Measure Theory
Professor E. K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 2 - Sigma algebras and measurability

So we will continue, we first define what a measure is on a sigma algebra, we look at some examples. Then we look at the class of functions called measurable functions on which we will be defining integrals.

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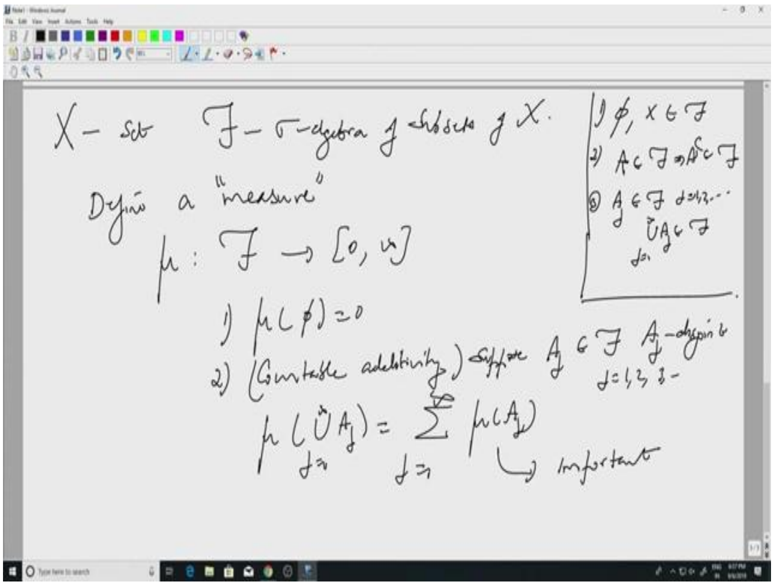

So let us recall what a sigma algebra is. So remember, we have a set X. This is a space. Script f is a sigma algebra of subsets of X. So remember there are 3 properties script f should satisfy. So let me write it down again. So, one is the empty set and the whole space should be in script f to which complementation should be it is closed under complementation phase in f, then a compliment is in script f and it is closed under countable unions. So A j are in script f, j equal to 1, 2, etcetera.

Then union A j is also in remember, it is a countable union of sets in f. On this sigma algebra we define a measure. So, measure is what is going to generalize the length function or the area or the volume which we are familiar with in dimensions 1, 2, and 3. So this would be a function mu which will denote generally by mu from script f to 0 infinity, infinity is included some sets might have infinite measure like the real line has infinite length. Well, it should satisfy two properties, one mu of the empty set is 0.

So, the empty set will not have any measure, two, this is the most important property, this is called countable additivity. Suppose, A j are in script f, A j disjoint. So, you take disjoint sets countably many, so, remember this is accountable collection 1, 2, 3, etc. Then the measure of the Union j equal to 1 to infinity, this is summation mu of A j, j equals 1 to infinity. So, we call that A js are disjoint so, when I take disjoint sets and take the union, measure of that should add up. So, this is what you see in the case of the real line.

If I take disjoint intervals, then length of union of the intervals should be the sum of individual components. That is precisely what those property is, this is analytically very important. Many of the properties you will see for integration is actually deduced because of this countable additivity property. So, let us look at examples.

So right now we will see very, we will essentially one example which can be generalized in some directions. A genuine example would be seen after we construct the Lebesgue measure which will be after few lectures, so what right now, what we will do is we will start with a sigma algebra measure and we will define what is meant by integration and prove certain theorems in integration. So, this would be a completely abstract approach for a while which will give you certain theorems in integration. Then we will construct what is known as the Lebesgue measure on R n where you will have a concrete example of a genuine measure. So, let us look at some examples.

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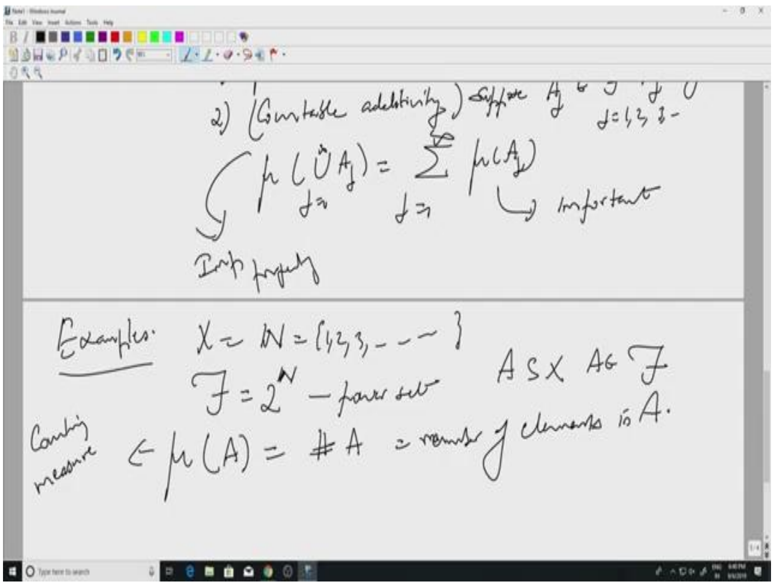

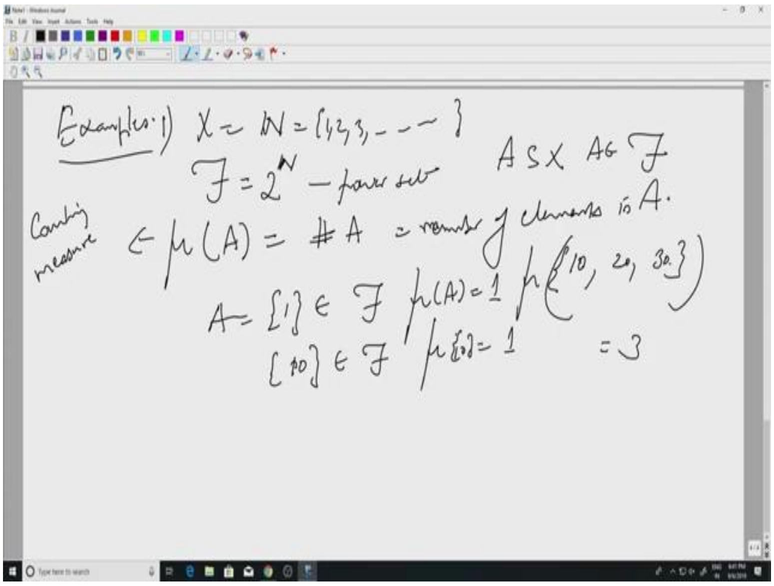

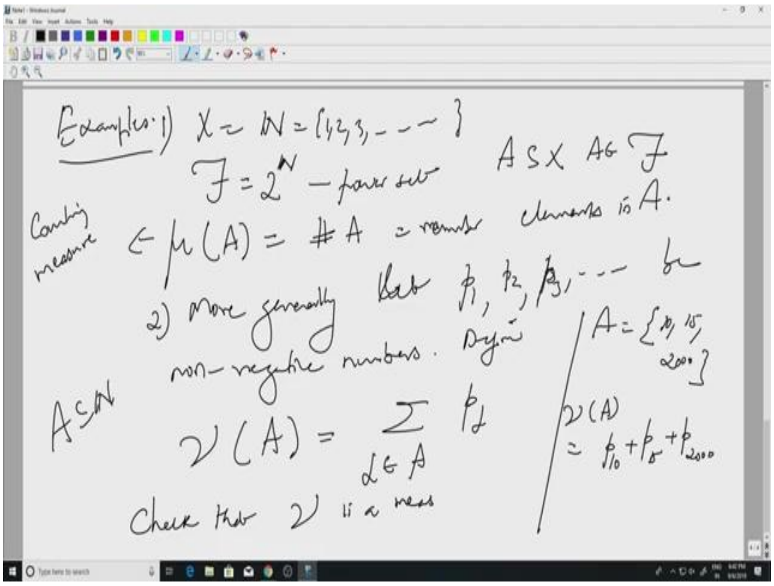

Let us say X is the natural numbers, we will call that say 1, 2, 3, etc. And we need as a sigma algebra script f. So, we will just take the power set. So this is the power set, I want to define measure mu, so this is called the counting measure. This is easy to understand, you simply count the number of elements in any set A. So this is the power set. So any set contained in X is actually in the sigma algebra. And I am trying to define its measure. So this is simply number of elements in A, so this is called the number of elements in A, this is obviously a measure, remember those properties which you need.

Countable additivity is the important property, that is what you should check. Everything else that it has measure 0 for empty set is a trivial thing. If you have disjoint sets then of course the number will add up. If one of them is infinity, some is infinity, etc. So you can look at various cases which will tell you that this is a measure. You can generalize this more generally, here when you look at number of elements in A, measure of each point is 1. So let us do that first before going to general stuff. If I take A to be the Singleton 1, so remember Singleton 1 is a subset of n. And so it is in script f.

What is mu of A? This is 1 or I take any other Singleton, let us say 10. This is also in script f, and mu of the Singleton 10 is simply 1, each element has measure 1. So if I take a set of element, let us say 10, 20 and 30. If I take this set, the measure of that the counting measure of that is 3. Right, you are adding this, this and this so, that gives me 3. So, instead of giving one for each element we can give some numbers.

So that is the example I want to look at. More generally, let p 1, p 2, p 3, etcetera be positive numbers, non-negative numbers. Define, so this is a new measure, define, let use some other letter for that nu of A, so you take a subset A contained in N, remember that is in script f because script f is the power set. This is simply summation so, you look at all the elements in A, correspondingly you add all the p i. So you look at P j summation P j, where j is in A. So this is also a measure. So for example, if A is 10, 15, 2000 then nu of A is simply p10, so that is a non-negative number. Plus p 15 plus p 2000 whatever those numbers are, so this gives me and it is easy to check that, so check that this is a measure. Check that nu is a measure, so this is what measure is, so we will continue with this.

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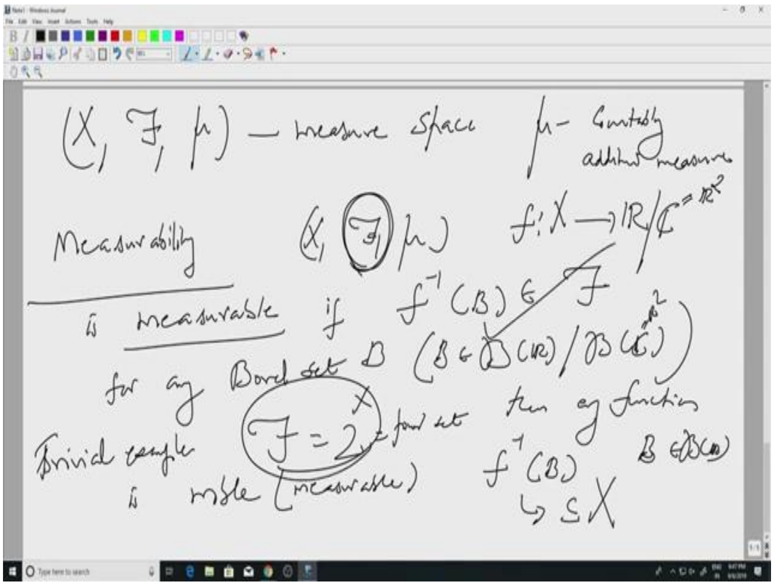

So counting measure, now so what we have is space, sigma algebra and a measure. Such a thing is called a measure space. So, mu will be always accountably additive measure, you remember that, countable additivity is very important, double additive measure. So sometimes I will say countably additive measure or I will say simply a measure. So, this is the first step. So, right now we have abstractly defined a measure mu. Remember this is supposed to generalize the length function or the area measure or the volume measure and so on.

So mu is supposed to have the countable additivity. If you have disjoint sets, then the measure of the union would be the sum of individual components. Now, we need to go to integration. For that, we need to get a collection of functions for which the integral can be defined. So, remember in Riemann integration we looked at lower sums, upper sums and took the limit and we said that if they were equal then the function is Riemann integral.

So, that is our next aim. So, we look at a collection of functions for which the integral can be defined. So, the suitable concept is measurability. So, I have a measured space and let us say I have a function from X to the real line or to the complex plane. So remember complex plane is simply R2. So for various reasons, we will write C. Given this sigma algebra, we say f is measurable. So, keep in mind that measurability will depend on which sigma algebra we are looking at, we have a sigma algebra script f given to us, we are trying to define measurability. f is measurable if f inverse of any Borel set B belongs to script f for any Borel set B, so Borel set meaning B is a member of the Borel sigma algebra.

So, remember C is simply R2. So, either you look at C as R 2 or C as a usual matrix space, you have the smallest sigma algebra generated by open sets that is your Borel sigma algebra. So, you pull back any Borel set that should land up in your sigma algebra script f then you say the function f is measurable. So simple example or a trivial example. If script f is the power set then any function is measurable, mble meaning measurable. Why is that? Because if I take f inverse of any Borel set B, this is of course a subset of my space X, but my Sigma algebra is the power set. So, this belongs to the sigma algebra and so this condition is easily satisfied, that is because you are looking at the biggest sigma algebra and the…

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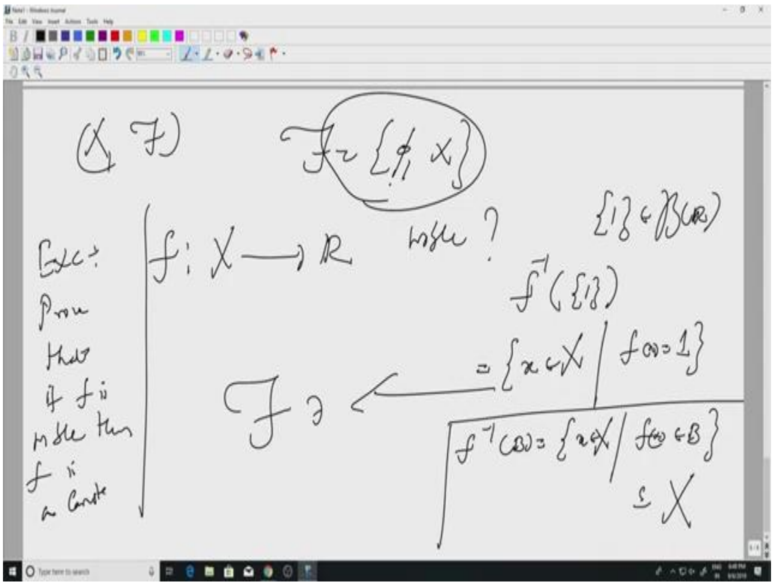

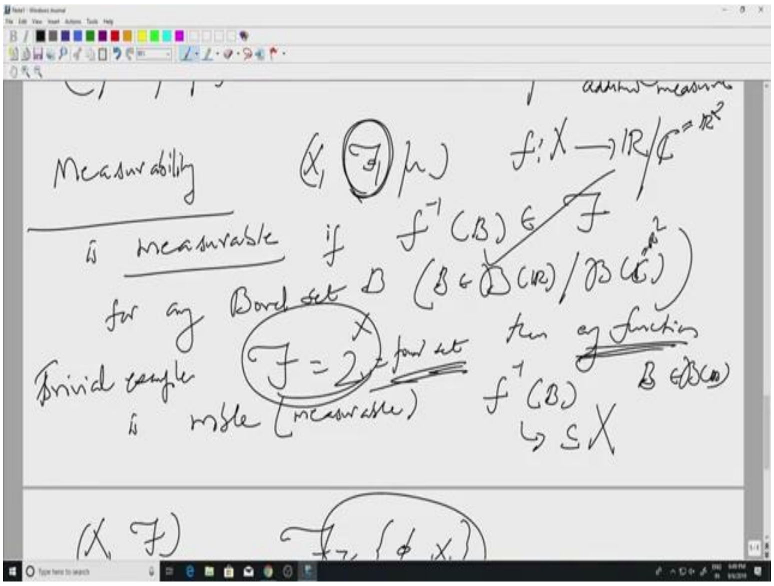

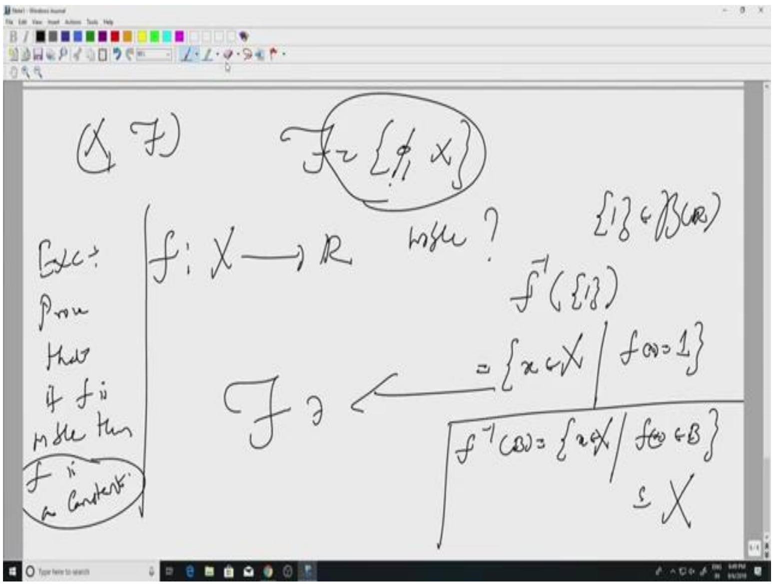

If I take script f to be simply the whole space and the empty set, this is also a sigma algebra, when will the function be measurable? Well, the inverse image of anything should be here in this collection, but in this collection there are only 2 sets. So, either it is empty or the whole space. So, for example, if I take Singleton 1 in the Borel sigma algebra of R, f inverse of Singleton 1, so what is f inverse of Singleton 1?

So, this is all those points x in capital X such that f of x is equal to 1. So more generally, so maybe I should have mentioned this more generally, f inverse of set B could be all those points in X, which are mapped into B. So this is a subset of X. So this, this is a subset of X and this should be in script f if f is measurable. So, this set should be either empty or the whole space. So, for exercise, from these considerations prove that if f is measurable, then f is a constant.

So, these are two easy examples. In the earlier case, when you look at the power set, we had any function to be measurable that is because you had the biggest sigma algebra here. When you have the smallest sigma algebra which is empty set and the whole space then any measurable function is a concept. So, these two are extremes, any genuine theory will be in between. So, we will continue with the Borel sigma algebra and Borel functions which are measurable and so on. So sometimes it is difficult to check f is measurable by looking at the inverse image of all the sets, if all Borel sets because Borel sigma algebra is quite huge, they have all the open sets, closed sets, intersections and so on. So it is desirable to have a smaller collection of sets for which one can check the measurability property.

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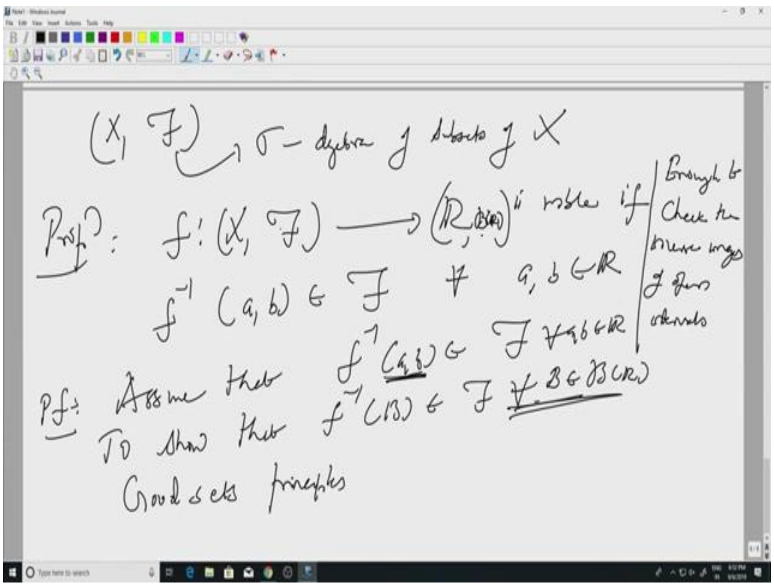

So let us look at a space X and f. I am not talking about the measure now. So this is simply a sigma algebra of subsets of X. So here is a proposition f from, so I will write the script f here just to indicate that the Sigma algebra under consideration is script f to R is measurable if f inverse of any open interval is in script f for every a, b in the real line. So we will prove that. So this is the first proof you are seeing in this course.

I am looking at a function from X to the real line, it is measurable if I look at inverse image of open interval, so the point here is that enough to check the inverse images of open intervals. You do not need to check the inverse image of all Borel sets, it is enough to check whatever is generating the Borel sigma algebra. So remember this on this side we are always looking at the Borel sigma algebra and you are looking at inverse images of things here and see if it is here.

So, Borel sigma algebra is generated by the open intervals. So, that is the reason this is happening, but we should write down a proof of this. So let us assume that this is true. So assume that f inverse of a, b is in script f for every a, b in the real line. So you take any open interval, look at its inverse image, that is a measurable set that is in script f. I want to show that, so to show that f inverse of b is in script f for every Borel set in the real line, right. So from all the open intervals, I want to go to all Borel sets in the real line, that is what I want to do. So what do you do? This is called Good sets principles, so what is this? So you look at a collection of good sets.

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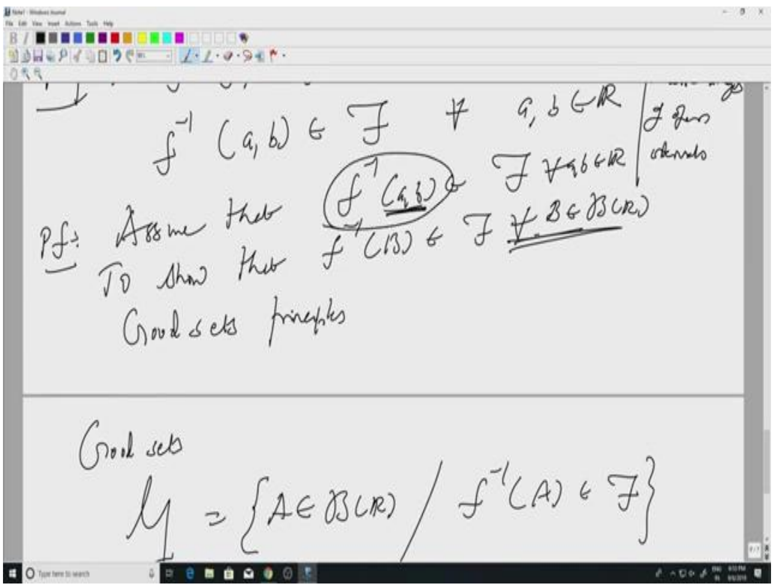

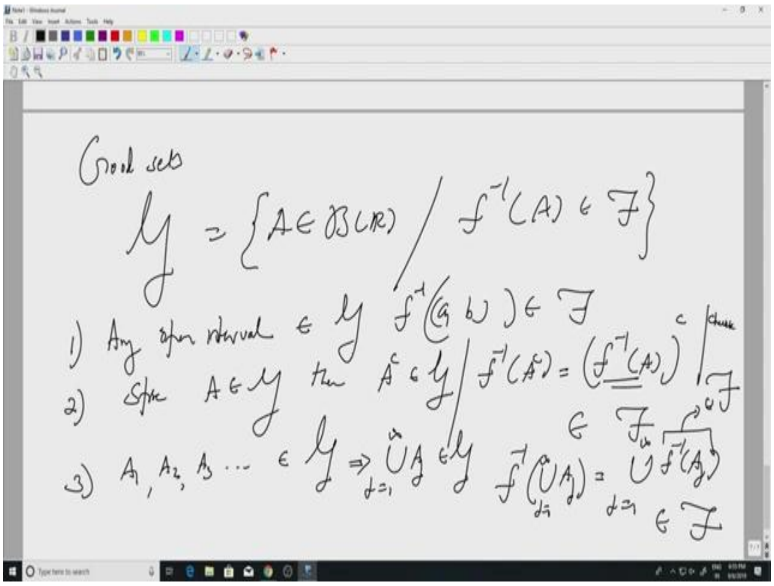

Let us look at good sets first, so what are the good sets? Good sets are sets which satisfy the property you want. Let us call that script g. This is all those sets, subsets of the real line or subsets in the Borel sigma algebra, such that f inverse of A is in script f. So, our assumption, our assumption is that if I look at inverse image of open intervals, they belong to script f. Here I am looking at a slightly general condition, I look at all Borel sets such that inverse image of that under small f will be in script f. What are the properties of g we know.

So, let us look at, are there any sets here? Well, any open interval belongs to g because if I take a, b then I know that f inverse of a, b belongs to script f. So, this property is satisfied. So, any open interval is here. Two, suppose A belongs to script g, then A compliment also belongs to script g. Why is that? Well, what do I want to check? I want to check f inverse of A compliment. I want to know if that does not script f. Well, f inverse of A compliment is f inverse of A whole compliment.

So this you have to check, that is a trivial exercise. But I know that f inverse of A is in script f, but script f is a sigma algebra. So its compliment is also in script f. Third property, I want to say it is closed under countable union. Suppose I have A1, A2, A3, etc. all belonging to g. So they are all good sets. Now I want to say union A j, j equal to 1 to infinity also belongs to g, so I want to say this is true. Well, how do I check this is true? I look at f inverse of union A j. Well, I know that this is equal to because f inverse respects all set theoretic operations. So this would be j equal to 1 to infinity, f inverse of A j. I know that each of them belongs to script f. But script f is a sigma algebra. So if I take countable union of sets in script f, this will also belong to script f.

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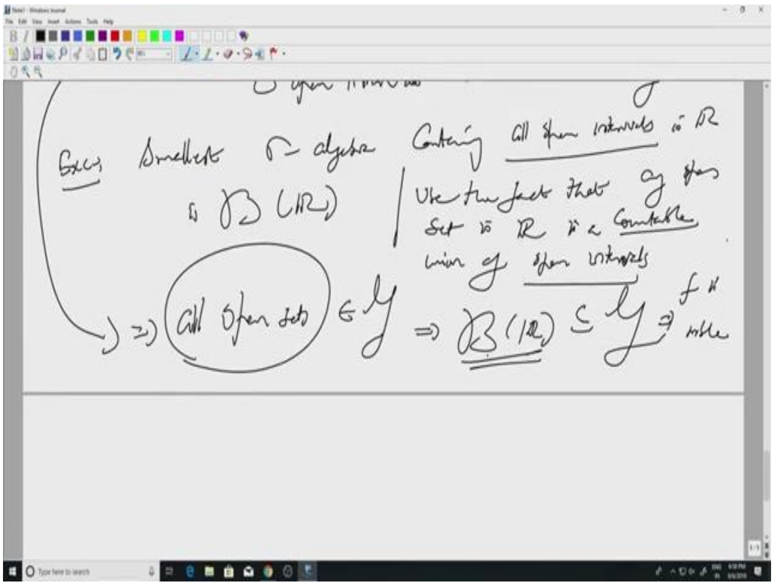

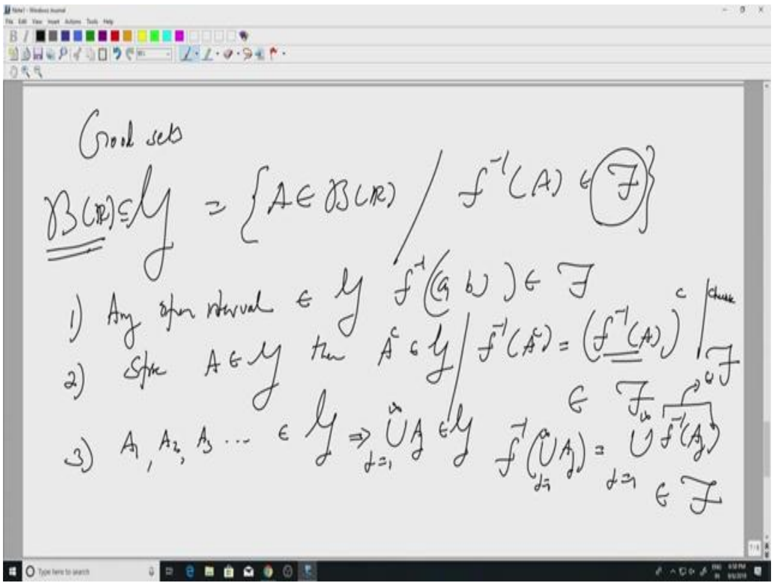

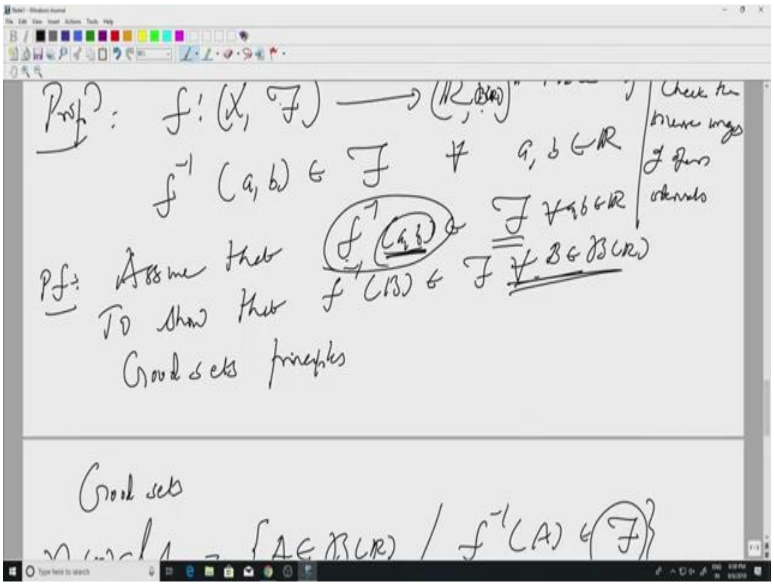

So these 3 properties tell me so, 1, 2 and 3 implies well, script f is a sigma algebra and open intervals are contained in script g. So, a simple exercise here, smallest sigma algebra containing all open intervals in R is the Borel sigma algebra. Remember Borel sigma algebra is the smallest algebra containing all open sets, but any open set in the real line is a countable union of open interval. So use the fact that any open set in R is a countable union of open intervals. So countable is important and so if you have all open intervals in a sigma algebra, all open sets will also be there.

So, let us use that, we will get that this implies all open sets are in script g, but script g is a sigma algebra, so sigma algebra generated by all open sets will also be in script g. So, implies the Borel sigma algebra of the real line is contained in g, remember because this is the smallest sigma algebra containing all open sets. I have another sigma algebra g containing all open sets and so Borel sigma algebra should be contained in this. Well, so, what did we prove?

We proved that the good sets, the collection of good sets here, I have the Borel sigma algebra contained in this, which means, if I take any set from here, its inverse image will be here, which is same as f is measurable. So, we started with the assumption that f inverse of open intervals are in script f. We proved that the collection of good sets which satisfies the assertion we want is a sigma algebra, it contains all open intervals, so, it will contain all Borel sets and so, what we have proved is f is measurable. So this implies that f is measurable.

So the, well, there is nothing surprising here because f inverse is something which satisfies which, well, f inverse is, will respect all the set theoretic operations. So, if I take A and look at A compliment, f inverse of A compliment is f inverse of A and taking the compliment, f inverse of the union is union of f inverse of those sets and so on and so forth. Because of that if I pull back a sigma algebra, I am going to get a sigma algebra and if that sigma algebra contains all the open intervals then of course it will contain the Borel sigma algebra as well and that is how we proved that the good sets contain all the Borel sets. And so, to check measurability of a function it is enough to look at inverse images of collection of sets which will generate the Borel sigma algebra. So I will stop with some exercises so that you get used to this concept.

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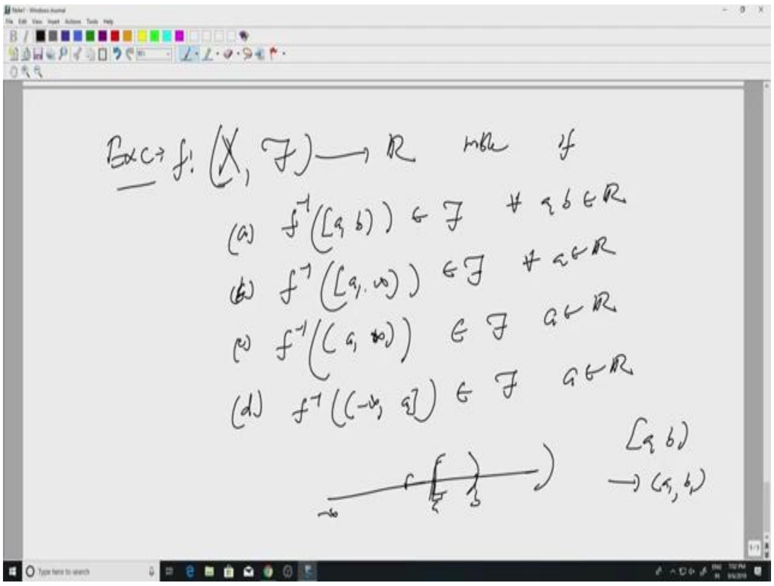

So I have a measure space, right? Well, I do not need the measure right now, we will simply say that I have a sigma algebra and I want to know when is it measurable. So, if I look at f inverse of all sets of this form, then also it is measurable. That is because the collection of closed at a, open at b intervals generate your Borel sigma algebra, so you form good size and prove that it is a sigma algebra and prove that your Borel sigma algebra is contained. Well, you could even look at sets of this form.

So, that is much smaller collection or you could look at in the real line all open into one sided open interval or the other way you can look at f inverse of minus infinity to a closed, any such collection will generate the Borel sigma algebra. So, the idea of the proof for all this is precisely what we did. Instead of open intervals, you look at these collections, prove that they all generate your Borel sigma algebra. That is not difficult because for example, if you start with sets of this form a, b, then taking countable union you will get any interval of this form a1 b1 and if you get all open intervals, you will get all open sets by taking countable union and similarly for the other.

For example, if you look a infinity, open and if I look at say, compliment of that, so that will give me some set like, let us say b here, minus infinity to b. So if I intersect, I am going to get this piece. So it is enough to have one sided intervals, even that will give me the whole Borel sigma algebra. So, what we have done so far is to define a measure on a given set with a sigma algebra, that is one. So that remember is supposed to generalize the notion of length function, volume, area and so on and so forth.

So, that gives us the measure, then we want to get a class of functions for which the integral can be defined. The first notion is the measurability we define which pulls back Borel sets, two sets in the Sigma algebra where you are interested. Remember the measure is defined on the sigma algebra. So we want to define the integrals of such functions. So we will start with that in the next lecture.