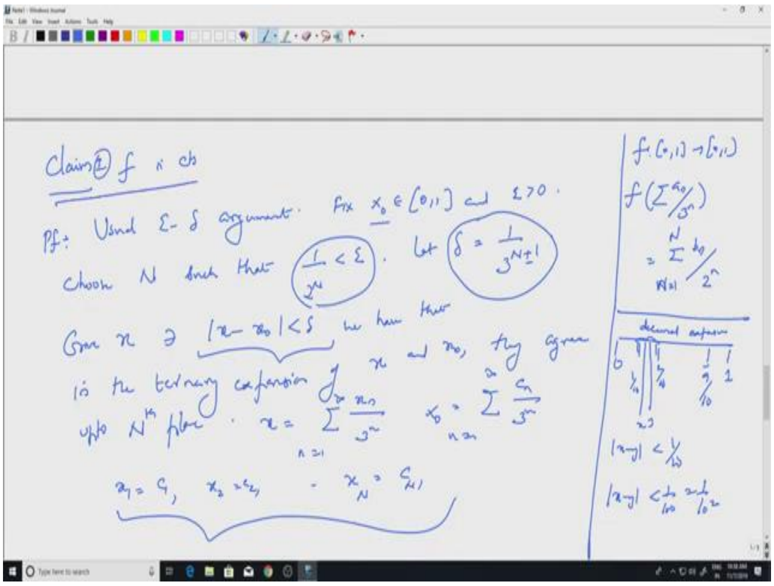

So because of that, if I look at f of x and f of y, so f of x naught, let us see the difference between them. So, we have f of x and we have f of x naught. So, they have binary expansion. So, let us say they are given by y n by 2 to the n and f of x naught is summation d n by 2 to the n. Remember the range has binary expansion, domain has ternary expansion. Well, because x n and c n agree up to nth place, y n and d n will also be agreeing. So, y n will be equal to d n because they are divided by c n divided by 2 or 1, depending on how the definition is.

So, y n equal to d n for every n equal to 1, 2, etc. up to capital N. So, the first capital N terms here and here are same. So, now let us estimate the difference between f of x and f of x naught. So, I want to find out what is f of x minus f of x naught, and I want to say this is less than epsilon, provided mod of x minus x naught is less than delta. Well, so this is, we have the definition, which is n equal to 1 to infinity, y n by 2 to the n minus summation n equal 1 to infinity, d n by 2 to the n.

Of course, the first n places are same, so they get canceled, and I will have 1, n equal to 1 to infinity. Now, I am going to write down the term in one go. So, first n places get canceled, I will have n equal to capital N plus 1 to infinity, because first n places are same because of this. And of course, I can write this as y n minus d n divided by 2 to the n, correct. Okay. So mod of f of x minus f of x naught, this is less than or equal to summation n equal to n plus 1 to infinity modulus of y n minus d n divided by 2 to the n.

So yn's and dn's are less than, what are yn's and dn's? So yn 's and dn's take value in the 0 and 1, because of the binary expansion on them. So this is surely less than or equal to summation, summation n equal to capital N plus 1 to infinity, 1 by 2 to the n, which of course sums to 1 by 2 to the n. So, that is a geometric series you can sum. But we chose this capital N so that one way to design is less than epsilon.

So, this is less than epsilon and that is all we want, right. Because we started with mod x minus x naught less than delta, so given an epsilon, we found a Delta, so that this implies mod of f of x minus f of x naught is less than epsilon, that is precisely f is continuous. So hence f is continuous at x naught. But x naught was an arbitrary point chosen in 0, 1. So the same, so that implies that f is continuous on 0, 1.

Okay, so the next claim is there will be several claims, so we have one claim where f is continuous. So maybe I can put 1 here, so we will go to claim 2.

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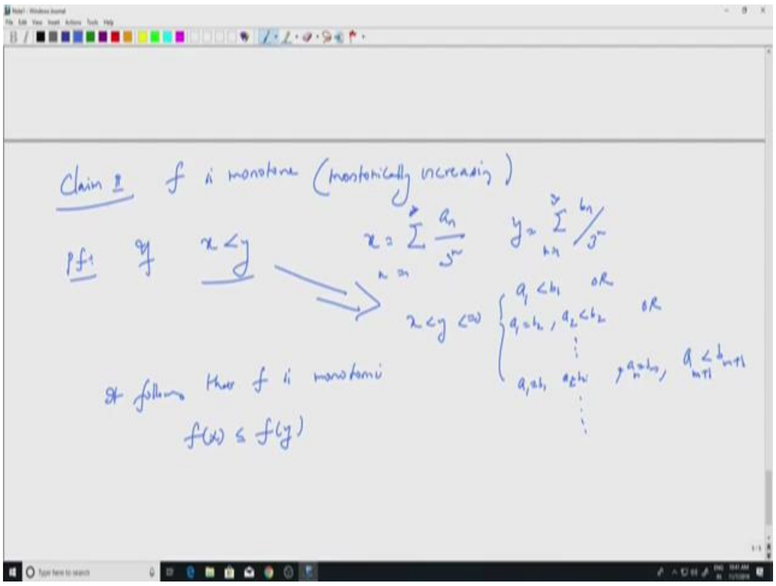

f is monotone, so f is monotone, so it is the same as, well, monotonically increasing, monotonically increasing. Of course, I am not saying it is strictly increasing it is not, we will see that. Okay, proof. So this proof is very easy. So if x is less than y, what does that mean? So let us go to the ternary expansion. So, I can write x as summation a n by 3 to the n, n equal 1 to infinity and y equal to summation b n by 3 to the n, n equal to 1 to infinity. X equal to y, x is less than y, meaning, well, so x is less than y if and only if I can write down this as an infinitely many conditions, but I start with the first place.

So a 1 is less than b 1, okay. Or if they agree, a 1 equal to b 1, they may they have to disagree on, I mean they have to satisfy this monotonicity in the second place. So, a 2 is less than b 2. So that is 1 condition or there so you can go on. So, a 1 equal to b 1, a 2 equal to b 2, etc. a some m equal to b m and the next place they have to not to be equal, but you will have monotonicity like this.

That is what this implies. Okay. So, x is less than y implies this, if and only if in fact and this condition is gone. So, now it is clear because all that you are doing is dividing an's by 2 or terminating the expansion. So it is clear that, so it immediately follows that, it follows that f is monotonic, f is monotonic. So either f of x is equal to y or f of x is less than or equal to y.

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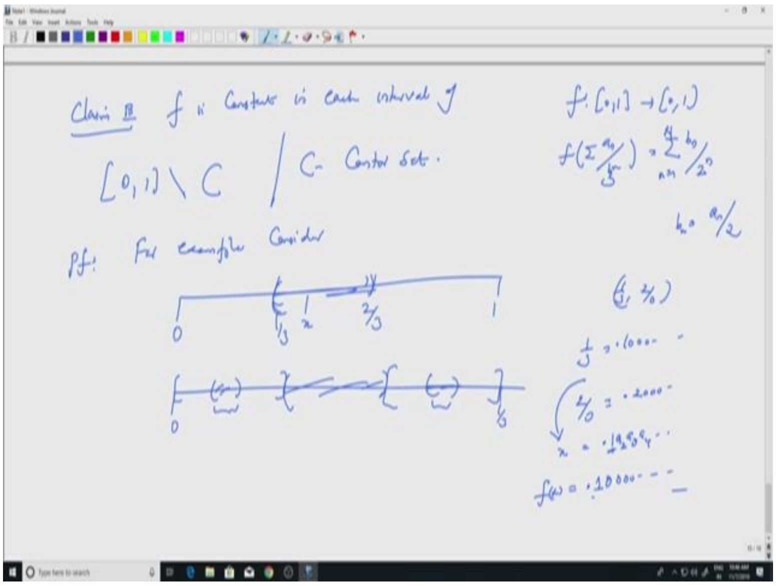

Okay, so let us look at one more claim. So that is claim 3. Claim 3. So let us recall the definition again, f was from 0, 1 to 0, 1 and f of summation a n by 3 to the n was submission b n by 2 to the n, n going from Capital 1 to, 1 to capital N. So remember b n was a n by 2. Claim 3 is that f is constant, f is constant in each interval of 0, 1 minus C, C is the Cantor set. So, C is the Cantor set. So, this is not difficult to see, but let us look at the proof, let us look at an trivial example just to justify that this is happening.

For example, consider the first step in the Cantor set section. So, that is I have 0, I have 1, I have 1 by 3 and 2 by 3, we are throwing out this interval. And in the next step we are throwing out the remaining one-third. So, this is already gone and this interval you divide into 3 parts and you throw away these things. So, all these intervals are disjoint, this is one interval,, this is another interval, this is another interval, they are all disjoint.

So, let us look at the easy case in the first step in the Cantor set. We have thrown out the interval one-third to two-third, this is the first interval we threw. I want to say on this interval, the function, Cantor function is a constant. Well, why is that? Well, let us try to understand what is the ternary expansion of one-third. One-third is written as 0.10000. And two-third, both of them are not in the interval, they are the endpoints of the open intervals.

2 by 3 is 0.2000, it is 2 into 1 by 3, so that is the ternary expansion. So any point in between, if I take a point here, let us say x in between, it will have to be 0.1 and something, it cannot be 0.2 and something because it will go out of that interval. So it has to be 0.1 and something. But then 0.1 so let us write a 2, a 3, a 4 etcetera, we do not know what that is. But what is the

image of this f of x? By definition is what do you do, you go until you hit 1 first and divide by 2 until then.

So, here you are hitting 1 on the first place itself. So just 0.1 then you put 0s. So, for any x in between this interval, in this interval, it is 0.1000. So that is the constant there. So f of x is a constant in this interval. So this proves that f is a constant in these intervals. So let us stop here. So we defined the Cantor function, we have proved that it is continuous, it is monotonic, and it is constant in these intervals which are thrown out in the construction of the Cantor set.

So we will look at some more properties and then use this function to construct the non-Borel but Lebesgue set. Okay.