Measure Theory

Professor E.K Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru

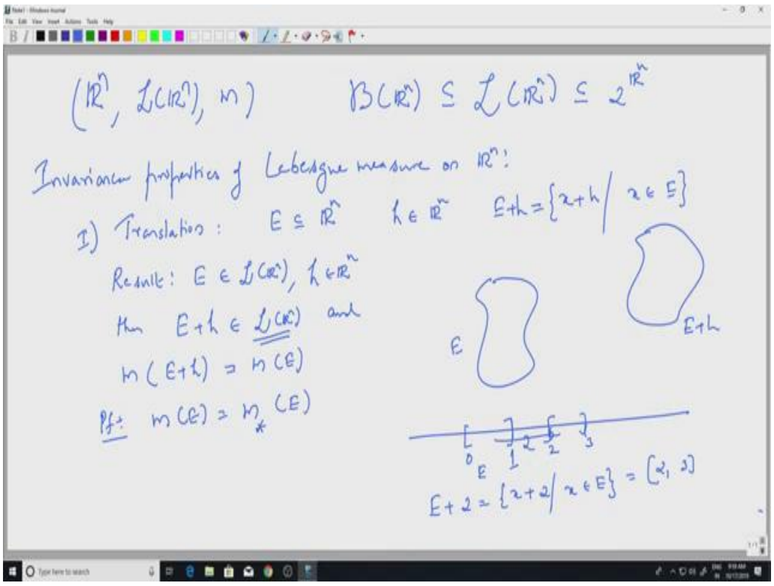
Lecture 19

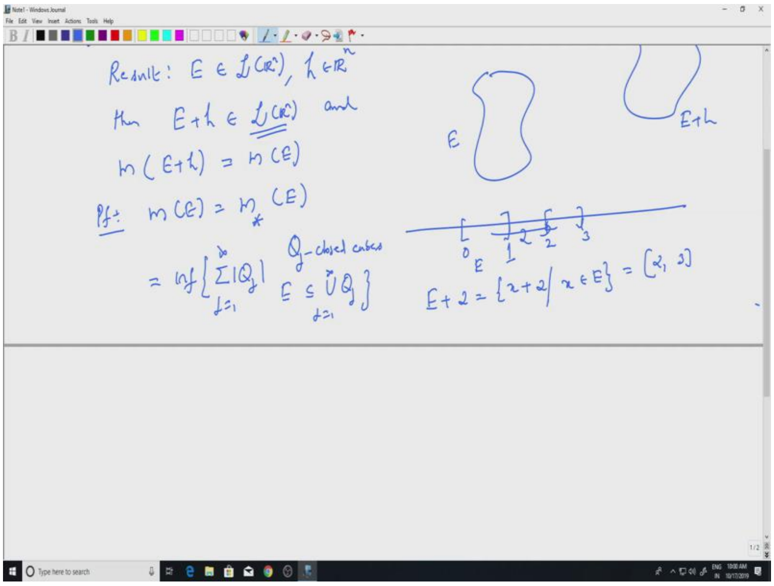
Invariance Properties of Lebesgue measure

So we have constructed the Lebesgue measure completely now. So, we have Rn we have the Lebesgue sigma algebra over Rn and the Lebesgue measure on Rn and we looked at some finer properties of the measurable sets.

Today will first look at some properties, some invariants properties of the Lebesgue measure itself and then we will, we will see that any Lebesgue set is can be approximated by Borel set very closely in fact, the Lebesgue sigma algebra is the completion of the Borel sigma algebra with respect to the Lebesgue measure. So that is what we will do in the first session, then we will go on to look at non measurable sets and things like that. So we will start with some invariants properties of the Lebesgue measure.

(Refer Slide Time: 1:19)





So, recall we have the space Rn, we have the Lebesgue sigma algebra and the Lebesgue measure. So, also recall that we have the Borel sigma algebra B of Rn and that is of course contained in the Lebesgue sigma algebra we know that, and which is of course, contained in the power set.

And yesterday I had mentioned that the inclusions are strict, okay. So, we will construct examples to show that these inclusions are strict. So, we start with some invariants properties of the Lebesgue measure, invariance properties of Lebesgue measure on Rn. So, this gives an idea about how the Lebesgue measure interacts with various natural operations on Rn.

So the first one is translation invariant, translation invariance. So, let us just look at translation. So what is translation? Translations, is if I take a set E, let us say subset of Rn.

So, we are not assuming it will be measurable now, and some vector h in Rn then we can define E plus h this is translating E by h.

So, how is this defined, this is defined to be all x plus h, where x belongs to E. So, we are simply translating the set. So, if the set E is like this and depending on where h is you will get E plus h to be somewhere here have the same shape. So, when I drew the shape will change, but shape change, but it is exactly the same.

So, an easy example would be on the real line. So, let us look at the interval 01 and translate… So, let us say this is E, translated by a number 2. So, let us see what is E plus 2 is. So, E plus 2 will be all x plus 2, where x belongs to E, E is the close interval 01. When I add 2 it, I am going to get the interval 2 to 3.

So, this is simply translating E to this interval to you, you translate this by two and when you translate of course, you see that the length of the interval does not change and this is true for cubes as well in Rn. So, when you translate the measure should not change. So, that is a first assertion.

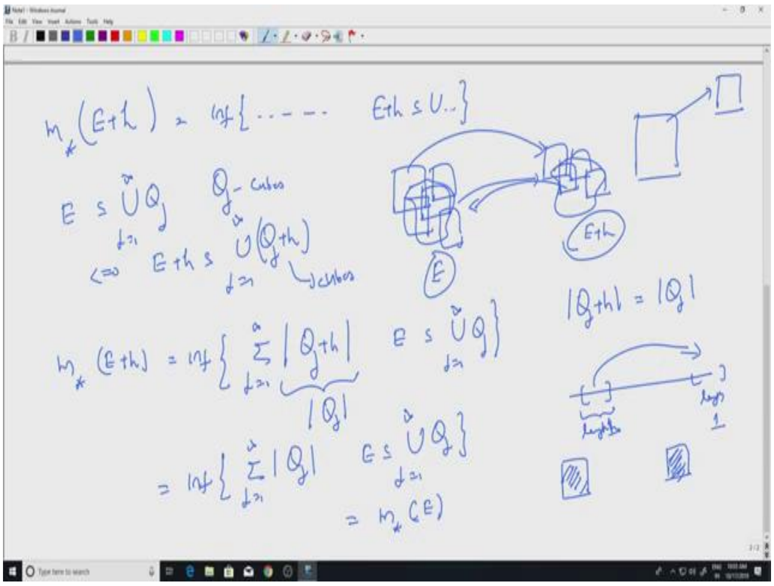
So, let us prove that. So, let me write this as a result these are easy results but it is, these are very important results as well. So, if I take a measurable set L of Rn, E in Lebesgue set then and so you fix a vector in Rn fix a vector h in Rn, then first of all E plus h is also measurable.

That is one assertion and the measure does not change. So, measure of E plus h, so translated set and the measure of E are same, so to say m of E plus h, one needs to know that E plus h is in L of Rn and. So, that is the first assertion and then we say that the measures are same. Well, proof is very simple, but let us, let us do this in a slightly elaborate manner, because the other invariance properties will be of the similar kind.

So, first thing is let us recall what is the measure. So, m of E is m star of E. So, it is the outer measure of E only thing E is, if E is in the Lebesgue sigma algebra, we say m of E. So, let us keep that in mind, which is equal to the infimum of various quantities which we have seen so many times by now mod Qj j equals 1 to infinity Qj are closed cubes, closed or open does not really matter, closed cubes and more importantly E is covered by Qj.

And then you take the infimum of these quantities. Well it is very easy to see that. So, for m star, m star is defined on the power set of Rn. So, m star of E is defined for every set E we do not need to bother about any measurability properties.

(Refer Slide Time: 06:30)



So if you look at E plus h and m star of that, well, what is this? This is of course infimum of various quantity, so I am not going to write this, so the only condition is E plus h is contained in whatever union you are taking.

Well, so how does it look like? So let us say E is here and E plus h here. So, if I have cubes, covering E like this and I translate E to get E plus h. If I translate these cubes, I will get a cover for E plus h and similarly, if I translate back by minus h, I will get a cover for E. So, covers for E plus h can be translated to covers for E and vice versa.

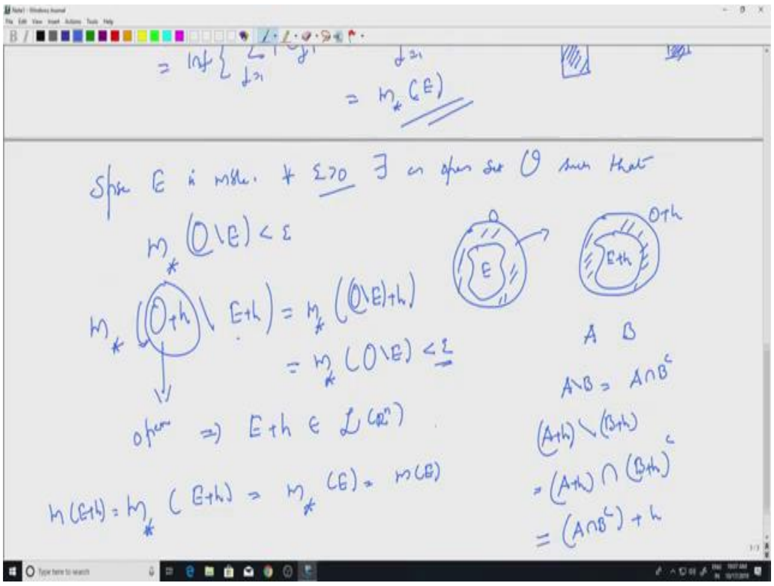
So, what I want to say is, if I have E contained in union of Qj j equal to 1 to infinity to Qj are cubes. So, if you translate a cube you are going to get another cube. So, if I translate this by some vector, I am going to get something similar. So, that is that is a cube. Then, well you can say if and only if E plus h is covered by the cubes translated.

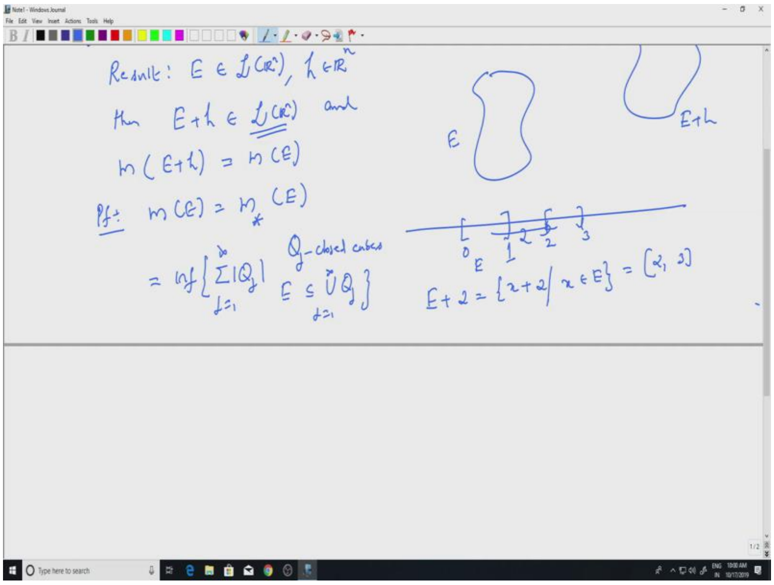
So, these are also cubes Q plus Qj plus h cubes. So, because of that m star of E plus h is equal to infimum of summation j equal to 1 to infinity volume of all those Qj which cover h. So, you translate them. So, now you get a cover for E plus h such that E is contained in union Qj, j equal to 1 to infinity.

But what is Qj plus h, the volume of Qj plus h, volume of Qj plus h is same as volume of Qj, if you translate you do not change the volume. So, look at the easy example, if I have an interval of length 1, length 1, I translate it to any place I want. I will still get a interval of length 1. So if I translate, I am not going to change the volume of the cube. If I have a square in R2, I translated I am going to get the same square, but the area of the square will position will change, but the area is going to be the same.

So, this is simply mod Qj. So, this is the same as infimum of j equals 1 to infinity, mod Qj where E is contained in union j equal to 1 to infinity Qj where Qj are cubes, but this is the definition of m star of E. So, this is a trivial computation we have done, but it shows that m star of E plus h is same as m star of E. So, m star is translation invariant if you translate the set you are not going to change the value of m star. So, but it does not prove that m that E plus h is measurable. So, we need to we need to follow that as well.

(Refer Slide Time: 10:10)





So, suppose E is measurable, suppose E is measurable. What does that mean? So for every epsilon positive there exist an open set open set, O, such that, such that the outer measure of

O minus E is less than epsilon. This is what measurability makes. So, what we have is E a set here and some O ones around it. That is my O, I am trying to prove that E plus h is measurable.

So I simply translate this open said O, by it, so I will get O plus h and whatever is the difference over the differences is simply translated by h. So, m star of O plus h, O plus h is a set minus E plus h well, what is this, this is m star of O minus E plus h well, you need to verify this, this is a trivial, trivial thing, but remember the minus here is the set theoretic minus.

So, let me, let me write it down for A and B, suppose I have two sets A and B I look at A minus B, what is A minus B? That is A intersection B complement. Now, A plus h as a set minus B plus h. So, another set well what will this be? So this is A plus h. By definition intersected with B plus h whole compliment.

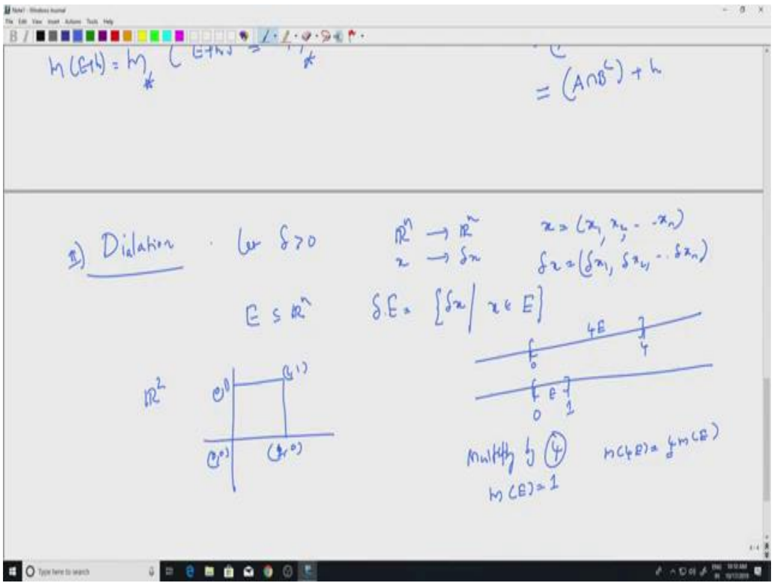
So, you need to check that this is actually equal to A intersection B complement translated by h. So, that is what of trivial from the, from the pictures, but you can write down a proof if you want. So, go back to m star of O plus h minus E plus h, that is m star O minus E plus h, which is equal to. So, we just proved that m star is translation invariance.

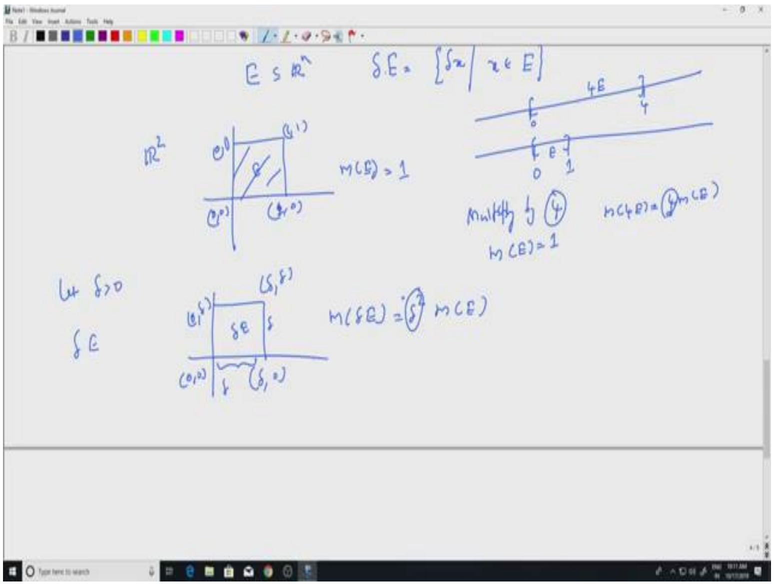
So, that is same as m star of O minus E, which I know is less than epsilon and the fact that O plus h is open. If I translate an open set, I am going to get an open set. So, what we have proved just now is given an epsilon, there is an open set O plus h such that m star O plus h minus E plus h is less than epsilon.

So, that implies that the set E plus h is also measurable, E plus h is measurable and so m star of O plus h is m of E plus h. That is simply the definition, so m of E plus h that makes sense to write m of something, that something has to be measurable sets. This is equal to, I know it is equal to m star of E because m star is trivially invariant and the translation and this is equal to m of E.

This is precisely the result asserted here. So let me yeah, so here, we claimed that E plus h is measurable and the measures are same. So Lebesgue measure is translation invariant if I if I translate something, I am not going to change the measure of the set. So that is the first property we have seen.

(Refer Slide Time: 14:10)





Second, second property, second invariance, this is with respect to dilation, dilation. So, what is dilation? So you start with let some take some positive quantity delta, greater than 0 and you multiply by delta. So, you are looking at the map from Rn to Rn x going to delta x, that makes sense. So, remember x is a tuple x 1, x 2, etc, etc x n delta x would be, you multiply all of them delta x 1 delta x 2 usual vector space operations.

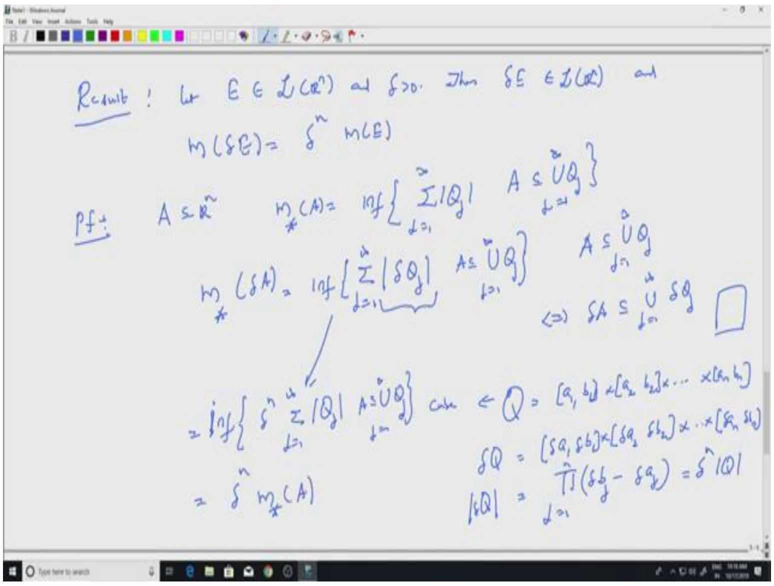
This one and this you can do for sets right if I take a set E contained in Rn, I can define delta times E to be delta times x, where x belongs to E just like what we did for translation. Now we want to say a similar, similar result hold for dilation of course Lebesgue measure will not preserve dilation that is easy to see. So let us, let us look at some example to guess what should be the answer.

So let us take the interval 01 in R. Let us multiply it by four. So, multiply by, let us say 4. So, I will get, if I multiply by 4, I am going to get so I will draw it here 0 to 4 that is the interval. So, if this is E, this would be 4E. What is the Lebesgue measure of E? That is one because it is an interval, so the length is 1 and if I look at the measure of 4E, I am waiting 4 times measure of E.

So, whatever you are multiplying by it comes out. But let us look at R2 and look at another example. So, let us take this unit, unit square 01 here. So, this is my set E, if I multiply it with the delta. So, let delta be positive and we multiply this with delta. So, how will delta E look like? Well that is also a square, it is cornered at 00 of course, but these coordinates become delta zero, delta delta and zero delta.

What is measure of E, the two dimensional Lebesgue measure of E? This is 1 this is the area of this square. Measure of delta E this is delta E, measure of delta E is the area of the square. So I have delta here, I have delta here. So, that is delta square delta square into measure of E. So, this time so instead of 4 here I have delta square, it is not delta because the two depends on the dimension. Similarly, if you got to R3, you will see that delta cube comes out.

(Refer Slide Time: 17:33)



So the result we want the state is, result, let E be a measurable set L of Rn and delta be positive, then, well, first of all, delta E is measurable. So, that we can talk about its measure and L of Rn and the measure of delta E is actually equal to delta to n. So, the n comes from the dimension of Rn times measure of E that is what happens. So, let us prove this. So, as earlier, we will look at m star of something and see how the multiplication by delta acts on it or the effect of multiplication by delta on m star.

So, let us go back to general set A subset of Rn, m star of a by definition is the infimum. We have seen this so many times now mod Qj, j the equal to 1 to infinity A is contained in union Qj. Now what is m star of delta A? Well, this is infimum of some collections, what are those collection? So, just like what we did earlier, we see that A is contained in union Q j, j equal to 1 to infinity if and only if delta A is contained in union j equal to 1 to infinity delta Qj.

So, all you have to do is to dilate the cubes accordingly to cover delta A. So, this would be infimum of summation j equal to 1 to infinity modulus delta Qj. Where Qj is covers A. So, A is contained in union Qj j equal to 1 to infinity. So, you take any such cover multiply by delta you will get a cover of delta A you take any cover of delta A divided by delta you will get a cover of A.

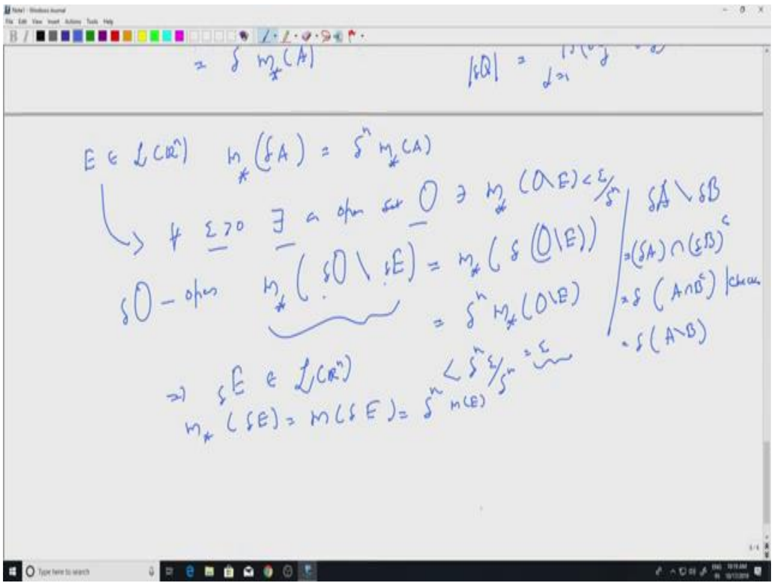
So, that is why this is 2, the collection of cubes have to be the same essentially apart from multiplication by delta well. But what is this? Well, for cubes, it is easy to compute. So, let us let us see, if I take a cube Q, so cube let us say closed cube this of course is product of intervals a 1, b 1, a2, b2 etc, etc a n and b n.

So, it looks like something like this in R2. You multiply by delta. But of course, you are going to multiply each of them a1 b1 cross delta a2 delta b2, delta is positive remember, cross delta an delta bn. So, what is the volume of delta Q? Well, you look at the product of the sides. So, that is delta bj minus delta aj j equal to 1 to m.

So, from each component that delta comes each side has got dilated by delta. So, the side changes by delta and from each component you get delta. So, this is delta to the n, times whatever remains is the measure of Q. So, for cubes this is trivial and that is why it becomes true for all other sets as well.

So this from here, I will get that this is equal to infimum of I will get delta to the n outside summation j equal to 1 to infinity mod Qj and A is contained in union Qj, j equal to 1 to infinity. But the delta to n will come out, it is a positive quantity and you are taking the infimum of the remaining which is simply delta to the n times measure of E measure of A m star of A, the outer measure of A.

(Refer Slide Time: 22:10)



So we apply all this this, to the result we want to prove. So, we started with E belonging to L of Rn. Again, we do not know what happened to delta A, but we know that the m star the outer measure of delta E is delta m to m star of E, this we know and this we know and well let me write it for general set. So, that we can use it for different sets, so A and A this we know. Now E is measurable implies for every epsilon positive, there exist an open set open set O such that the measure of O minus E is less than epsilon this we know.

Well, what should we do to get an open set around delta E well we look at delta O. So, delta times O is open and m star of delta O minus delta E, well, what is this this is m star of. So, again trivial verification to be checked is this. So, what did we use, we use that delta A minus delta B, well this is delta A intersected with delta B whole compliment, this is nothing but delta times A intersection B complement.

So, this is what you have to check the trivial thing equal to delta times A minus B. So, that is what we have used and we know that m star when you dilate m star as the property that delta to the n comes out. So a measure of O minus E, which is less than delta to the n times epsilon. That is also enough, so if you want to be very, very precise, for every epsilon there exists and open set O such that m star O, m star O minus E is less than epsilon by delta to the n, you can start with that.

So that will give me delta O minus delta E is delta n times this is equal to epsilon. So this inequality implies that delta times E is measurable and of course, if it is measurable, then we know that m star of delta E well, this is actually m of delta E equal to delta to the n times m of E. This we know, because that is a property of m star.

So, that proves this result. So, this result is proved. So, Lebesgue measure is translation invariant with respect to dilation, it gives me a factor of delta to the n. That is expected and well there is one more result. So, I will not prove this in as much details as we just did, but because this is much more easy.

(Refer Slide Time: 25:23)

Three reflection well, what is reflection? Reflection is the map x going to minus x. So, this is a map from Rn to Rn. So, the result we want to state is, suppose, E belongs to L of Rn and then its reflection is minus E, minus E equal to well what is this this is minus x where x belongs E.

That is the reflection, so in the case of real line, suppose I take an interval here, let us say a b, this is my set E then, minus E will be minus b2 minus a 1 this would be minus E and of course, the length does not change, if I take a cube Q in Rn cube then minus Q is also a cube, this is also a cube more importantly, minus Q has the same side length.

So, the volume will be the same and so m star will respect this. So, m star will, will not change when you take the negative of minus of Q. So, m star of E and m star of minus E will be same. So, easy to verify. So I will leave it to you. Easy to verify that m star of E is same as m star of minus E.

So, this is because any cube Q and it is negative minus Q, what it means is will have the same volume. So, when you take the infimum it is not going to change. So, any cover of E you can take minus of that will get a cover of minus E and the corresponding quantities are same.

So, because of those, if E is measurable if E is measurable then of course, so for every epsilon positive that exists and open set O such that m star of O minus E is less than epsilon. So, you can change everything to minus E now.

So, this would imply that you look at minus of O, minus O is open set this is an open set and m star of minus of O minus minus E. So, remember there are two minuses the slanted minus is the set theoretic minus. So, if you look at minus A minus minus B minus A intersection minus B whole compliment and you prove that this is actually equal to A minus B minus of that.

So, this is equal to m star of minus of O minus E and we have seen that when you take minus nothing happens to m star. So, that is m star of O minus E which is less than epsilon, which is precisely the statement that minus E is also measurable implies minus E belongs to L of R. So, I started with E in L of R I have minus E in L of Rn and of course, m star of E and m star of minus E are same. So, m of e and m of E and m of minus E are same.

So, let me stop with some remarks. We have just seen three invariants properties of the Lebesgue measure it is translation invariant with respect to dilation, we have a certain different kind of invariants and it is reflection invariants. You will notice that these the last two examples, the dilation and the reflection. These are linear maps from Rn to Rn. And so you can ask the question how does the Lebesgue measure deal with? Or how does it behave with linear transformations on Rn?

So a general result we will state later and improve if I, because there are certain things to be established before we can state the, state the result if you, if you translate or if you apply a linear transformation to a measurable set, we have proved that it has measurable and its volume or it is, it is a Lebesgue measure would be the determinant of the linear transformation times the measure of the original set.

And that is precisely what do you have seen in the case of multiplication by delta. So, multiplication by delta is a linear transformation whose determinant is delta to the n, and x going to minus x is another linear transformation whose determinant is one. So that does not show up here. So we will stop here, continue.