Measure Theory
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Lecture 26
Lebesgue measure via Riesz representation theorem

So, now, we start the construction of the Lebesgue measure using Riesz representation theorem, which means we have to define a positive linear functional on R n and Riesz representation theorem will give us sigma algebra and a measure.

So, the sigma algebra is going to be the Lebesgue sigma algebra which we have seen earlier and the measure is going to be the Lebesgue measure. And along with it all those properties which some of which we already know like outer regularity and inner regularity for sets with finite measure and sigma algebra is complete.

Since, R n sigma compact and every open set is sigma compact, we know the measure is going to be regular. So, in particular the Lebesgue measure will be a regular Borel measure on R.

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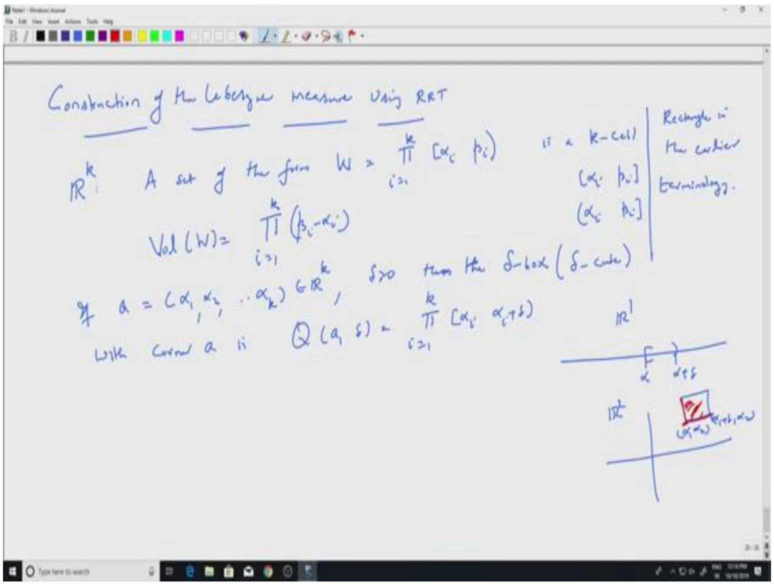

So, let us start the construction, construction of the Lebesgue measure using Riesz representations theorem. So, this is the second construction you are seeing there is another approach due to caratheodory, which we will not be doing. But I will give you some references for that that actually works in a much more abstract setting. So, let us look at R to the k. So, we change the notation slightly instead of R n, we will use R k. So, some simple definitions first, which you have seen already, we have used the term cubes, rectangles, et cetera. Now, we slightly change it because I am following Rudin's book "Real and Complex Analysis".

So, a set of the form, a set of the form say W equal to product of intervals alpha i beta i open, i equal to 1 to k is a k cell. Of course, sometimes one may change this to alpha i beta i or alpha i open beta i closed, et cetera. So, this is simply a rectangle in our earlier rotation. So, this is a rectangle in the earlier notation for terminology and of course, it has the volume which we already know.

So, volume of the W is simply the product of the length of the sides. So beta i minus alpha i, i equal to 1 to k. Now, if I take a point, if a equal to let us say alpha 1, alpha 2, et cetera alpha k, so this is a point in R k and delta is positive then the delta box or the delta cube with corner a is denoted by, so Q a delta. So this is simply the product of i equal to 1 to k because we are in R k, so there are k intervals, alpha i to alpha i plus delta.

So, you simply look at delta, so this is a cube with sides delta and the left hand bottom corner is a that is what is meant by this. So, let us look at R 1, in R 1 so there is only one coordinate, so I will have some alpha and alpha plus delta, so it this. In R 2, so if I take a point here, alpha 1, alpha 2, well what do you do?

You go this way to alpha 1 plus delta to alpha 2 and similarly here and similarly here, so you are only choosing the, so this the right hand side is open. So, you are only choosing this portion. This is contained and you have all the interior points, but not the boundary on the top right hand corner. So, the corner portion is here that is the, that is a for you. And you look at a delta cube cornered at a, that is what we are doing.

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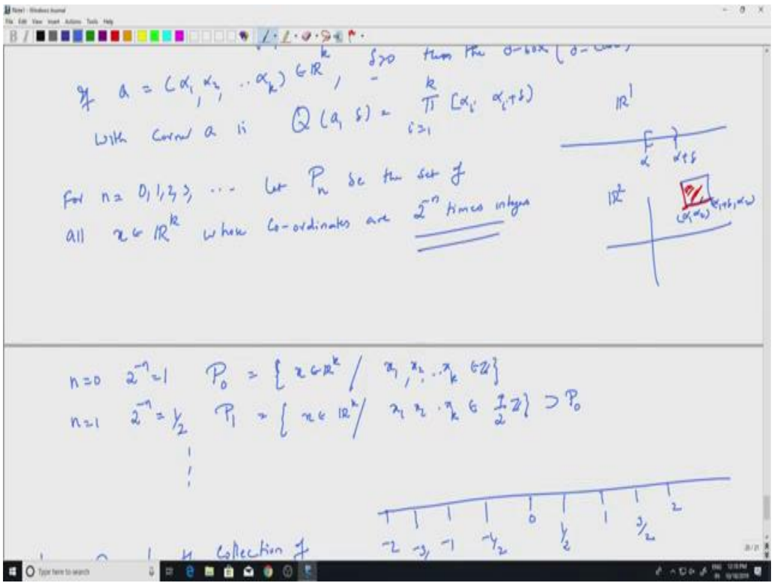

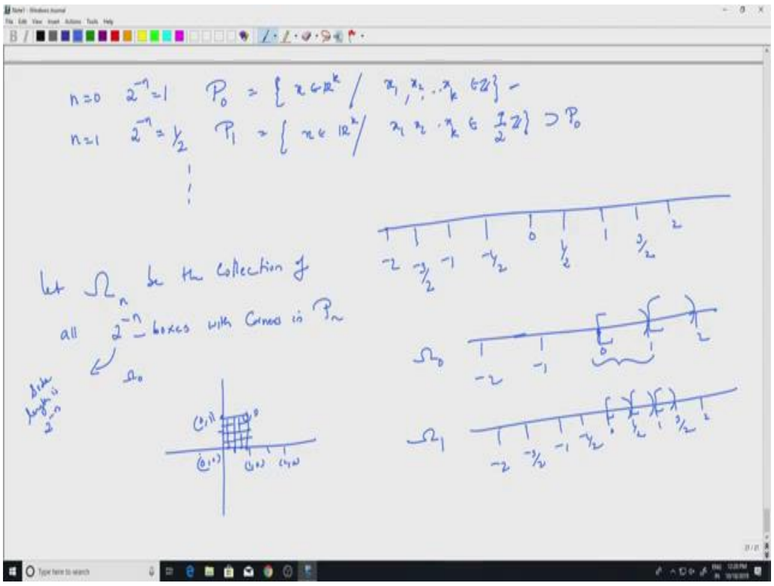

So, now, some notation for n equal to 0, 1, 2, 3 and so on. Let P n be the set of all points, all x in R k, so remember k is the dimension whose coordinates are, so I am fixing an n for P n. P n is the set of all points whose coordinates are 2 to the minus n times integers. So, let us try to understand this a bit more clearly.

So, when n equal to 0, 2 to the minus n is 1. So, P 0 is all those points x in R k such that x 1, x 2, et cetera, x k they are all integers, n equal to 1 well 2 to the minus 1 is half. So, P 1 is all those points such that x 1, x 2, et cetera, x k will be in 1 by 2 times z. So, 1 by 2 times z, so in particular this will contain P 0.

So, let us look at P 0 and P 1 in the dimension 1 case because that is a easy case, so 0, 1, 2, minus 1, minus 2 and so on. So, P 0 is just integers. P 1 is you divide all of them by half et cetera. So, you have divided everything in P 0 by 2. That is what you are doing and you continue.

Let omega n, so we have P n, Pn's are simply coordinates. Omega n is going to be collection of box be the collection of all 2 to the minus n boxes, so what is 2 to the minus n boxes? Remember the delta boxes. So, if I take delta I know what is the delta box, so 2 to the minus n box meaning the side length is 2 to the minus n. So, 2 to the n minus meaning side length is 2 to the minus n.

Well, so what are the corners for these boxes? Corners are in P n with corners in P n, P n is all those coordinates with this property. So, let us see, what is omega 0, so omega 0, what is omega 0? Collection of all one boxes with corners in P 0, P 0 is just integers. So, we have 0, 1, 2, minus 1, minus 2, et cetera. This is P 0, so box with corner at 0 of side length 1. What is that? That this interval, a box at corner 1 and side length 1. So, that is this et cetera et cetera. So these partitions, we align into intervals.

Well, what will be omega 1? Well, first of all corners are P 1, which means you divide all of them by 2. So, you will get minus 2, minus 3 by 2, minus 1, minus half, 0, half, 1, 3 by 2, and 2, et cetera, et cetera. And what is omega 1? You look at 2 to the minus 1 boxes, that is half boxes, so side length is half with corners in P 1, so corners are here and length half. So, all that you are doing is the first one which was this, you divide that into 2 and you have this one, this, this and so on. So, all the previous ones are divided by 2, this is what is happening in the real line.

Similarly, if you look at R 2, you will see omega 0, well omega 0 is first of all you have to find corners. So, you have 0, 0, 1, 0, 2, 0 and 1, 1, 0, 1 et cetera, et cetera. So, it will be boxes of the kind that will be omega 0, omega 1 well you have to find the corners first. First you divide these things by 2 and so on and then you will be dividing this by 4 that is because you are in dimension 2. In dimension k, you will divide this into 2 to the k boxes and you continue, next time you will have further divisions like this and so on and so forth.

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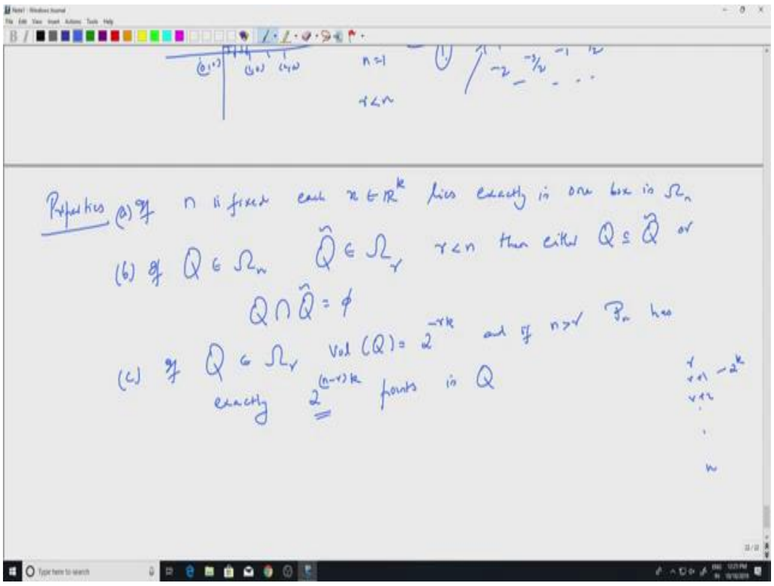

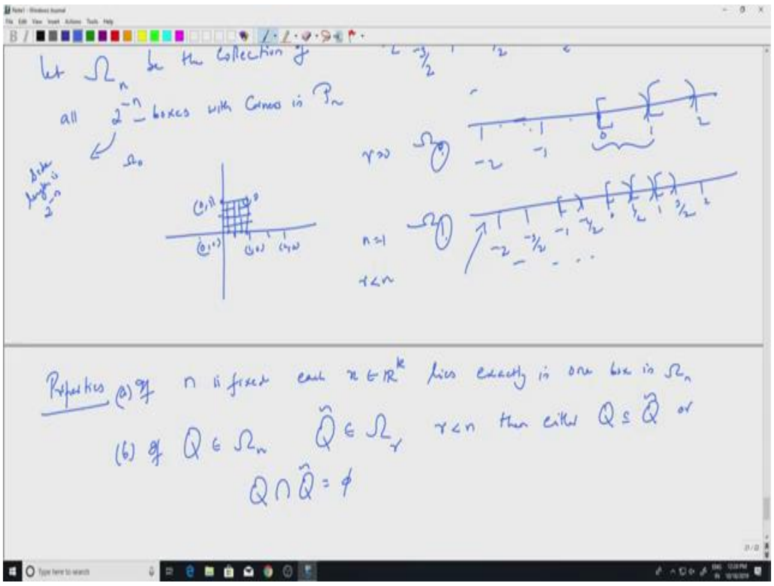

So, that is the collection of boxes we are looking at. So, there are properties of this. So, let us write down some properties if n is fixed, so remember n is not the dimension, k is the dimension, n gives you the corners and the boxes and so on, n is fixed. Each x in R k lies exactly in one member of omega n exactly in one box in omega n. So, that is clear from the picture right each time.

So, first time you have partition, so if you fix n equal to 0, any point will be in one of the intervals. Similarly, if you fix 1 that is your n, then any point will be just 1 interval because these are all partition, each time you are dividing the earlier intervals by half. In R 2, you are dividing each square into 4 squares. In R 3, you will be dividing into 8 cubes and so on. So, in R k. you will divide each box into 2 to the k boxes to go to the next level.

So, in each time it is a partition of this whole space and so, any point will lie exactly in one box, so that is 1. Second, if Q belongs to omega n and Q tilda belongs to omega r, where r is less than n then either Q is contained in Q tilda or they are disjoint, Q intersection Q tilda is empty. So, this is a pretty trivial observation even though it might sound very complicated.

Let us take omega 0 and omega 1. So, n is 0 and let us take R equal to 1. So, R is sorry the other way, so r is 0 and n equal to 1. So, r is less than n. What this says is if I take any box from here, so let us say this interval this will be of course contained in one of the intervals here or it is disjoint from any of this intervals.

That is because the intervals here are obtained by dividing the intervals in the bigger set, in the previous level. So at each level, you are dividing the intervals into 2 equal halves to go to the next level. So, when you compare it with previous box level, boxes in the previous level, either they are contained inside one box or they are disjoint, so that is very easy to see.

The next one requires a little bit more thought I will leave it to you, if Q belongs to omega r, so that is at the rth level. Volume of Q, this is trivial is 2 to the minus rk because you are in dimension k. So, each volume, so you have k intervals and each side has length 2 to the minus r, so it is 2 to the minus r k that is trivial.

And if n is greater than r, P n has exactly 2 to the n minus r k points in Q, so if n is greater than r, you will be dividing Q again and again to go to n. So, you will start from r, you go to r plus 1th level, r plus 2th level et cetera et cetera and you will end at n. At each time, you are dividing each Q by 2 to the k cubes so, at each time you will get 2 to the k cubes. So, there are n minus r levels and that will give me 2 to the n minus r k points. So, do it for k equal to 1, it will be really very-very clear to you.

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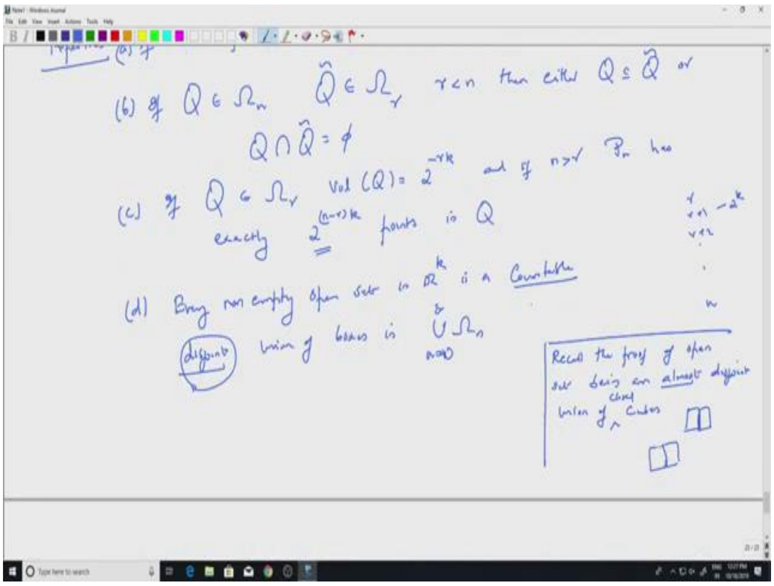

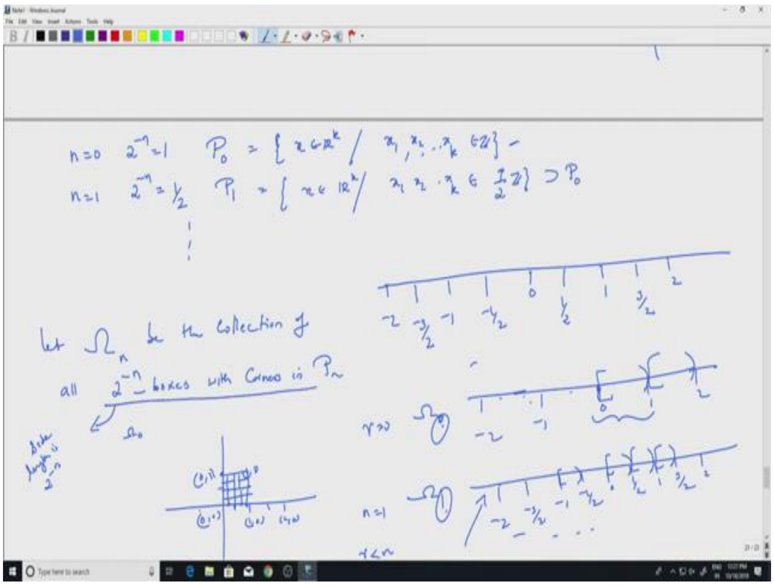

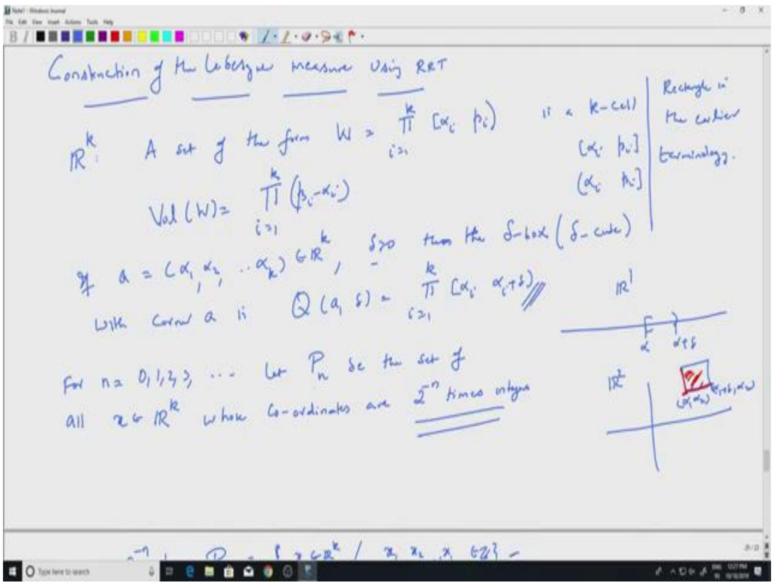

So, let us see, we need one more property d this is something which you have seen already. Every non-empty open set in R k is a countable, so that is important, disjoint union of boxes in omega n, of course, you have to take the union for 0 to infinity, you look at all the omega n, so these are boxes.

And you can write any open set as a countable union of boxes from the, so that is a, that is very easy because we have done this before for writing an open set as a countable, so let us recall that part. So, recall that, recall the proof of open set being an almost disjoint union of cubes. So, we did that closed cubes.

So, we had cubes like this and there was intersection at this middle, but the boxes are such that we are only looking at this part, the top part is not there in the box. So, when I take an adjacent cube they are sort of disjoint. So, that is what you should understand. So, that is why we get a disjoint union.

If you take the closed cubes, then you will get almost disjoint, but the cubes we are looking at omega n these are 2 to the minus n boxes with corners in P n. When I say delta box, remember the, this right hand side is open. So, because of that these boxes are disjoint. So, that is the reason you get disjoint ones.

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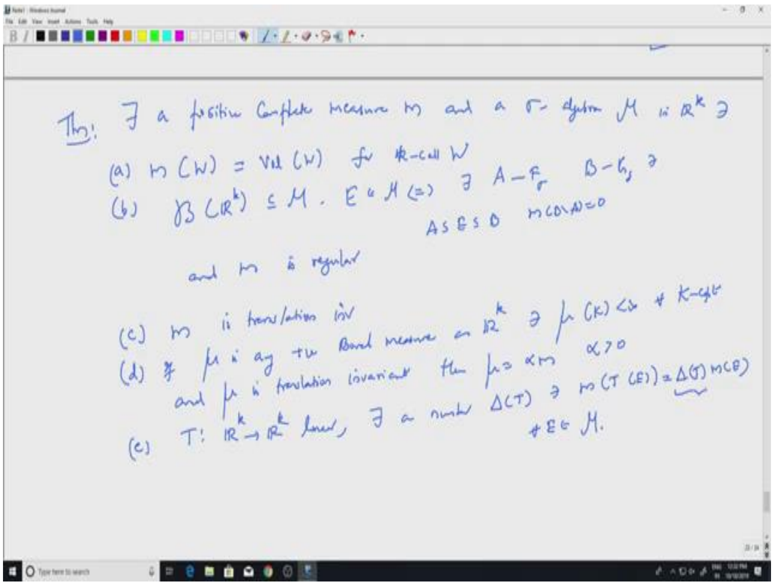

So, we will stop with the statement of the theorem and which actually gives us the Lebesgue measure. So, keep in mind all the omega n and so on. So, there exist a positive complete measure m, this is the Lebesgue measure and a Sigma algebra, and a Sigma algebra script M, which is actually the Lebesgue algebra sigma algebra in R to the k such that several properties so, all of those you most of you already know.

Measure of W's, a k cell W is the volume of W for k cell W, b, the Borel sigma algebra R k is contained in M. Also a set in M if and only if there exist, so let us use E because that is the standard thing we have been following. E in M , if and only if there exists A which is an F sigma and B which is a G delta such that A is contained in E contained in B and measure of B minus A is 0 and the measure M is regular, so we have seen this regularity.

Now, it is completely regular, it is not outer regular for all sets and inner regular for sets with finite measure. It is regular for all Borel sets. M is translation invariant, we know this but more importantly if mu is any positive Borel measure on R k such that mu of k is finite for every compact set k and mu is translation invariant, mu is translation invariant, invariant then mu equal to some alpha times M, it is a multiple of M. So, this is like the uniqueness of Lebesgue measure any reasonable translation invariant measure is a multiple of the Lebesgue measure there is nothing else.

So, the last property, if I have a linear transformation, linear there exist a number which we call delta T this is actually going to be the determinant of T, we will compute this later such that the measure of T E, so that is the image of E. So, you have to prove that T E is actually measurable is actually equal to delta T times m E for every E in M.

So, recall that we did something like this by multiplication by delta, when we looked at dilation of Lebesgue measure. We saw that delta to the n comes out which is precisely the determinant of the linear map, x going to delta x so that is the delta T here. So, we will stop here. So, we have just written down the theorem which constructs the Lebesgue measure out of Riesz representation theorem.

In the next lecture, we will look at the proof. So, this would be by constructing a linear functional which is positive on cc of R compactly supported continuous functions on the, on R k, in fact, and that will give us the Lebesgue measure along with the properties which we have already seen.