**Measure Theory**
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**Lecture 20**
**Non-Measurable Set**

So, in the previous session we saw invariants properties of the Lebesgue measure. Now, our aim is to prove that the Lebesgue sigma algebra is the completion of the Borel sigma algebra with respect to the Lebesgue measure, which means that any Lebesgue set is actually a Borel set union a set which has measures 0. So, if you recall the completion properties, what we do is we take a sigma algebra. We have a measure on the sigma algebra.

We just put all the subsets of the sets of measure 0, we get a bigger sigma algebra, which does not change much, but that becomes a complete sigma that is called the completion of the original sigma algebra. So, that is what we will do no. So, let me state the theorem, this is actually finer properties of the Lebesgue sets, which we need to actually prove, which will lead us to this proof of the theorem of completion. So, let me write the statement of the theorem first. So, let us say theorem.

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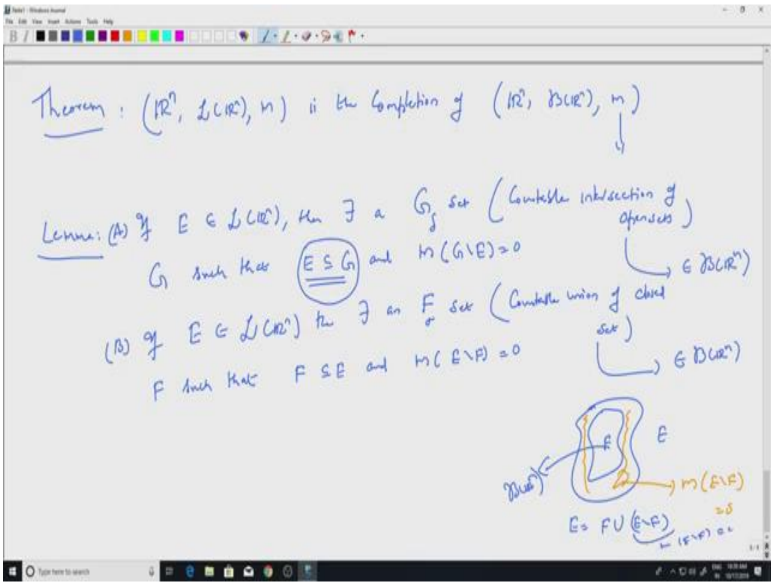

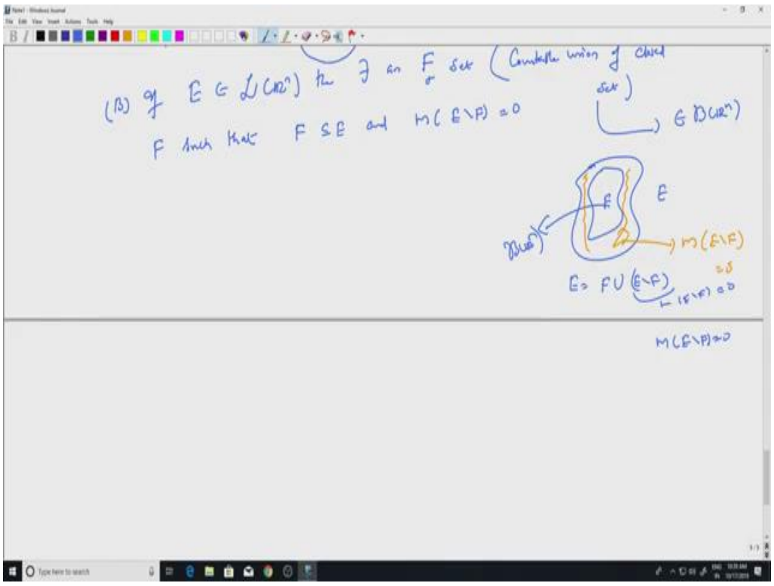

So, we have R n, we have Lebesgue sigma algebra of R n. And the Lebesgue measure is the completion of R n. You look at the Borel sigma algebra of R n, so that is a smaller sigma algebra and the measure. So, m is the Lebesgue measure. So, the Lebesgue measure, Lebesgue measure restricted to any smaller sigma algebra is also a measure. So, there is no confusion here, B of R n is a smaller sigma algebra than Lebesgue sigma algebra.

Of course this, it assumes that they are not equal, so we will construct some sets to prove that they are not the same, but there are other cardinality arguments which will tell me that tell you that these 2 sigma algebras are not same, because this has this is what is known as accountably generated sigma algebra. So, it is it has the first uncountable cardinality and this has much more. So, that is one way of seeing that the inclusion is strict. But let us not bother about it right now.

We will just prove that, this is the completion. Well how do you prove that this requires some results? So let us start with a lemma, some lemma, so if E was a Lebesgue set, then it is very close to being a Borel set. So, that is a statement we want to make. Then there exist a G delta set, G delta set, well, what is the G delta set? So, G delta set is countable intersection of open sets. So, this is a countable intersection of open sets right of course.

So, this would belong to the Borel sigma algebra. Because opens sets are in the Borel sigma algebra and so countable intersection of open sets will be in the Borel sigma algebra. So, I want to say there is a G delta set, let us call it G such that E is contained in G and the measure of G minus E is 0. So, let us say A and similarly B. So, this tells me that E and G, they differ only by a set of measure 0 that is the, that is a assertion. So, E differs from the Borel set G only by a set of measures 0.

So, and any such statement will have a counter statement from inside because E is contained in G. Similarly, we can find something inside E, so recall the definition of measurability and so on, we had our open set which is bigger than the set. Similarly, you had a closed set which is smaller than. So, this, the statement B follows from statement A is simply by the same proof. So, if A belongs to L of R n then there exist an F sigma set, what does an F sigma set? So, this would be a countable union of closed sets. It is a complement of G delta set.

And of course, closed sets or Borel sets, so countable union of closed sets will also be Borel set. Well, so F sigma set F such that F is contained in E. So, compare it with this one F is smaller than E. And of course, measure of E minus F is 0. So, let me, let us draw some pictures. So, if this is E and let us say this is F, the F is a Borel set. And whatever is remaining, the remaining portion that has measure 0, E minus F is 0.

So, what did we, so the lemma asserts that E can be written as F union E minus F where E minus F has measure 0. So, measure of E minus F is 0, which is precisely what we mean by completion. So, any Lebesgue set is the union of a Borel set and a set which has measure 0. So, that is why we get the completion. So, I will and by construction, the Lebesgue sigma algebra is complete, so, I let me, I will come and draw it after the proof the dilemma.

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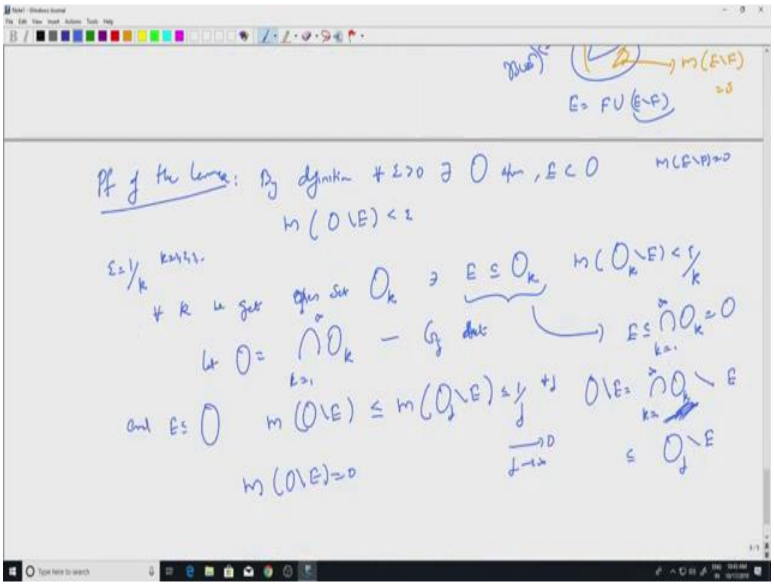

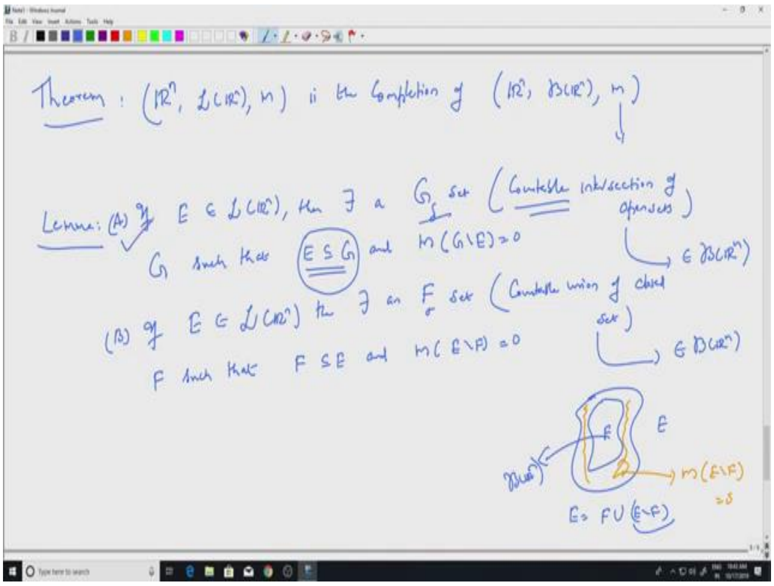


So, let us start with the proof of dilemma. Proof of dilemma, so well, we know that for every epsilon, so by definition, so we know that by definition, by definition, for every epsilon positive, there exist an open set to O such that m of O minus E is less than epsilon. So, and E was of course contained in, so this we know. We do this for epsilon equal to 1 by k so, and k runs from 1, 2, 3, et cetera.

For each 1 by k we can do this. So, remember our idea is to get open sets and countable in this section. So, we run epsilon over a countable collection of set of numbers going to 0 that is a standard technique. So, we will get for each k, so for every k, we get an open set, open set O k such that E is contained in O k and the measure of O k minus E is less than epsilon then 1 by k, epsilon in 1 by k. So, what is the countable interception you can take? You simply take the intersection of O k. So, let O be equal to intersection O k, k equal to 1 to infinity.

So, this is a G delta set, G delta set. And of course, each for each K is contained E in O k. So, this of course tells me that E is contained in intersection O k, k equal to 1 to infinity. So, the set O, so this is overwrite, so O contains E. So, we know that and we want to look at measure of O minus E. Well, what is measure O minus E? Well, O minus E is equal to intersection k equal to 1 to infinity O k minus E, which is of course contained in, you take anyone set from here let us say O j minus E for any j.

So, this would be less than or equal to measure of O j minus E by monotonicity, which I know is less than 1 by j. And this is true for every j, I can here, I can choose any set from here and that will give me a 1 by j. And that of course goes to 0 as E go infinity, which means measure of O minus E is 0, which is precisely the statement of the lemma, write the first statement. So, if I have a G delta set such that difference has measure 0. Similarly, I have a F sigma set well what do you do? You use the other definition of measurability or the equal and definition of measurability.

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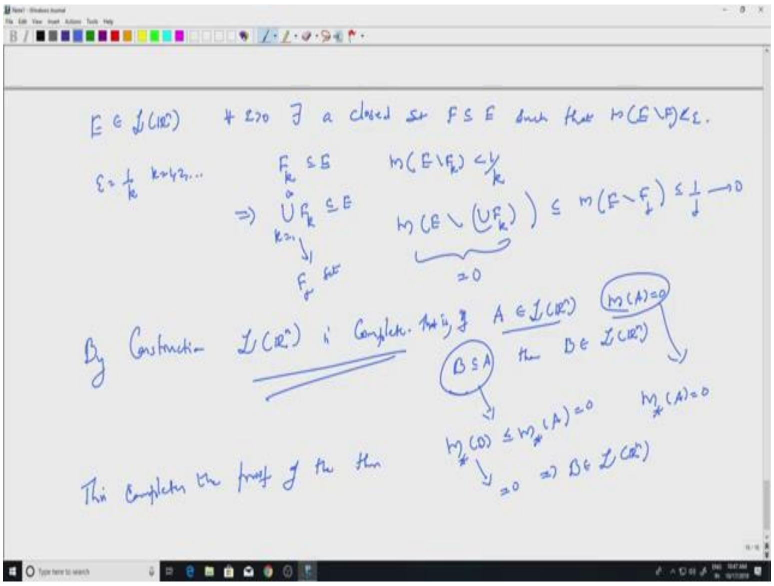

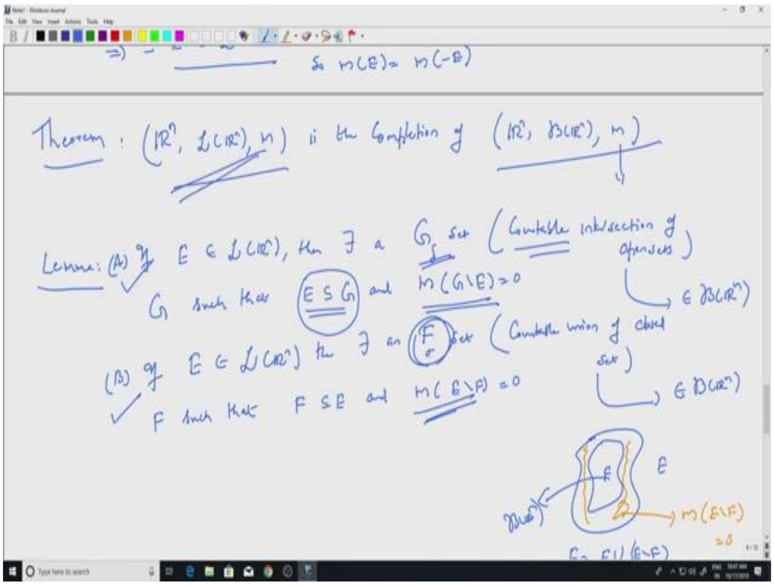

So, we have, so E belonging to L of R n tells me that for every epsilon there exist a closed set, there exist a closed set F, but now F is contained in E such that the measure of E minus F is less than epsilon, this we have. So, we can write epsilon to be 1 by k as usual k equal to 1 to et cetera. And we will get closed sets F k contained in E such that measure of E minus F k is less than 1 by k and of course because also this is a same proof.

So, because of this F k union F k is also contained in E and this is a F sigma set and m of E minus union F k this is less than or equal to measure of E minus any of them, when you throughout bigger junk union F k and that will be contained in E minus F j for every j which is result 1 by j which goes to 0. So, this is equal to 0, so it is a same proof. So, these are statements sort of complement to each other. We have seen this before.

So, this tells me that any Lebesgue set is actually a Borel set and a set of measure 0 and by construction, by construction Lebesgue sigma algebra E is complete. Well, why is that? Well, what does that mean? That means if A is a Lebesgue set such that measure of A 0 and B is contained in A then B is also a Lebesgue set, this is what completeness means. So, let us take that, so if I take A in Lebesgue set, Lebesgue measurable and measure of A 0. So, measure of A 0 is same as saying the outer measure of A is also 0.

But B is contained in A. So, by monotonicity we know that B star, m star of B is less than or equal to m star of A. But m star of A is 0. So, that implies this is 0. So, if m star of B is zero then B is measurable that we have seen any set which has outer measure 0 is a measurable set. So, B is in L of R n, which means that L of R n is complete. So, you put together these two, this completes the proof of the theorem.

So, this completes the proof of the theorem. So, we stated a theorem in the beginning that, that this is the completion of this. What we have just proved from the lemma is that any Lebesgue set is the union of an F sigma set which is a Borel set and something which has measured 0 and this Lebesgue sigma algebra is complete by construction. So, that put together those two will give me the proof of the theorem.

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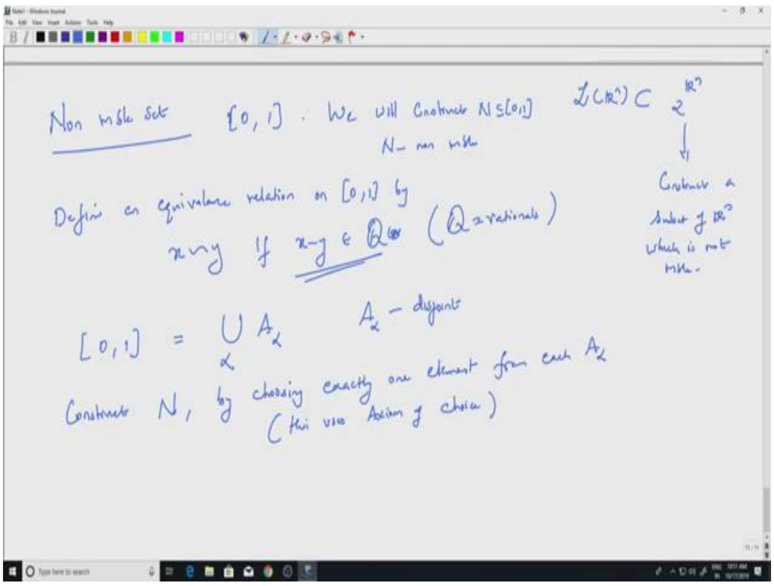

So, let us construct an example of a non-measurable set. So, just to conclude that, so remember one of the inclusions we had L of R n sitting inside the power set, we want to say this is actually strict inclusion that means, we need to construct, so this is the power set. So, we need to construct a subset of R n which is not measurable that is what we will do, so non-measurable set. So, this requires what is known as axiom of choice, I will comment upon it when the time comes.

So, what we do is first define, so look at closing double 0, 1. So, we are going to construct a subset of 0, 1 which is a non-measurable. So, we will let me write down that we will construct a set N which is contained in 0 and N non-measurable. So, how do we do that? First define an equivalence relation, define an equivalence relation on 0, 1 by x is equivalent to y, if x minus y belongs to Q, Q remember as rational. So, these are rationales. So, here the not the Q, Q here is rationales.

So, if x minus y is the rational then we say x is equivalent to y. So, using the equivalence relation, we can partition 0, 1 you put together all those numbers, which are equivalent to each other, by this definition, we will get one equivalence class and so on other equivalence classes. So, by equivalence classes, we can partition 0, 1. So, we can write 0, 1 to be equal to union alpha, A alpha, where A alpha are equivalence classes. We do not know anything about the indexing set alpha, so alphas can be countable and well it will be uncountable that is clear.

But you know how big it is, how small it is, et cetera, we do not bother about it. 0, 1 is partition, A alpha are disjoint. So, this is something which happens with any equivalence relation. An equivalence relation partitions a set and any partition gives you an equivalence relation. So, construct N from this, well what do we do? By choosing, so this is the tricky part actually even though it will sound very obvious this use a axiom of choice, choosing exactly 1 element from each A alpha. So, this is what uses, so this uses axiom of choice, so this uses axiom of choice.

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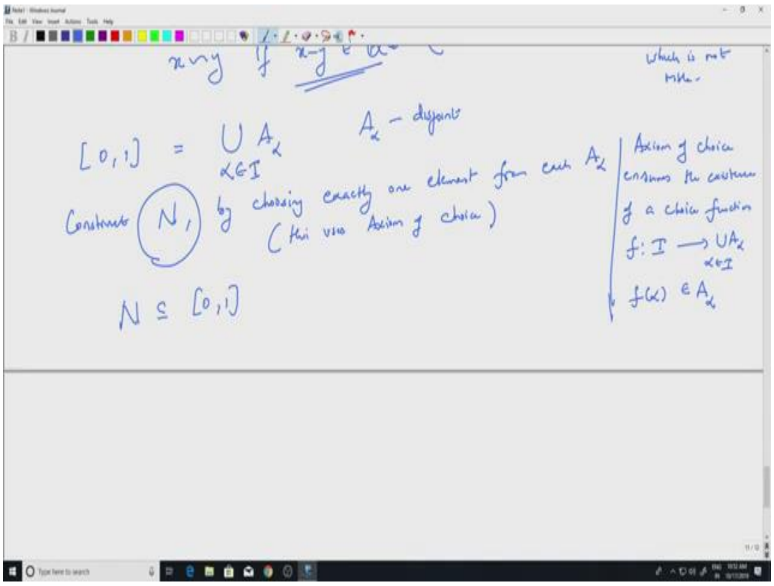

So, let me tell you what it is. So, I am not going to get into what is axiom of choice. So, what are we trying to do? So, let us say we have an indexing set I, capital I. So, axiom of choice axiom of choice ensures the existence of a choice function of a choice function. Let us say F from I that is the indexing set to union A alpha, alpha NI, well what is the property?

F of alpha will be in A alpha. So, you can pick 1 element from A alpha for every alpha, that is what axiom of choice tells you that it is possible to do that. And that you put together to get the set N, so that N is of course contained in 0, 1. These are numbers in 0, 1 we do not know how it looks like inside 0, 1.

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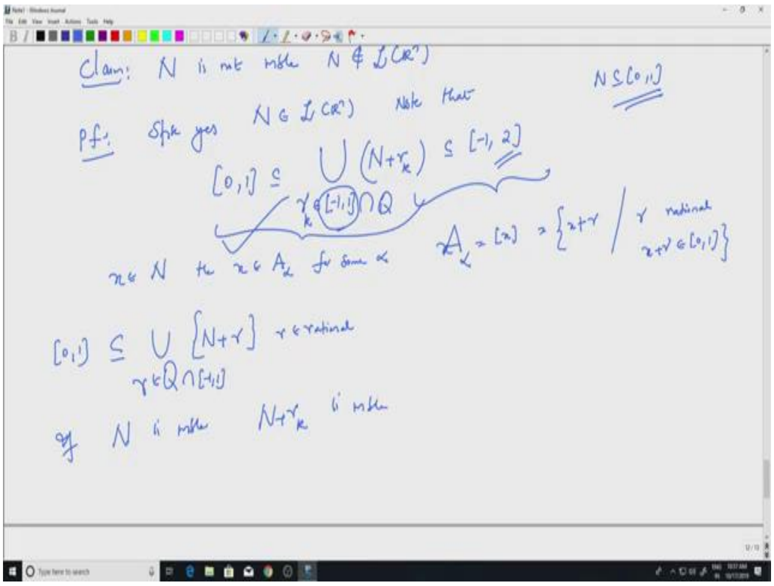

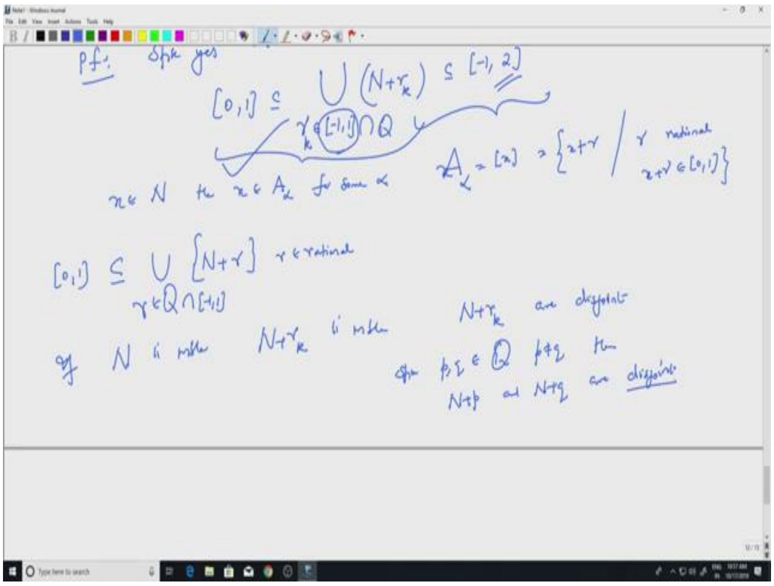

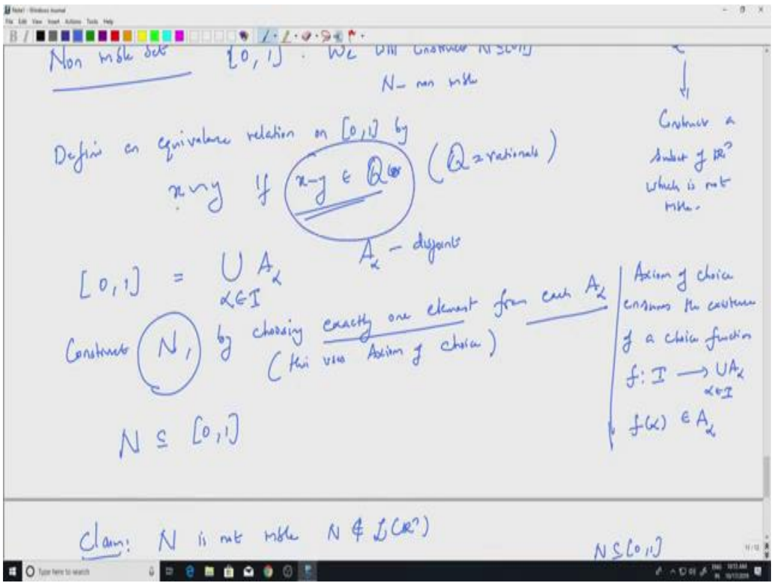

So, claim is that m is not measurable, claim N is not measurable. Well why is that? So, let us N is not a element of L of R n. So this is proved by contradiction. Suppose S, suppose N is actually a measurable set. So, measure of N will make sense, it is a measurable set. Now, what you need to notice is that, note that, so let me write down those and then explain. Closed interval 0, 1 is contained in union m plus, let us say R k, what are R k? R k are rationales inside minus 1, 1 intersected with so Q, Q is rational.

So, you look at all rationales in between minus 1 and 1 and look at N plus R k. So, remember N plus R k would be, you simply add R k to L every element in N, is contained in minus 1, 2.This is easy to see, to the N is contained in 0, 1, N is contained in 0, 1, So, if I add anything from minus 1 to 1, I am going to end up inside minus 1, 2. And of course, 0, 1 is contained in N plus R k. So let us try to prove that, prove the first part.

So let us see this. So N is chosen by exactly one element from each A alpha. So, if I take 1 element from N, so let us say x belongs to N, then x belongs to some A alpha for some alpha. So, what would be x? What would be A alpha? So, A alpha is the equivalence class corresponding to x. So, this is simply x plus all rationales. So, this is simply x where r is a rational and such that x plus r belongs to 0, 1 that is all is needed.

So, overall, if I know 1 element, all the other elements in the equivalence classes was given by simply adding rationalists because that is how the equal insulation is defined right x minus y belongs to Q, so x and y differ only by a rational. So, all elements which are equal into each other will differ by a rational. So, that is how you get these rationales. So, if I look at N plus r where r belongs to rationales, I am going to get the union of this will give me every element in 0, 1, so intersected with minus 1, 1 will do.

So, all elements in 0, 1 will be inside this, because I have equal, I am choosing N to be exactly 1 element from each equivalence class and if I have 1 element in 1 equivalence class, and I add rationales to it, I will get every other element in the equivalence class. So, if I put together of them, I will get 0, 1. And of course, this part is fine. So, this is trivial because N is contained in 0, 1. And if I add things from minus 1 to 1, I am going to L end up here. So, we have this.

So, if N is measurable, if N is measurable, then N plus r k is also measurable, that we have seen. So, translation does not tell me, translation will preserve measurability. In this measurable, so N plus r k is also measurable. Well, it is not just that they are measurable, they are disjoint, so N plus r k are disjoint are disjoint. Well, what does that mean? If I take two rationales suppose, p and q are rationales and p not equal to q then N plus p and N plus q are disjoint. Let us see why is that?

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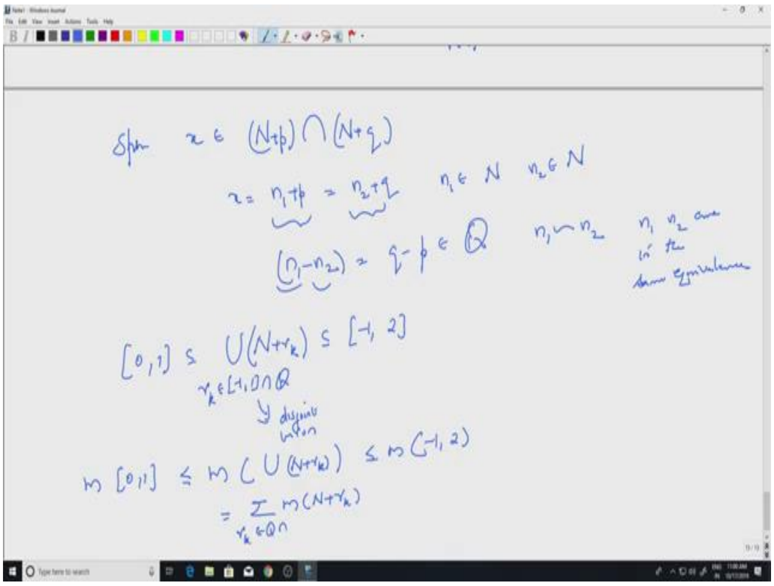


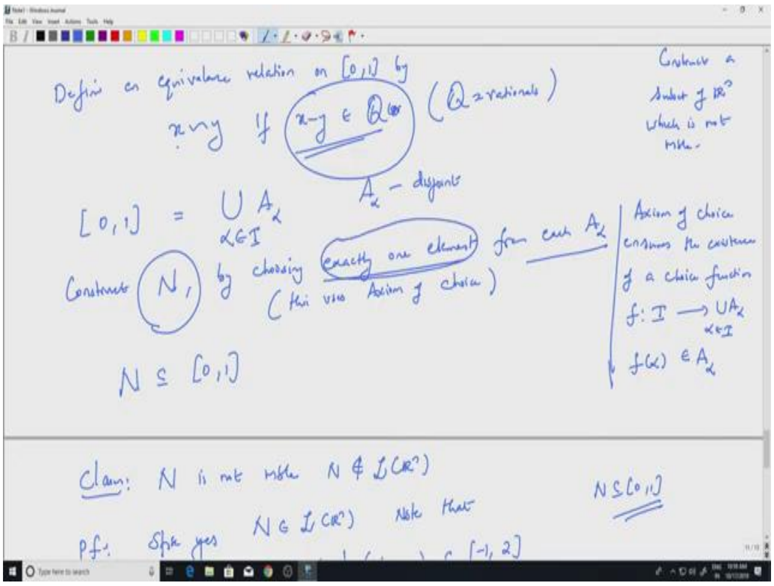


Suppose, they are not disjoint, so I take some element x inside N plus p and inside N plus q, intersection is not empty. What does that mean? That means x can be written as some n 1 plus p and it can also be written as some n 2 plus p, where n 1 belongs to N and n 2 belongs to N that is the meaning. But, what does, so this tells me n 2 plus q sorry this means n 1 minus n 2, equal to q minus p which is a rational number. So, n 1 or n 2 are 2 elements, but there are difference is in q.

So, n 1 will be equivalent to n 2, correct? But that is a contradiction because n 1 is equivalent to n 2 then n 1 and n 2 are in the same equivalence class, are in the same equivalence class. But that is not possible because n 1 and n 2 from each equivalence class, we have only exactly 1 element in N. So, if I take 2 elements in N, they cannot be in the same equivalence class. So, because of that these are design.

So, now let us go back to the situation at hand. So, we have 0, 1 contained in union N plus r k where r k is rational inside minus 1, 1. And of course, this is also contained in minus 1, 2. So, if this is measurable, these are, so now we proved that this is a disjoint union, disjoint union. So, this is measurable, N plus r k are measurable.

This is a disjoint union, we can take the Lebesgue measure, so Lebesgue measure of 0, 1 by monotonicity, this will be less than or equal to measure of union of N plus r k over that set less than or equal to measure of minus 1, 2. But the middle term because it is a disjoint union is the sum, sum of N plus r k, where r k belongs to rationales intersected with whatever intersected with minus 1, 1.

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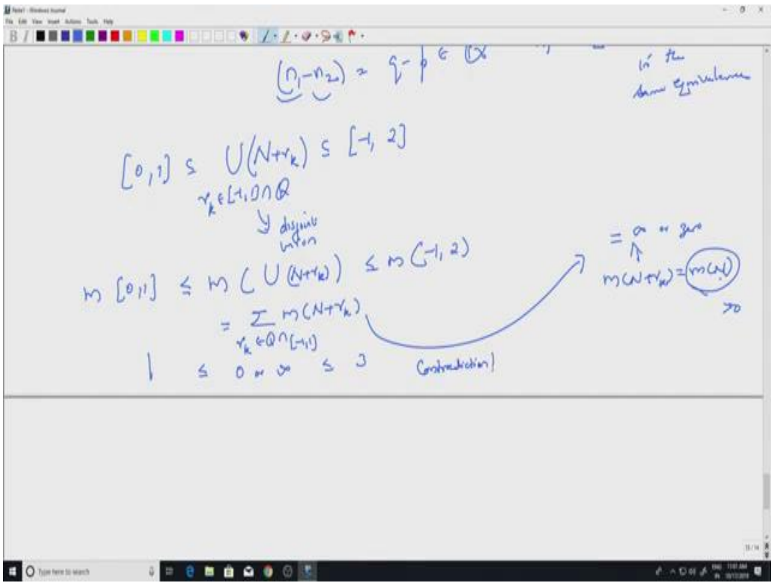

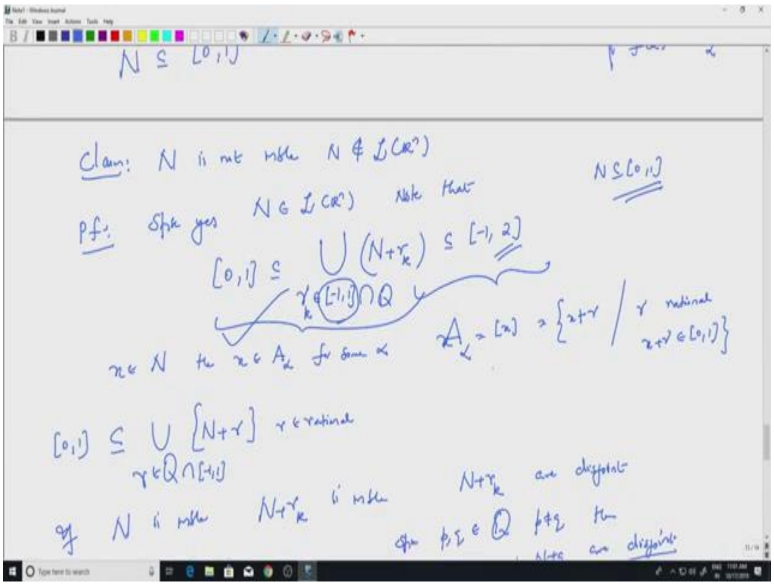

But what would be this, this will be infinity or 0, because N plus r k is m of N plus r k is, same as m of N, this is the translation invariance of the Lebesgue measure. So, if this is positive, then I will get infinity or if it is 0, I am get it 0. But both will lead to contradiction because I have 1 here less than or equal to it is either 0 or infinity, result to 3 here. So, that is a contradiction. And that is why the set is not.

So we started with the assumption that this is measurable, but it is not. So, that is how we construct a non-measurable set. So, we will stop here. So, we have just seen an example of a non-measurable set, which is contained in 0, 1, which proves that the Lebesgue sigma algebra is smaller than the Borel sigma algebra. At some point of time, we will also comment upon why there are Lebesgue sets which are not Borel.

We will continue with Lebesgue measure for the next few lectures as well. The next topic, we to compare the remand integration and Lebesgue integration and see that we are not losing anything, we are only looking at a much more general theory than remand integration.