Measure Theory

Professor E. K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru

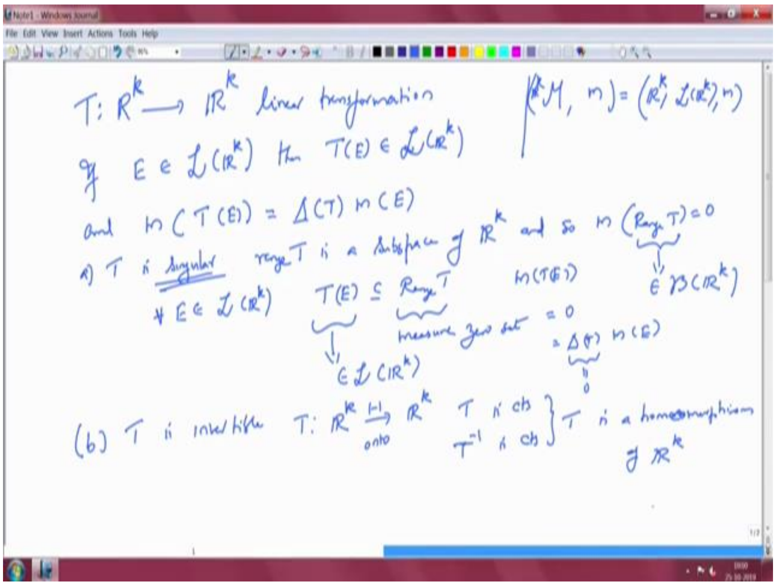
Lecture 29

Linear transformations and Lebesgue measure

So, we will continue with the properties of the Lebesgue measure. We saw some of them in the last few lectures. So, we ended up with proving this result that if you have a linear transformation T from Rk to Rk, then the measure of TE where E is a Lebesgue set, is some constant times the Lebesgue measure of E itself, the constant depends only on the linear transformation T. That is what we proved.

Today, we will start with the result that the constant which comes out is actually the determinant of the linear transformation. Okay, so I will also clarify why T of E is miserable whenever E is miserable. So, we will start with that assertion which I should have clarified in the last lecture. And then we will go on to prove that the measure of TE is actually determinant of the linear transformation T times measure of E. So, let us start.

(Refer Slide Time: 1:31)



So, we have a linear transformation T from Rk to Rk. So, this is a linear transformation, linear transformation and what we have proved is if E is a Lebesgue set, so before that, let me clarify that we constructed the Lebesgue measure and we got a sigma algebra and the measure. So, this is same as the triple we already had, okay. So, this is Rk and we have the Lebesgue sigma algebra Rk and the Lebesgue measure M, okay.

So, in one case we constructed using the outer measure, in the other case, we constructed using Riesz representation theorem. But both of them give us the same sigma algebra and same measure, okay. So, if E is a Lebesgue set, then T of E is also a Lebesgue set and the Lebesgue measure of T of E is actually equal to some constant which depends only on T times the Lebesgue measure of E.

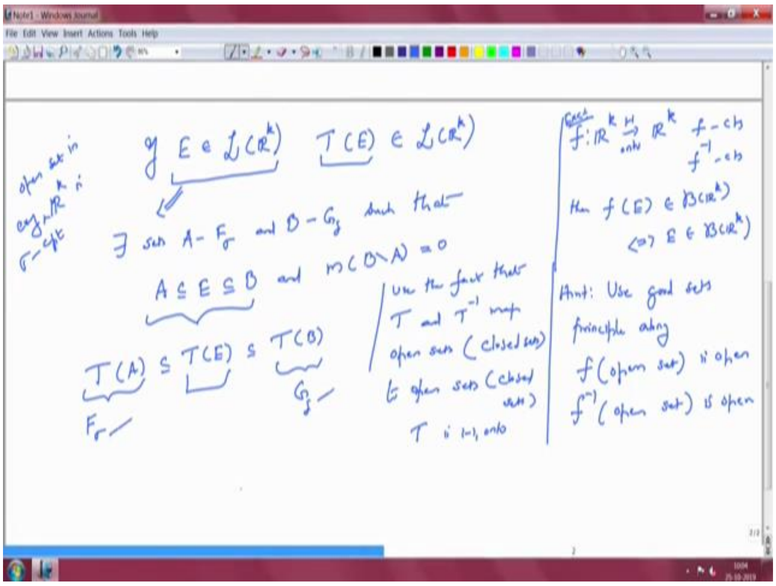
So, we had two cases, one was T singular. If T is not invertible then range of T. So, this is the easy part range of T is a subspace of, range of T is a subspace of subspace of Rk and has measure 0. And so, measure of range of T is 0. Subspace is close, so it is a measurable, so range of T this is a Borel set in fact.

So, there is no problem in looking at measure of that set. And so, if I take any E, so for every E in the Lebesgue sigma algebra of Rk, T of E of course sits inside range of T. But range of T has measure 0, so this is a measure 0 set, measure 0 set. And so this is a subset of measure 0 set. And so will belong to Lebesgue sigma algebra Rk.

So, it makes sense to look at M of TE. And this is of course 0, which is delta d times M of E, where delta T 0. Okay, so this is the determinant of T, T singular, so, its determinant is actually 0. So, we will, we will prove that delta T is actually the determinant. But before that, let us look at the remaining case. So, B, T is invertible, if T is not singular, then T is invertible. Well, what does that mean? So, that would mean that T is a linear map from Rk to R k, which is one-one onto.

Okay, and so T well any linear map is continuous and so inverse also is continuous because one-one onto linear so T inverse is also continuous. In other words, T is a homeomorphism, T is a homeomorphism of Rk. So, it is a very nice map.

(Refer Slide Time: 5:36)



So, first let us justify why if E is a Lebesgue set, if E is a Lebesgue set, let us justify why TE is also a Lebesgue set. So, this follows from various properties of T. So, T is a homeomorphism. So, let us let us give a simple exercise here to use that. Suppose, I have a homeomorphism from Rk to RK. Okay, what does that mean? I have it is one-one onto, f is continuous and f inverse is also continuous. Then f of E is a Borel set if and only if E is a Borel set. So, consider this as an exercise, exercise.

So, the hinders used, it is not difficult hinders used good sets principle along with the properties of the homeomorphism. F of an open set is open because it is a homeomorphism and f inverse of an open set is open. So, it takes open sets to open sets and so it will take Borel sets to Borel sets, so use that. Now, I know that E is a Lebesgue set and I want to prove TE is also a Lebesgue set.

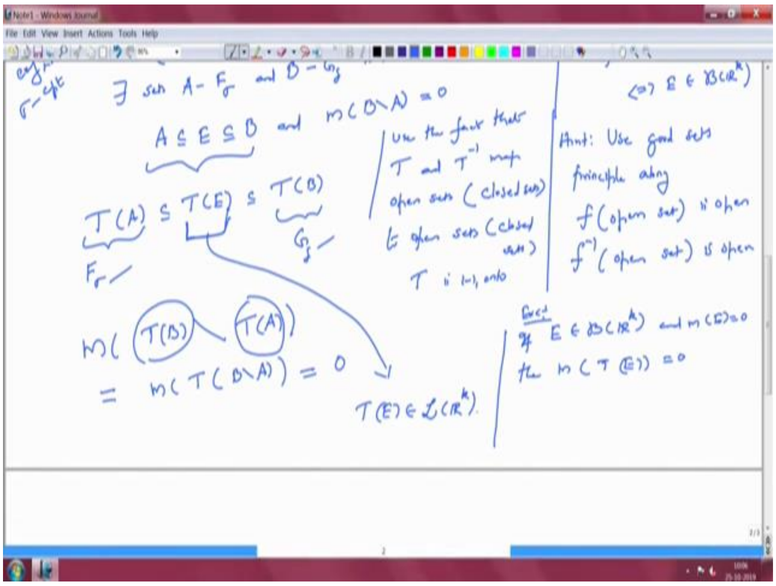
E is a Lebesgue set tells me, because Rk, so, this is a consequence of the Riesz representation theorem, Rk has a property that every open set in Rk is sigma compact. So, that tells me as a consequence of the Riesz representation theorem, there exists sets A which is F sigma and B which is G Delta such that A is contained in E contained in B, and that is the important part B minus A has measure 0.

So, this is a property of the Lebesgue sets, which we already know. And it was also proved using the Riesz representation theorem. Okay, so because of this, so now if you look at I am looking at TE so, so apply T to this, so I will get to T of A is contained in T of E contained in

T of B this is trivial of course. But T of A, what is T of A? A is F sigma. T of M a is also phase also F sigma and this is G Delta.

F sigma remember is the union of close sets, G Delta is the intersection of open sets and T is linear and one-one onto and all that all the set theoretic operations will be respected and T will take open sets to open sets and close sets to close sets. So, that is all is used there. Okay, so use the fact that, use the fact that T and T inverse map open sets respectively, close sets to open sets and respectively close sets. And the fact that T is one-one and onto. All set theoretic operations will be respected essentially. So, TE is sandwiched between an F sigma and G delta, which is what we want.

(Refer Slide Time: 10:04)

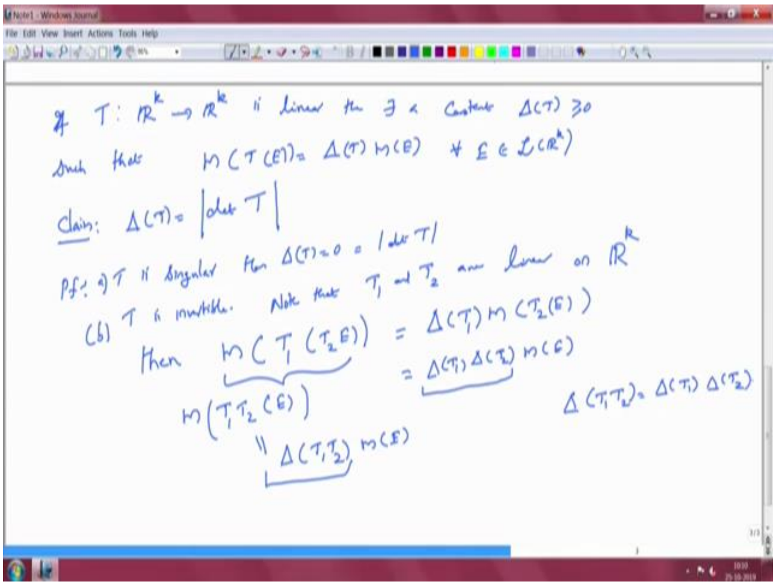


But we want a little bit more, so we look at the measure of TB which is a G delta set minus TA, which is F sigma set. Well, what is this? This is M of T of v minus a. So, this is something I will leave it to you to check it is the set theoretic equality TB minus T is linear TB minus T A is T of B minus A. This is equal to delta T times M of B minus A, this we already know.

Okay, so, maybe I should explain this. So, maybe instead of this I will leave this as an exercise to you. If E is a Borel set and measure of E is 0, then measure of TE is also 0. So, T has a property that it takes sets of measure 0 to sets of measure 0. Okay, so this is another exercise, it is true for much more general functions which are Lipschitz. But for the time being linear T will do for us, so this will give us 0.

So, I have a G Delta set, I have an F sigma set, such that the difference has measure 0 and TE is sandwich between them. So, this tells me that TE is measurable, so TE is a Lebesgue set. So, it makes sense to talk about M of TE. So, I wanted to justify this. So, that is what we did. And from yesterday's proof, we know that M of TE will be delta T times.

(Refer Slide Time: 12:17)



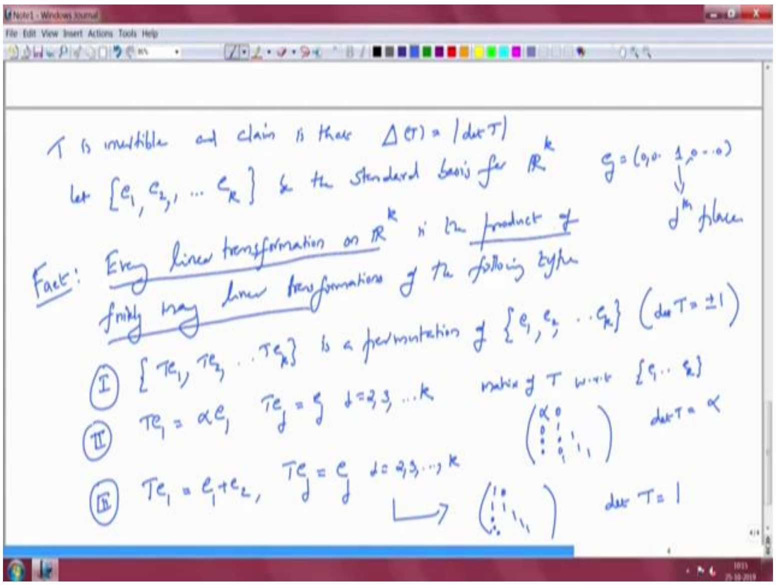
So, we have a constant. So, if, let me write the statement, if T from Rk to Rk is linear, is linear, then there exists a constant delta T such that M of TE is equal to delta t, that is a constant times M of E, for every E Lebesgue set. Our aim is to compute delta t. So, claim the constant delta T is actually equal to determinant of t. So, T is a linear transformation, so you can choose a basis, write down the matrix of T and then calculate the determinant, that would be your delta T. Well, modulus of determinant of T, because it is a positive quantity.

Okay, so let us, let us prove this, so again the singular part is trivial, T is singular, that is the first case, then delta T we know is 0 is 0, which is equal to the modulus of the determinant of T. T is singular, so the determinant if T is not invertible, so determinant of T is 0. So, we need to look at only the case where T is invertible, T is invertible. So, before we go further, notice that.

So, note that if I have, if T1 and T2 are linear on Rk then M of T1 acting on T2 E. So, T1 composed with T2 is another linear map. So, this is equal to, because of the way delta is defined delta of T1 times M of T2 E. But M of T2 E is delta T2 into a M of E, so delta T2 times measure of E. That is the right hand side, left hand side I can write as a measure of the linear transformation T1, T2 acting on E. T1 times T2 is the composition of T1 and T2 and that is another linear transformation.

Which I know is the delta of T1 T2. So, now my linear transformation is T1, T2. So, I have this. So, these two are equal, which means delta T1, T2 is delta T1 into delta T2, so, that property is there. So, delta T1 T2 is equal to delta T1 into delta T2.

(Refer Slide Time: 15:44)



So, let us prove the claim. So, claim is that delta T is. So, we are assuming T is invertible, T is invertible and claim is that, and claim is that delta T is in, delta T is the modulus of determinant of T. So, we will use some results from linear algebra, I will not prove that. Let E1, E2, etc. etc Ek be the standard basis for, be the standard basis for Rk. So, Ej is simply 0, 0, 0, 1 at the jth place and 0, 0, 0 so, this is the jth place.

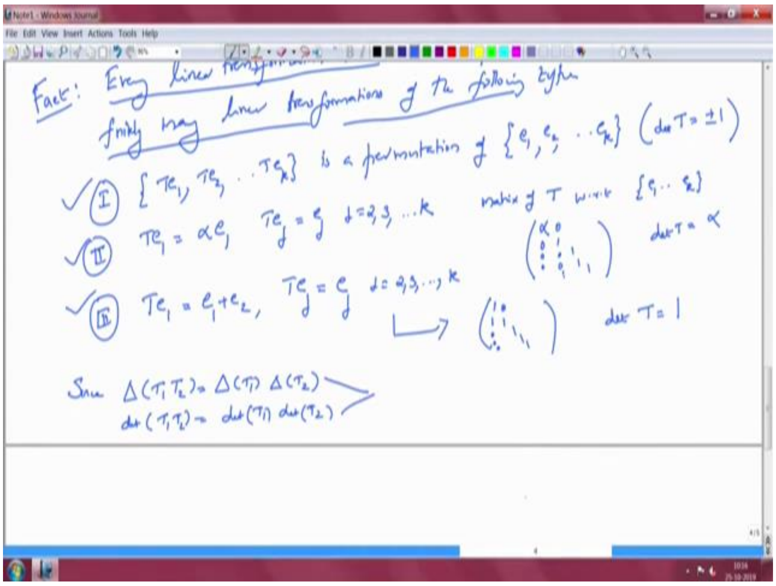
So, we use the following fact, so I will write it as a fact, every linear transformation, every linear transformation on Rk is the product of, is the product of finitely many, finitely many linear transformations, linear transformations of the following type, of the following type. So, what I am saying is, any linear map is the product of the following type. So, I will write down three types, one, TE1 and so we have fixed the basis E 1, E 2 , Ek which is the standard basis.

TE1, TE2, etc. TEk is a permutation of, is a permutation of E1, E2, etc. Ek, so such a T is of course invertible and so on. And you will have the matrix of T would be a permutation matrix. So, the determinant of T is plus or minus 1. In this case, determinant of T is plus or minus 1, so modulus is 1.

Second case, TE 1 equal to alpha times E1, where alpha is some constant and the other ones are fixed. So, TE j equal to Ej for j equal to 2, 3, etc up to k. So, then the metric of T, so matrix of T with respect to the standard basis, with respect to the standard basis, E1, E2, Ek will take the form, so let E1 goes to alpha E1. So, I will have alpha here 0, 0, 0 and well, everything else is fixed 0, 0, 0. So, I have 1 here, 1 here and so on. So, the determinant of T is, determinant of T is equal to alpha.

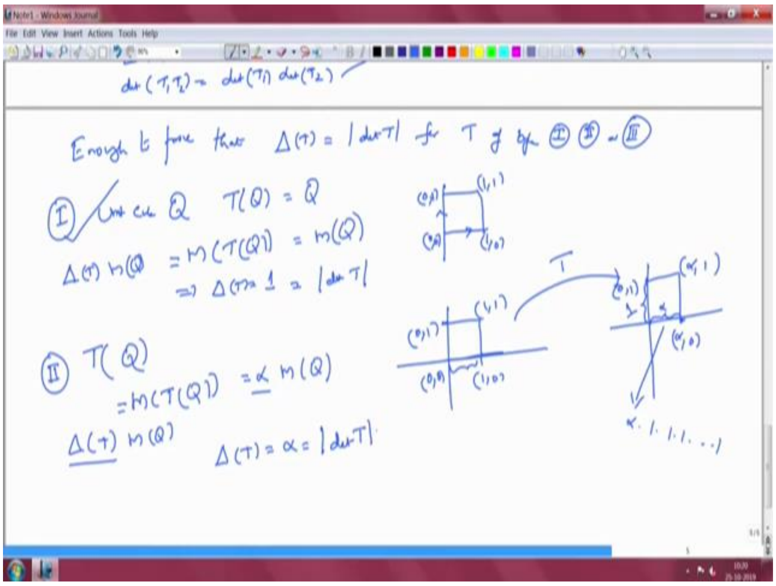
The third one, is the slightly complicated one, TE1 equal to E1 plus E 2 and other ones are fixed. So TE j equal to Ej for j equal to 2, 3, etc, etc. k. So, how will the matrix of T look like? Well E1 goes to E1 plus E 2. So, I will have 1, 1 here and 0 then, then I have 1 on the diagonal. So, this is how it would look like. And the determinant is of course 1, determinant of T is equal to 1. So, this is a fact we will be using that every linear transformation on Rk is the product of finitely many linear transformations of the following type.

(Refer Slide Time: 20:27)



And since we know that delta is. So, let us write one more line here, since, since delta of T1 T2 or any product if you like is delta T1 into delta T2 and the same for determinants, determinant of two (matri) product of two matrices is the product of the determinant, determinant T1 and determinant T2. If we have inequality at these types 1, 2, 3 where delta T is equal to determinant then because every linear transformation is a product of these three things, we will and using these two facts, we can extend it to all linear transformations.

(Refer Slide Time: 21:15)



So, it is enough to prove. So, enough to prove, enough to prove that delta T is equal to modulus of determinant T for T of type 1, 2 or 3, because of the product property. So, let us start with that. So, in the first case what is T? So, let us go back to the T is a permutation of E1, E2, Ek. So, if I fix the cube, unit cube Q, so unit cube Q then T of Q is equal to Q. Because it is the, so in dimension 2, I am looking at this Q 00, 10, 11 and 01.

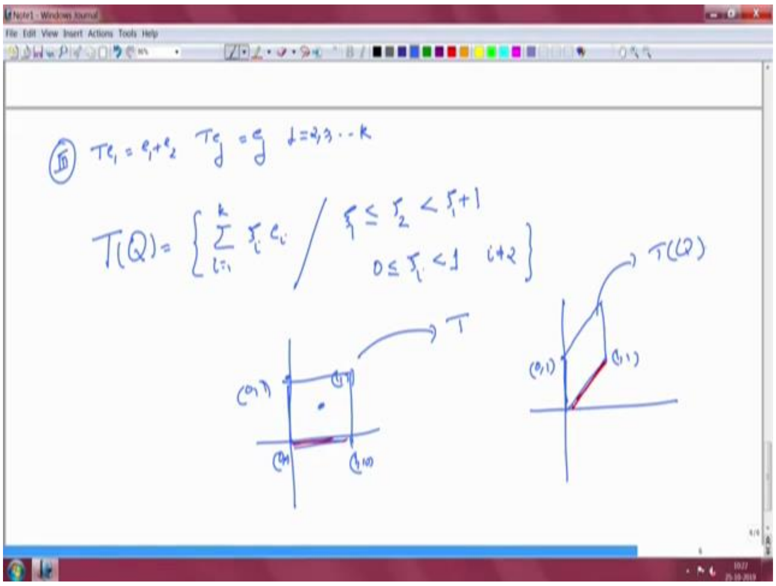
So, all that you can do is permute E1 and E2, this is E1 and E2. So, if you permute them, you will still the cube will be fixed. So, M of T of Q is equal to M of Q. To calculate delta T, it is enough to calculate the measure of 1 such set. But this we know is delta T times M of Q which implies delta T is equal to 1. Which is the modulus of the determinant, because in the first case with determinant is 1, plus or minus 1. So, modulus of, right, plus or minus 1, so modulus of that will be 1. So, this is equal to modulus of determinant of T. So, first case is done.

Okay, that was easy. Second case, well, in second case what happens to the unit cube? So, let us look at that. Second case, T E1 becomes alpha E1, everything else is fixed. So, if I take Q to be the unit cube, T of Q, what would be that? So, let us see, so, this is the unit cube, so, let us draw this. I have 10 here, 01, 00 here, 01 here and 11 here. So, this is my cube. Now, I apply t. Well, T multiplies E1 by alpha, so this will go to alpha 0. So, this goes to alpha 0 wherever that is, the remaining things are fixed.

So, this goes to 01 and then you add up. So, you will have something like this. So, this is alpha 1. So, all that has happened is, this side becomes alpha, and this side stays 1. So, that happens in all dimensions. Except one side everything else becomes alpha. So, T of Q, if I look at measure of T of Q, well measure of T of Q what happens, I have 1 side alpha, then I have sides 1, so, I multiply all of them.

So, I will get alpha, alpha into 1 which is measure of Q. But this we know is delta T times measure of Q which is 1. So, delta T has to be equal to alpha, which is the modulus of the determinant. So, delta T is equal to alpha equal to modulus of the determinant.

(Refer Slide Time: 25:22)

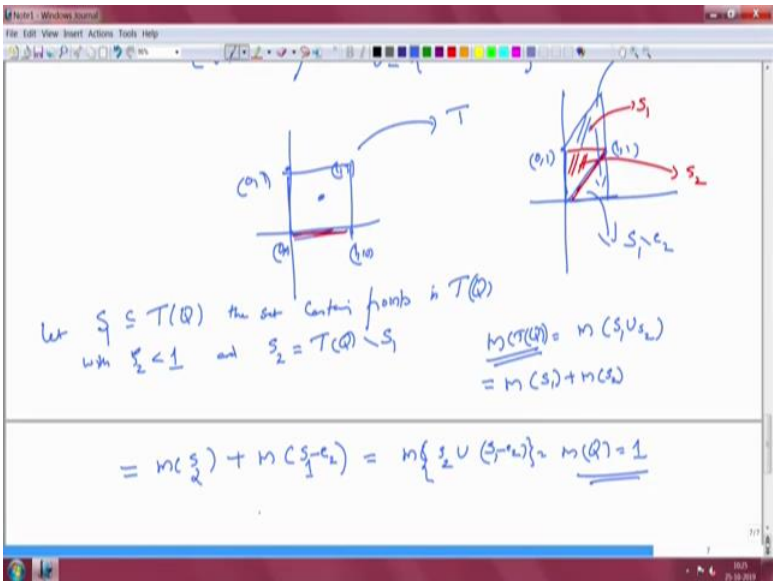


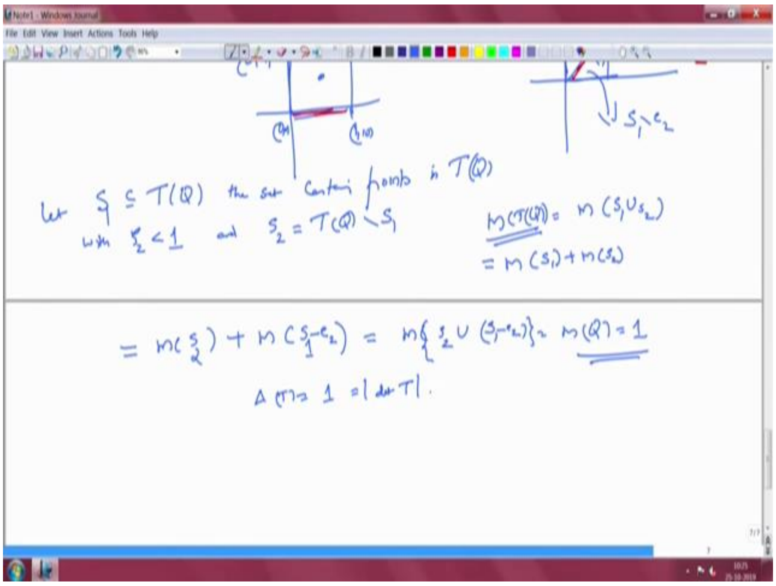
So, let us do the third case, third case is the one which is slightly complicated. So, remember the third case TE1 is E1 plus E 2 and other things are fixed TE j equal to Ej for j equal to 2, 3 etc up to k. So, let us write down what is T of Q, I will write down and then explain. So, T of Q is equal to all those points of this form. Xi Ei i equal to 1 to k where Xi 1 is less than or equal to Xi 2 less than Xi 1 plus 1 and the remainings are all fine. Whose Xi i less than 1 for i not equal to 2.

So, we will do this in R 2 and see how the picture looks like. So, I have 00, 10 which is E1 and 01 which is E 2 and 11 which is E1 plus E2, so I have this Q. I apply T to it, so I am going to get the image of this Q. Let us see where it goes. So, E1 goes to E1 plus E2. So, E1 plus E2 is this the vector 11, E2 goes to E2. So, that is again this vector. So, that is 01. So, I know this vector. So, let us use some other. This vector goes to this vector and this vector goes to this vector.

So, anything here is a sum of these two vectors. So, let us go to the corresponding sums. So, you complete the parallelogram, you will get exactly what is the image of Q under T. So, this would be your TQ okay.

(Refer Slide Time: 27:35)





So, some more steps to complete this. So, keep the picture in mind. So, let S 1 be contained in T of Q be the set containing points in T Q with Xi 2 less than 1. So, second coordinate has less than 1 and S2 the rest. So, let me denote it here, the Xi 2, the second coordinate is less than 1. So, what do you do so, you look at this line and whatever is here. So, this would be your S2 and this is S 1 okay. And of course, there are other coordinates in the higher dimension, so which I will not draw.

So, if I look at measure of TQ, this is equal to measure of S1 union S2, so that is a disjoint union, which is equal to measure of. So this is measure of S 1 plus measure of S 2 which is equal to measure of S 1 plus measure of S 2 minus minus E 2. So, this is the usual subtraction here, I am translating the set E 2 by, S2 by E2. So, what do I mean by that? So, let us see. Well, maybe I change the notation properly, I should, I should be subtracting the other way.

So, maybe S2 here and S1 here. So, the S1 part, if I subtract by E2, this portion will just come down here, and I will get this triangle. This would be my triangle S1 minus E2. But then these are still disjoint and so I have M of S2 union S1 minus E2, which gives me the original cube Q, which is 1. So, M of TQ, I started with M of T Q and I ended up with M of Q. So, delta T is equal to 1, delta T is actually equal to 1, which is the modulus of the determinant of T.

So, we stop here for the time being. We just showed that if T is a linear transformation, it maps Lebesgue sets to Lebesgue sets. And the measure of T of E were E is a Lebesgue set is modulus of the determinant of T time times measure of E. We will see that these properties of the Lebesgue measure which is translation invariance or the behavior with respect to the linear transformations etc, actually gives you some change of variable formula. So, in the next lecture we will start with that, and then we will go on to look at other sets with some special properties. Okay.