# Measure Theory

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## Lecture 23

## Locally compact Hausdorff spaces

So we have seen the construction of the Lebesgue measure and various invariance properties, in particular it is invariant under translation, invariant under dilation and we have invariance under reflection. So these were the three properties which we saw in the last few lectures. There are more such properties which we will see later on but for the time being we will stop the concrete example of Lebesgue measure and we will move to some abstract setting.

So in the abstract setting instead of Rn we have a locally compact Hausdorff space X. So remember when we started we always had a triple. There was X, the space the sigma algebra and the measure mu. So this space X will have more structure now, more topological structure. So it would be a locally compact Hausdorff space. So this is sort of anticipated in the sense that we already saw some such results for the Lebesgue measure like you can approximate measurable sets by sets which are G delta and f sigma and so on those, those come from topology.

So this was the interplay of topology with Measure Theory and we will be exploring that bit further in the abstract setting when the underlying space will have nice topological structure, namely, locally compact Hausdorff space. So the reason to look at locally compact Haudorff space is that it has a rich class of continuous functions because of Urysohn's Lemma. So some of these results I will state but these are results from topology which we will be using again and again. I will not be proving any of the topological results but I will state clearly what is needed.

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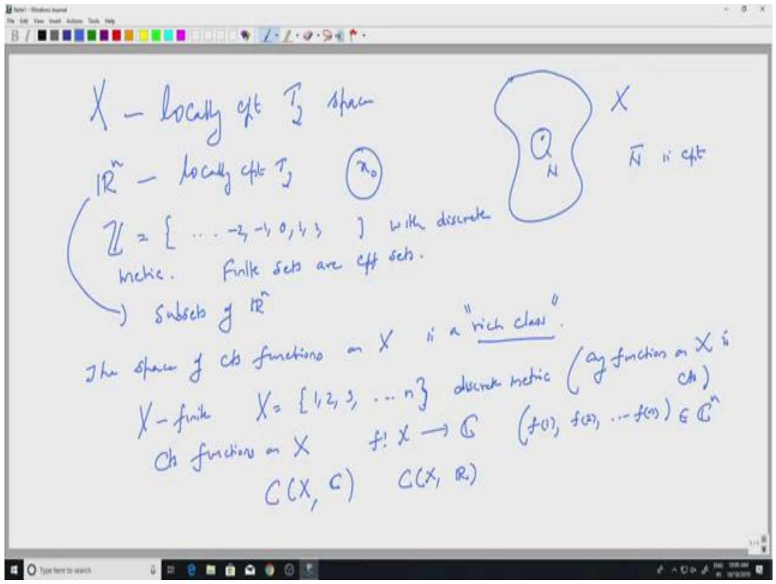

So let us start. So we have a space X which is supposed to be locally compact T2, T2 meaning Hausdorff space, so what does that mean? Locally compact means any point has a compact neighborhood. So if I have space X and if I take a point, there is a neighborhood around that point. Let us say N such that closure of N is compact. This is a very nice property to have usual examples work, so examples are Rn. So in case you are not familiar with such locally compact Hausdorff spaces, you can always replace this space X by Rn.

Whatever I say will be true for Rn as well because this is a locally compact Hausdorff spaces. Because you take any point x naught, you can take a ball around it whose closure is close and bounded, so it will be compact. So this is locally compact. Another example would be you can look at metric spaces. Let us say Z. So Z is all integers minus 2 minus 1, 0, 1, 2 etcetera, with discrete metric is also locally compact because any point is a neighborhood and that is also a compact set.

So in this space finite sets are compact sets. And of course from Rn you can also look at subsets of Rn. These are also locally compact. So there are lots of spaces of that kind where one can keep as a model example from which we will derive various results. So well, as I said one looks at a locally compact Hausdorff space because the space of continuous function is rich enough. The space of continuous functions on X is a rich class.

Well, I will explain what that means. Rich class, so it is a huge enough class to decide various properties of X. Of course, if X is finite this does not give you anything new. So let us dispose of that case. So if X is finite, we can think of X as let us say 1, 2, 3 and etcetera and with the discrete metric, and so continuous functions on X, so discrete metric. So any function on X is continuous. So, if I take a function on X, let us say complex valued, it is identified with its image, so f is entirely determined by f1, f2 et cetera, fn.

There are only n points, so you look at the image under f and that tells you what f is. So you can think of it has a tuple of n numbers and that will of course belong to Cn, n dimensional complex space. So continuous functions on X so we will denote it by C, taking values in the complex plane. Sometimes we may look at continuous functions taking values in the real life, so it will be clear from the context what is that we are using.

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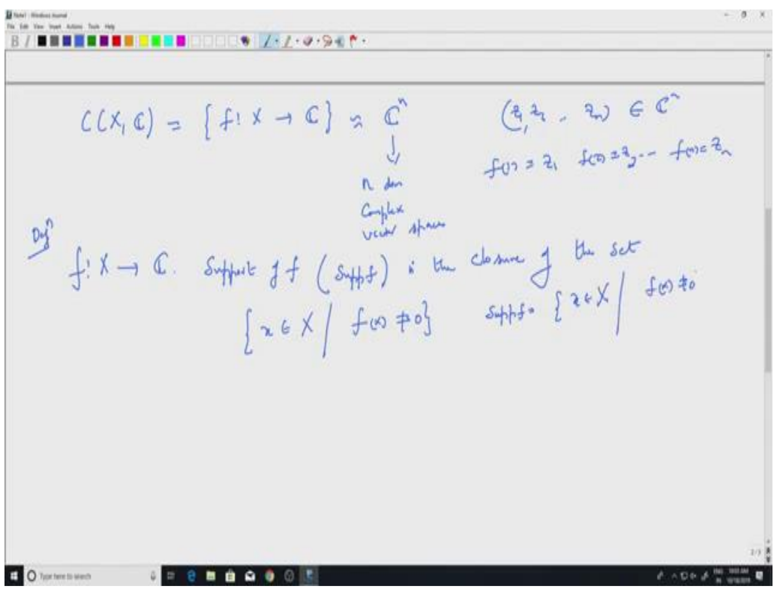

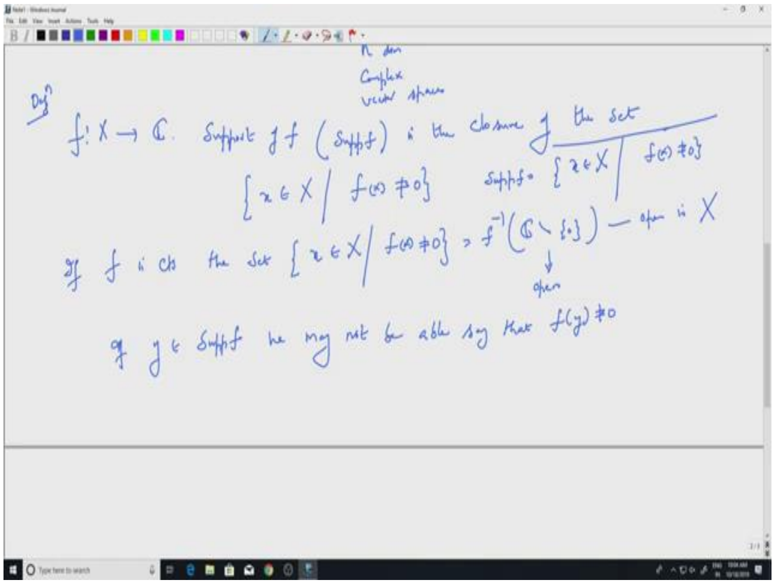

So these two spaces can be identified with, so C of X C that means continuous functions, so let me write it down so this is all those functions f from X to C because any function is continuous because X is a finite discrete metric space and this is clearly identified with Cn because any f is identified with the tuple, n tuple like this and any n tuple will of course define a function if I take Z1, Z2 etcetera, Zn in Cn, then all I have to do is to define the function f1 equal to Z1, f2 equal to Z2 and so on, fn equal to Zn.

So the continuous functions on X is identified with Cn. This is an n dimensional space, n dimensional complex vector space, so that is small enough. But if X is infinite, this is typically an infinite dimensional space and it is a rather huge class which determines various properties of X. So among continuous functions we would be looking at different class, subclass, so that is denoted by compactly supported functions, so let me define what support is. So let us take a function f from X to C.

The support of f, so call this a definition, support of f, so we will write support of f, is the closure of the set of all those points x in X such that f of x is not equal to 0. So you look at the space where f is, look at the set where f is nonzero and take the closure of that. That is the support of f , so support of f is the closure of the set. Now if f is continuous the set where f is nonzero is an open set, so this is f inverse of the complex plane minus 0.

This is open and so it is inverse image will be open in C in the space X  in the topology of X. So the closure would be a nice set, it is a closed set and it will have some interior  f inverse is open so when f is continuous you have this the support of f to be a nice set but what you should keep in mind that on the support of f so if I take some point from support of f we may not be able to say f of y is nonzero that is not possible.

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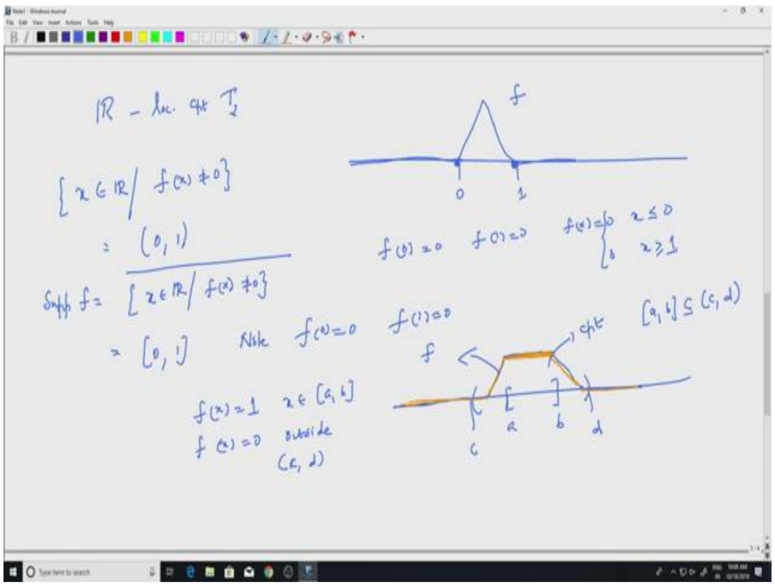

So let us look at some trivial examples to understand what this is, so let us look at the real line. This is of course locally compact, locally compact Hausdorff space and let us look at some function like this, so let us say this is 0, this is 1 and so on. Let us draw some function, some triangle like this. So it is, so f is f0, so this is my function f, this is a graph of f, f0 is 0, f1 is 0 and of course f is 0 here and here. So, f of x is 0 for x less than not equal to 0 and x greater than or equal to 1.

So the set where f is nonzero, so f of x is not equal to 0 that is the set open interval 0 at 0 and 1 it is 0. So support of f is the closure of this set that is the definition f of x is not equal to 0 and you take the closure of this and that is close interval 0 1. Notice that f is 0 at f 0 is 0 and F 1 is also 0 so but both 0 and 1 are in the support of f, these two points are in the support of f but f vanishes there, so that can happen.

But if I take an interior point perhaps, well, even then you may not be able to see. So let us not worry about it. So that is one thing to keep in mind that if I take a point from the support we may not be able to say this even though it does not cause any serious trouble. So let us look at the real line again. The, if I take some interval, so let us take some close interval like this a b. This is compact. Now if I take a neighborhood of this, so let us say this is c and this is d, so the close interval a b is contained in the open interval c d.

So open interval is a neighborhood of that. It is possible to draw a function which is like this, so which is 1 in a b and then dice down fast enough so that it is inside c d, so I am looking at some function like this. So, let us change the color and I have this function, this function here and then it is 0 that is my f. So this is the graph of f. So, what is the property f is 1 when x belongs to the compact interval a b and f of x is 0 outside the interval c d, outside the interval c d, c d is a neighborhood of a b. So this is very easy when I look at compact sets like this and open sets like this. But this is true in general for locally compact Hausdorff spaces.

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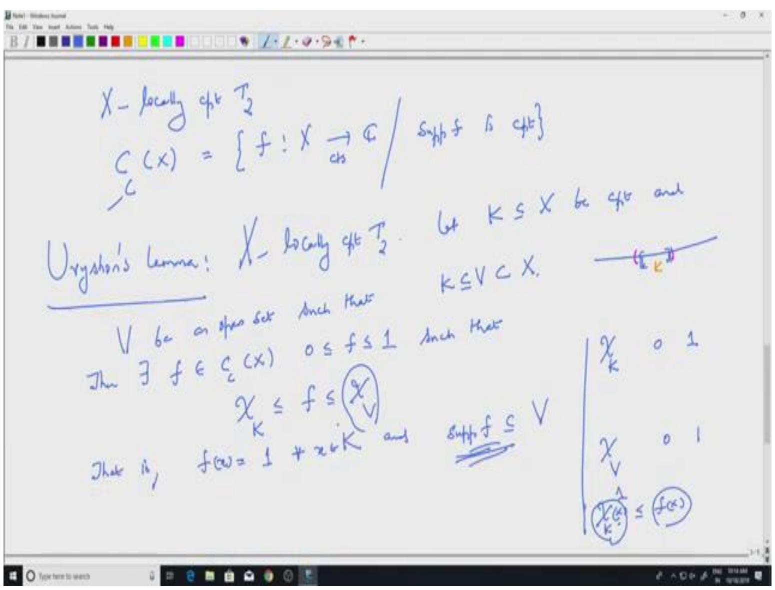

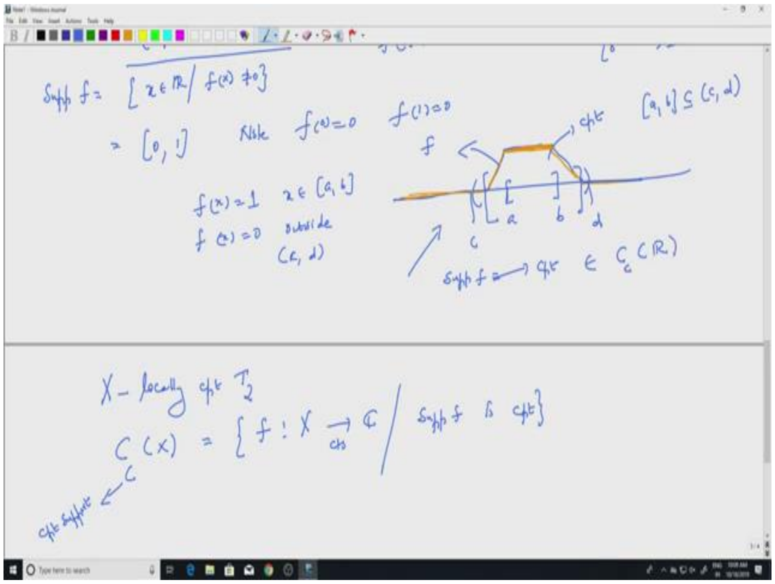

So, let us define a space first and then we will look at it more closely. So X is locally compact Hausdorff and I am looking at C c X. So the c here stands for compact support. Well, what is this? This is all those functions f from X to C continuous and support of f, so this is a set, this is a set. This is the closure of the set where f is nonzero, support of f is a compact set. So this example which we just looked at is one such f because the support of f is support of f.

So what is support of f? Support of f is close and bounded interval and so it is compact so that is a compact interval, so this is compact. So according to our definition this f belongs to continuous functions with compact support on the real line. So on the real line you draw lot of continuous functions with compact support. And this continues to be true for locally compact Hausdorff space. Well why is that?

So that is a rather useful and famous theorem due to Urysohn. So, let me write that. So, I will not prove the Urysohn's Lemma, it is a standard result from topology, so Urysohn's Lemma. What does it say? So X is a locally compact Hausdorff space. Let K contained in X be compact and V be an open set, such that K is contained in V. So, V can be a very small open set containing K, so look at the real line, for example, I can take compact interval here and my open set V, so this is K and my open set V can be very close to it.

So it can be very close to, but it does not matter, we have certain functions. So that is the conclusion of Urysohn's Lemma. So then the rest is f in C c X, so that is the compactly supported continuous function with the property that f is between 0 and 1, such that indicator of K is less than or equal to f, less than or equal to indicator of V. Well, what does that mean? That is just the way of writing the following that is f of x equal to 1 for every x in K and support of f is contained in V that is what it says.

Because indicator of K takes the values only 0 and 1, similarly indicator of V takes the values only 0 and 1, f is between 0 and 1 and so if I have chi K, chi K x greater than or equal to f of x then for every x in K this is 1, so f also will have to be 1 because f is less than equal to 1 anyway, so on K it takes the value 1 and outside V chi V is 0, so f will have to be 0 outside V which is why I am saying support of f is contained in V and this is what we have done here as an easy example.

So, whenever I take an interval like this and a neighborhood around it, it is easy to draw a picture like this which will tell me that there is a function which is 1 here and 0 outside and this can be done on any locally compact Hausdorff space that is what the Urysohn's Lemma tells you. So this already tells you that the collection C c X we are looking at is a rather rich class because you take any compact set and you take a small neighborhood of it, you have continuous function with this property.

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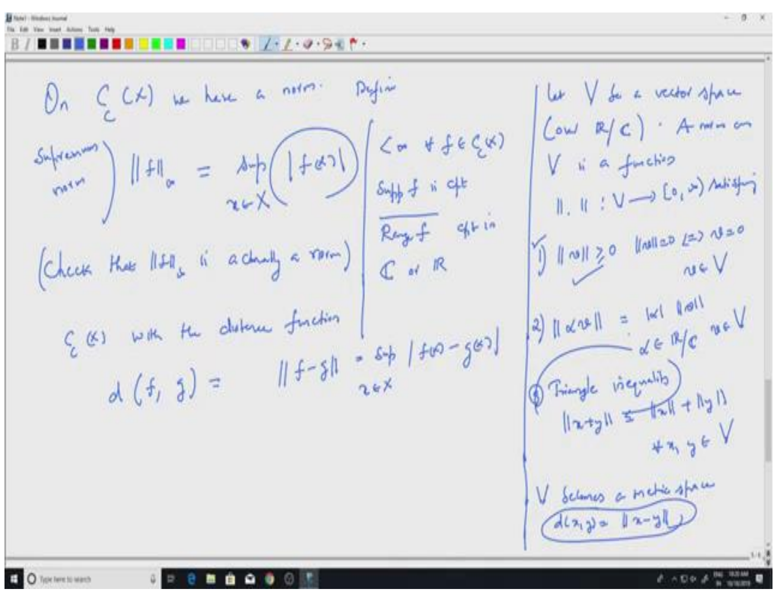

So we will look at this space once more so let me take a slight digression, so we will define things here. So let V be a vector space, it can be complex or real, so over the real or over the complex plane. So a norm on V is a function is denoted by these two lines norm from V to 0 infinity with the following properties satisfying 1, norm of any vector is greater than or equal to 0 and norm of V is 0 if and only if V is 0. So this is for V in the vector space V.

2, alpha v equal to mod alpha times norm V where alpha belongs to the field, real or complex, and v belongs to V. There is multiplication on the vector space mod alpha comes of. 3, is called the triangle inequality. It says norm of x plus y is less than or equal to norm of x plus norm of y for every x and y belonging to V. So that is called the triangle inequality. If you have these three things you say it is a norm. So this is exactly like the norm you have seen on Rn.

You can generalise it to get a definition on any vector space. Well more importantly V becomes a metric space so with the distance function d so d of x comma y you write as norm of x minus y. So to check that this is a metric you check three conditions, d is greater than or equal to 0 and d is 0 if and only if x equal to y that is the first condition here. The triangle inequality, d xy is equal to d yx that follows immediately and the triangle inequality here will give me triangle inequality for d.

So that is a trivial exercise. So any vector space with a norm becomes a metric space. So on C c X we have a norm. So, how do we define this norm? So define, we will denote it by f with an infinity there, that is called the supremum norm. So it is name is supremum norm or infinity norm. So this will be clearer when we define lp spaces later on. So, now just say supremum norm and as the norm indicates this is simply the supremum of x in X mod f of x. This is of course is finite for every f in, so this is finite for every f in C c X because the support is compact.

So, support of f is compact. So if I look at the range of f, range of f would be closure of range of f will be compact in C or R. So because of that the supremum is finite, so this is a finite quantity and this is clearly a norm because first of all it is greater than or equal to 0 is easy to see, and if I multiply f with the complex number alpha and that will come out and the triangle inequality is the triangle inequality for the modulus function and then you take the supremum.

So check that this is a norm is actually a norm. It is easy. Now, so it becomes a metric space so C c X with the distance function d well I take two elements from C c X so this is d f g equal to look at the definition of d xy equal norm of x minus y in the vector space. So this would be norm of f minus g which is simply supremum over x in X, modulus of f of x minus g of x. So this is a, this becomes a metric space.

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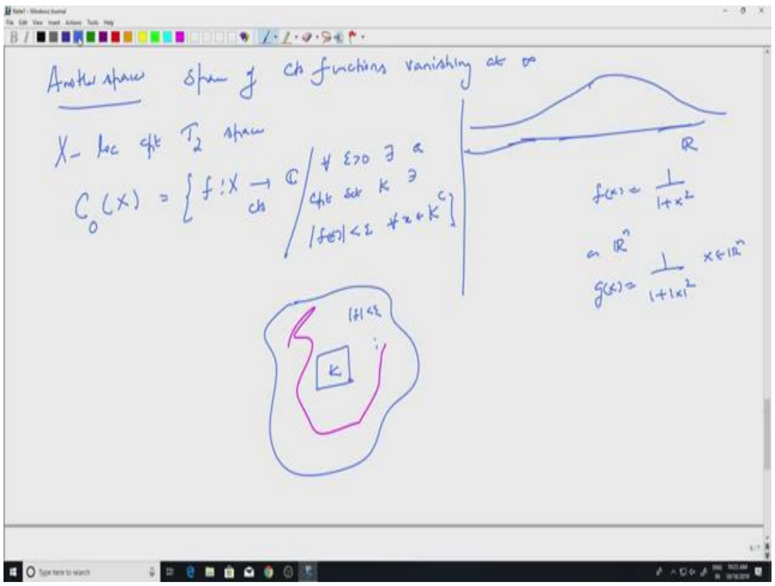

Now, let me define another space, so another space. This a space of functions vanishing at infinity. So space of continuous functions vanishing at infinity. So on the real line we are looking at functions which will vanish at infinity. Of course, we want to change real line to any locally compact Hausdorff space. So here, for example, you can take f of x to be equal to 1 by 1 plus x square. So this goes to 0 as x goes to plus infinity or minus infinity such functions would be.

Or on the complex plane on complex plane or more generally on Rn you can take, let us say g of x to be equal to 1 by 1 plus mod X square so x in Rn. So these are functions which vanish at infinity, as x goes to infinity this will go to 0. So we want to define something similar for locally compact Hausdorff space. So let us take X to be locally compact Hausdorff space and look at C 0 of X. What is C 0 of X?

These are functions vanishing at infinity but you know instead of defining what infinity is in this case what we do is we look at following definition. You look at all complex valued continuous functions such that so they should be vanishing at infinity which means that if you go towards infinity they becomes smaller and smaller. So for every epsilon positive the X is a compact set K such that modulus of f of x is less than epsilon for every x in K complement.

So, if x is like this, for every epsilon I should be able to find a compact set K such that modulus of f is less than epsilon in this portion. This is the portion K complement and there it should be less than epsilon. Well, why is it called functions vanishing at infinity? That is because there is something called the one point compactification of x, so I will write it down as an exercise for you because this is purely topology. This has nothing to do with Measure theory.

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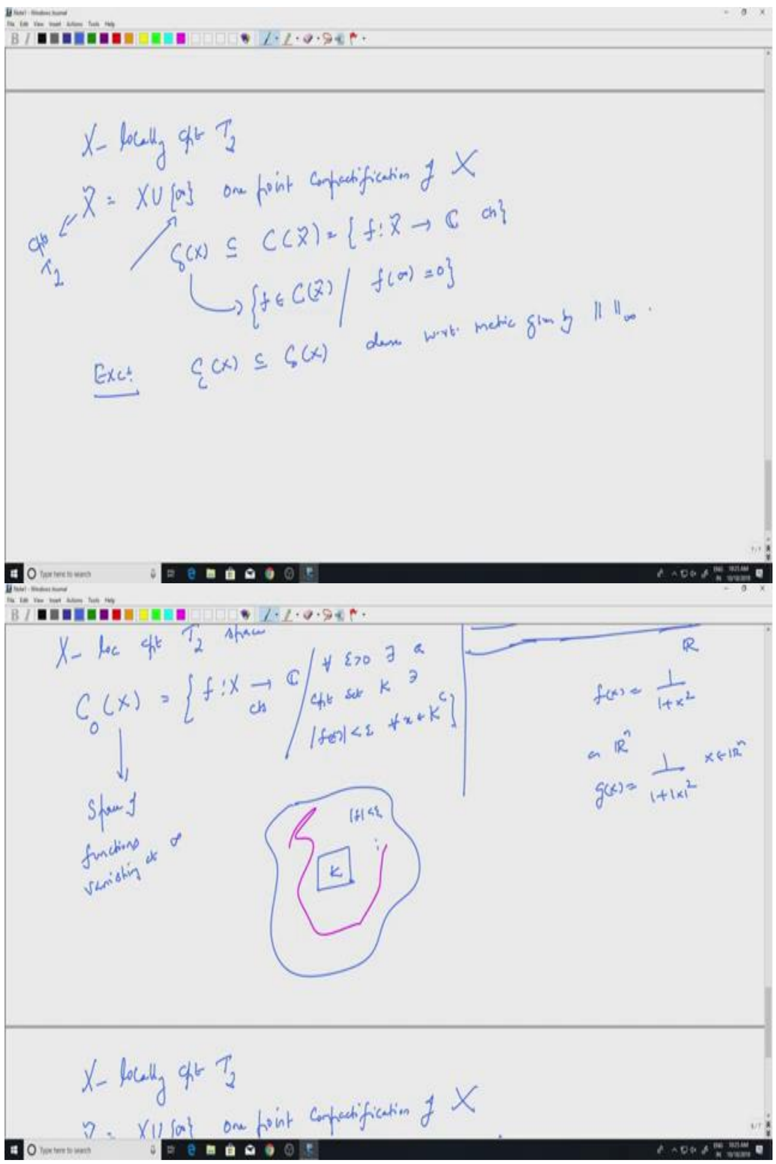

So, X is locally compact T2, then X tilde, so X tilde is X union infinity. This is the one point compactification of X and so this will be a compact Hausdorff space, so compact Hausdorff space and C 0 of X can be viewed as a subset of C of X tilde. So these are, what is this? This is all continuous functions on X tilde, continuous and C 0 X is identified with all those f in C of X tilde such that f at infinity, so infinity is the extra point we have added to make this space compact, f infinity is 0.

So that is why they are called functions vanishing at infinity. So this is the space of functions vanishing at infinity. This space will come up later so let me stop this with following exercise. So exercise, so purely analysis exercise or a topology exercise: C c X, so C c X is of course contained in C 0 X that is trivial. It is actually dense with respect to the metric given by the supremum norm. So in other words C 0 X is the completion of C c X with respect to the supremum norm.

So we will stop here. In this session we actually did not do any measured theory, we just set up certain abstract settings where the space is locally compact Hausdorff and we are looking at the space of continuous functions with compact support on such a space. In the next lectures you will see the connections with the Measure Theory. We will be looking at the linear functionals on the space. This is a vector space and so linear functionals will make sense and we will study those linear functionals and we will see that they are given by measures on X.