Measure Theory

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Lecture 18

Fine Properties of Measurable Sets

So, now we have constructed the Lebesgue Measure, we know that it is a countably additive measure. We saw some examples, we will be looking at finer properties of the measurable sets how to approximate it with nice sets like compact sets F sigma sets G delta sets and things like that. And we will also compare it with the Borel sigma algebra which we already talked about earlier. So, let me write down some things now as questions, and we will take it up later. And then we will go on to look at some finer properties of measurable sets.

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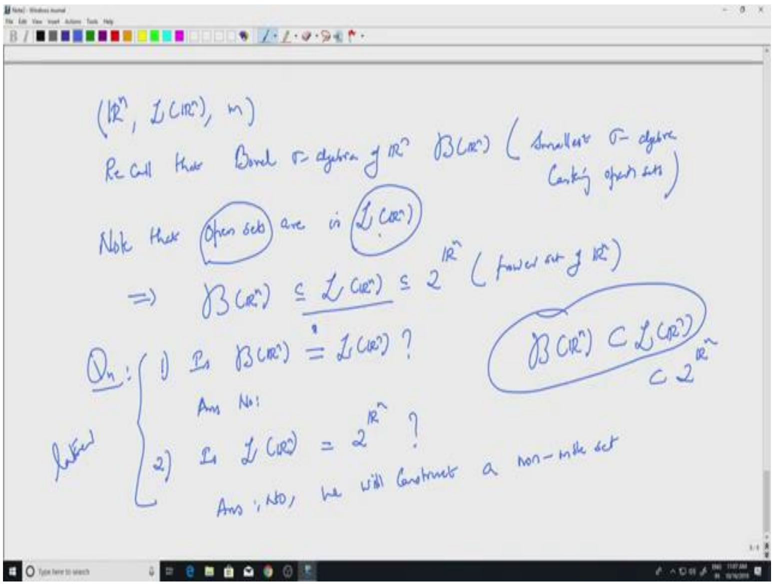

So, let us recall we have Rn we have the Lebesgue measure, the Lebesgue sigma algebra and we have the Lebesgue measure m. So, recall that we have, recall that the Borel sigma algebra. So, this is what we initially defined, Borel sigma algebra of Rn. So, what was this? This was the smallest sigma algebra, smallest sigma algebra containing open set, sigma algebra containing open sets.

So, we had looked at it, we had looked at examples in Borel sigma algebra and so on. So, note that, note that open sets are open sets are inside Lebesgue sets, inside the Lebesgue sigma algebra, that is by definition itself it will follow. So, since this is a sigma algebra it will also follow that the Sigma algebra generated by open sets will also be here, which means the

Borel sigma algebra which is generated by the open sets will be contained in the Lebesgue sigma algebra.

So, this is a rather large collection and of course, all these are subsets of Rn. So, these are all contained in 2 to the Rn, 2 the Rn by this we mean the power set of, power set of Rn we started with that. On, we defined M star on the power set of Rn, the outer measure was defined on the power set of Rn and then we restricted it to L of R. So, there are two questions whether these are equal there is equality at any level?

So, may be let me write it as a question 1 is B of Rn, that is the Borel sigma algebra, is it equal to the Lebesgue sigma algebra? So, answer is no. We will see examples later. 2 is the Lebesgue sigma algebra equal to the power set well, here also the answer is no, the answer is no. We will construct, we will construct a non-measurable set, a non-measurable set, probably in the next few lectures we will do that and of course, this so these things will be taken up later.

So, Lebesgue sigma algebra is something which is in between the Borel sigma algebra and the power set. So, B of Rn is actually strictly contained in the Lebesgue sigma algebra strictly contained in 2 of R, the powers. But these two, do not differ too much the Borel sigma algebra and Lebesgue sigma do not differ too much, we will see that the Lebesgue sigma algebra is actually.

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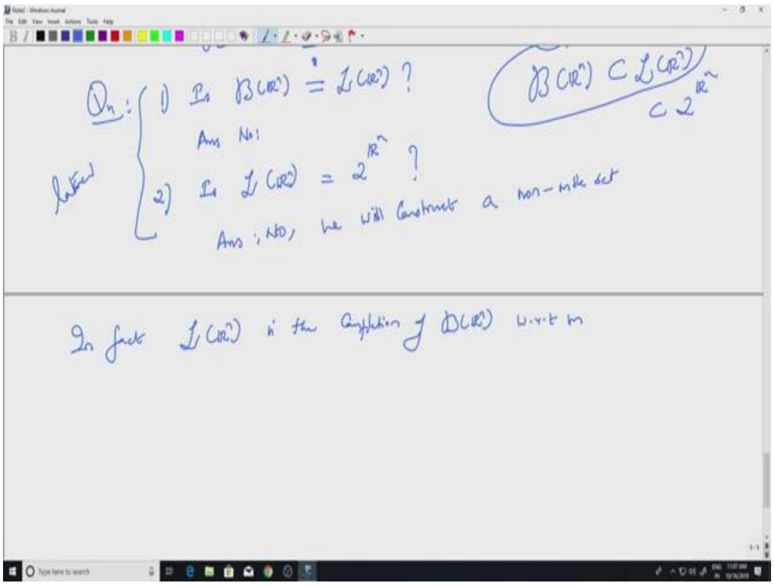

So, we in fact inside the Lebesgue sigma algebra is the completion of, is the completion of the Borel sigma algebra with respect to the Lebesgue measure. So, remember we define completion, this is simply by adding all the sets of measures 0 subsets of measures 0. So, if you start with Borel sigma algebra and add subsets of measures 0 you will get Lebesgue sigma algebra.

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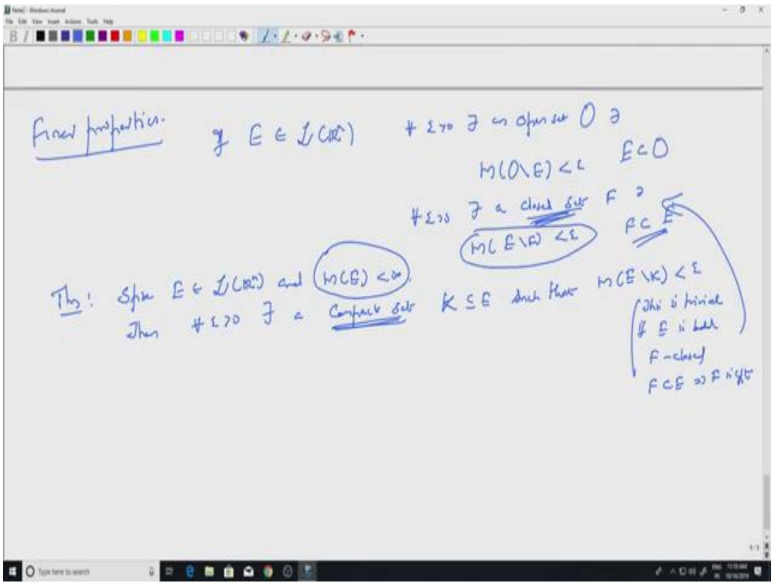

So now, now, let us look at finer properties of the Lebesgue sets. So, let us say finer properties finer properties, I can write it as a theorem. So, if I, before we state the theorem, so if, I have a measurable set. Well, I already know that for every epsilon positive, there exist an open set open set O, such that measure of O minus E is less than epsilon, I will not use M star anymore because these are all measurable sets. So, I will just use m, we also know that for every epsilon positive there exists so, the open set O is such that it is bigger than E.

So, there exists a closed set now, this also we know F such that measure of E minus F is less than epsilon, where F is a subset of subset of E. So, you can approximate E from outside by open sets and from inside by closed sets. So, this these are equal in statements we have seen that So, I can choose any of them for measurability.

So, let me write another theorem suppose, E is a measurable set and it has finite measure then for every epsilon positive there exists a compact set k contained in E such that measure of E minus the compact set is less than epsilon. So, that is a strengthening of whatever we have here whatever you have measurability you have a closed set contained in E such that we have this inequality.

Now, we are saying we have a compact set what is the assumption assumption remember is a set which has finite measurable. For example, if E is bounded this is trivial because if it is, so this is trivial if E is bounded because the closed set F contained in E implies F is bound F is compact. So, this will already imply that but E need not be bounded. So, that is a exercise.

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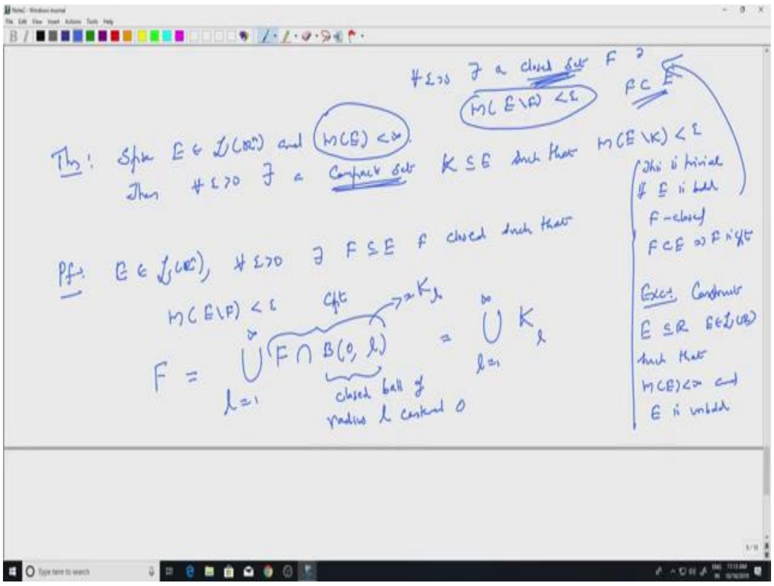

So, here you can convince yourself about this. So, simple exercise construct a set E contained in the real line measurable, such that measure of E is finite and E is unbounded think of union of intervals which are under unbounded so let us prove this.

So let us go back to the statement and If I have a set of finite measure, it can be approximated from inside by compact sets that is the statement. So, because of measurability for, for epsilon positive, we already have the closed set right. So, there exists F which is contained in E F closed such that, such that the measure of E minus F is less than epsilon. So, that is simply the definition of measurability.

So, what we do is we write F as union, let us say l equals 1 to infinity, F intersected with closed ball of radius l centered at 0. So, this is the closed ball, close ball of radius l, centered at 0, centered at origin. That is a compact set. So F is closed, this is compact so when I intersect, this is going to be compact. So, let us call that K l, so l equal to 1 to infinity I will write it as K l. So, K l is this this intersection this K l. So, I am writing F as a countable union of compact sets.

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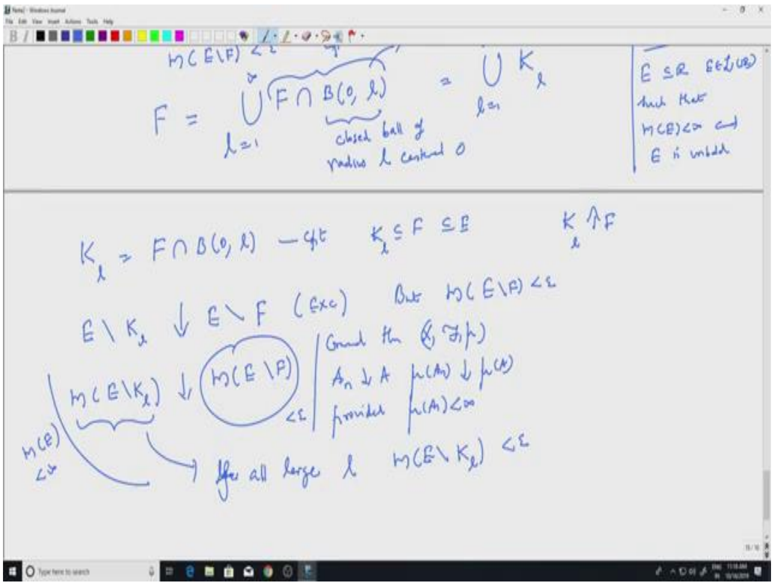

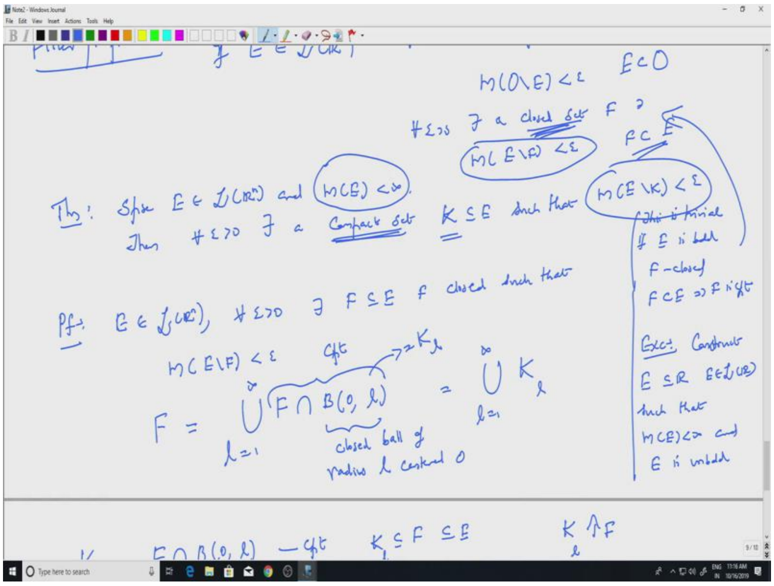

Now we want to approximate E from inside. So, you look at E minus K l, so may be one more. Let me write this a bit more clearly. So, what is K l? K l is not the ball it is F intersected with the ball. So this is compact. Look at E minus K l, remember F was a closed set contained in E I am looking at E minus K l, K l is contained in F because it is F intersection something I am looking at E minus K l.

Well what happens so this K l increased to F, K l increased to F. So, E minus K l will decrease to E minus F. So, this you can check this is a trivial theoretic exercise. But I know that, I know that measure of E minus F this is less than epsilon that I know given epsilon I have this F. So, measure of E minus K l will decrease to measure of E minus F.

This is from the general theorem, general theorem, what is the general theorem if I have a triple and if I know that An decreases to A then mu of An decreases to mu of A provided, one of them as finite measure right mu of A1 has finite measure here everything is bounded. So, all these things are finite. So, there is no problem E minus K 1 m of E itself is finite.

So, let us see why we can use this. So, m of E is finite. So, all these are finite quantities and so, this theorem applies. But m of E minus F is less than epsilon. So, this will have to be less than epsilon after some time. So, for all large l, m of E minus K l is less than epsilon that follows from the definition of the convergence.

So, that is all we need we need a compact set we need a compact set K such that m of E minus K is less than epsilon. So, we have proved so, that is one finer property so far I have said which has finite measure, I can approximate it from inside using compact sets so let us look at one more search result.

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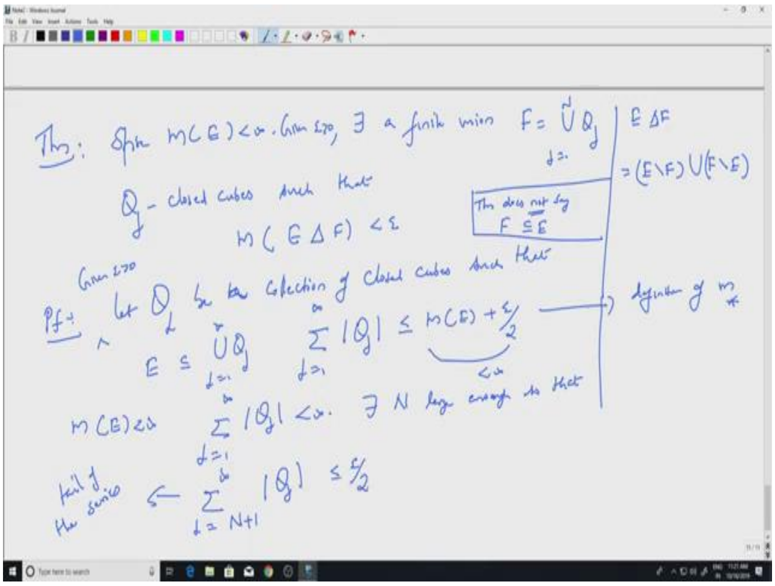

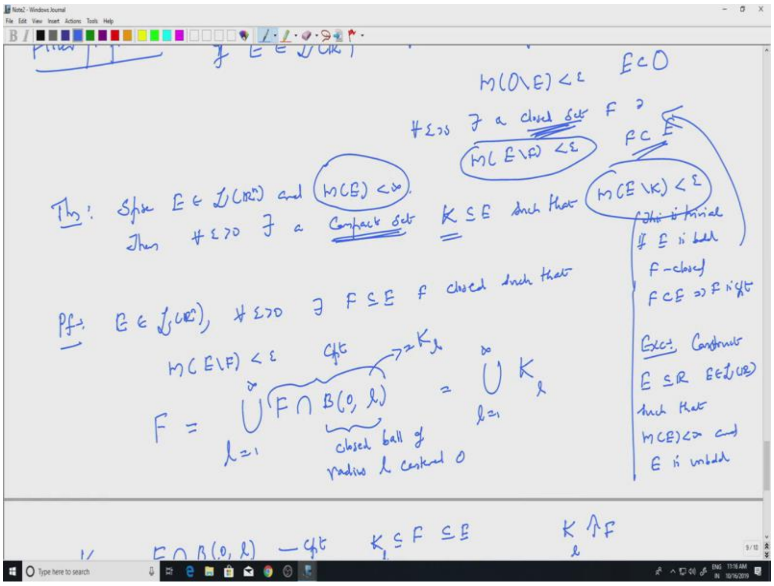

Suppose m of E is finite then there exist a finite union, finite union we will call F to be equal to union Q j, j equal to 1 to n, what are Qj? Qj closed cubes, closed cubes such that, such that measure of E delta F. So, that is a symmetric difference that is less than epsilon. So, of course then given epsilon, so let me write this part again given epsilon positive, there exist, there exist a finite union F such that E delta F.

So, what is E delta F? E delta F is a symmetric difference. So, this is E minus F union F minus E. So, difference between E and F is very small. So, E is sort of approximated by a finite union of closed cubes. So, that is slightly stronger than or finer than this property because we have here we have arbitrary closed sets, what we are, here what we are essentially saying that close, the compact set can be a union of close cubes.

So, let us look at the proof, proof follows from whatever we have just done well, not really. So, let Qj be the collection of be a collection, of closed cubes, closed cubes such that…So, here in the statement you should realize that F is not contained in E. So, so we are not unlike the earlier theorem, theorem does not say does not say that F is contained in E this need not be true. So F, because E may not have an interior at all. So, F is contained in E, E is not true may not be true does not say.

So, let Qj with a collection of closed cubes such that E is contained in union Qj j equal 1 to infinity. So, this is one of the first arguments we had right using the property of the infimum summation j equal to 1 to infinity mod Qj is less than or equal to measure of E plus epsilon by 2. So, given epsilon, so, given epsilon positive and Qj be the collection of closed cubes such that this happens, this this we know that exists.

So, that follows from the… So, this is from the definition of outer measure, definition of m star and m are same right for measurable sets. But m of E is finite, m of E is finite. So, this quantity is finite. So, this quantity also has to be finite the infinite sum of mod Qj will have to be finite. So, there exist some capital N large enough, large enough.

So, that the tail of the series is small, that means summation j equal to N plus 1 to infinity now. So, that is the tale of the series. So, this is the tale of the series, series converges. So, the tale of the series will have to be small less than or equal to epsilon by 2, I can do this. So, I can choose a large enough m, so that this happens.

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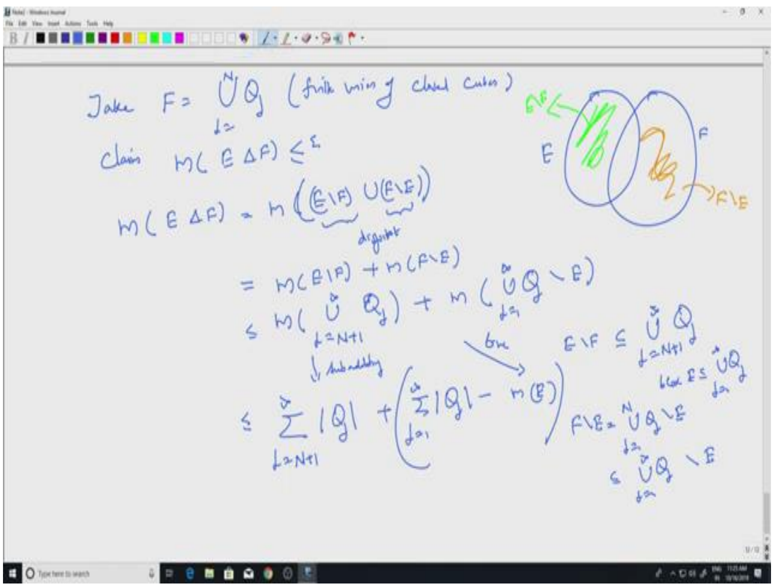

So, just take the remaining to be your F, take capital F to be union Qj, j equal to 1 to N. The remaining is sort of taken care here, whatever remaining you in the union you take it to be F. So, F is a, so this is the union of finite union of, finite union of closed cubes, closed cubes and you want to say F approximate claim, is that claim is that, measure of E delta F is small is less than epsilon less than or equal to. Let us see, why?

So, measure of E delta F, well what will be this? This is a measure of E delta F is E minus F union F minus E, that is the definition of the symmetric difference. But these are disjoint these are disjoint that is easy to see. So, if this is E and this is F, so let us say this is E and this is F, E minus F. This portion is E minus F and this portion is F minus E.

So, they are disjoint, the intersection part does not there. So, because m is a measure this becomes m of E minus F plus m of F minus E, which is less than or equal to measure of union j equal to N plus 1 to infinity Qj. Let me explain write down and then explain plus m of union j equal to 1 to infinity Qj minus E, let us see why?

So, I want to say that, so this is this is true because well. So, how does one prove this? This is true because E minus F is contained in union j equal to n plus 1 to infinity Qj. That is easy because E is contained in because E itself is contained in the full union j equals 1 to infinity Qj and you are throwing out the first N by subtracting F.

Similarly, F minus E, So, what is F minus E, this is j equal to 1 to N, Qj minus E, of course, will be contained in j equal 1 to infinity Qj minus E, you are taking a much bigger set. So, this is true and this is a of course less than or equal to I can apply sub additivity here. So, I will get summation j equal to N plus 1 to infinity m of Qj simply the volume of Qj. m of Qj m star of Qj equal to modulus, mod Qj plus summation j equal to 1 to infinity remember these are all finite Qj minus m of E. So, this has to be justified. So, I am, saying this portion is less than or equal to whatever is written here. So let me write down that.

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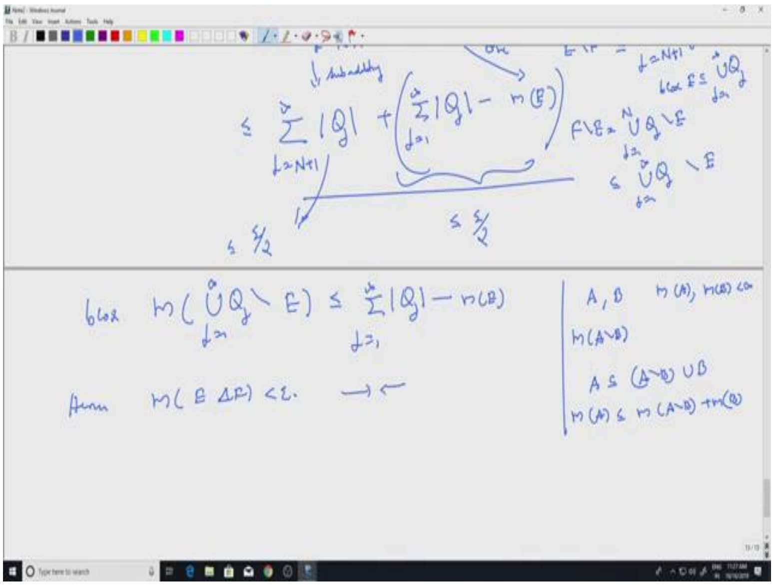

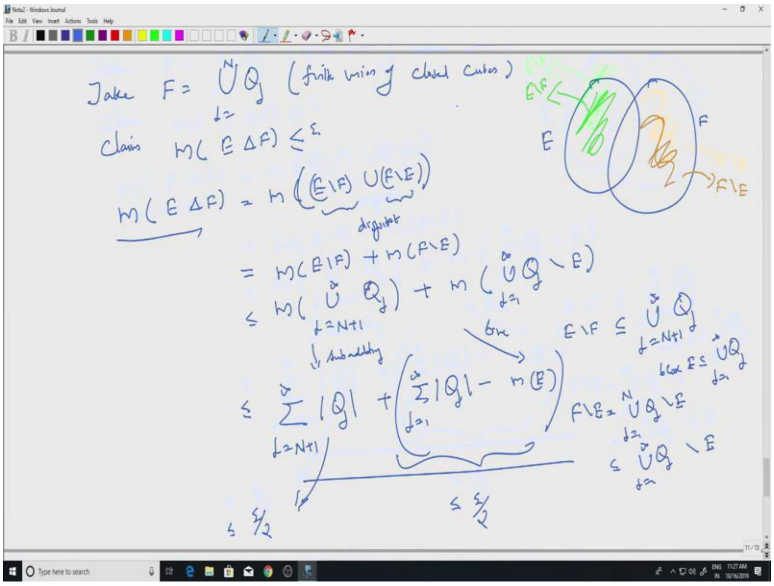

So, this is because m of union j equal to 1 to infinity Qj, Qj minus E is less than or equal to summation j equal to 1 to infinity mod Qj minus m of E. Why is this true? Everywhere we have finite quantities. So, let us take two sets A and B, let us say m of A, m of B are finite. Then if you look at m of A minus B, this I can write this as… So, start with m of A I write it as m of A minus B. So easy, easy things.

So, A is contained in A minus B union B and so m of A will be less than or equal to m of A minus B plus m of B. So, use that you will get this one. So, now from here it follows that. So, if I look at the LHS and the RHS remember this portion is less or equal to epsilon by 2. And this is also less than or equal to epsilon by 2 because that is the way we have chosen Qj.

So, we have chosen Qj with this property. So, both of them put together will give me that hence we get m of E delta F, to be less than epsilon, so that finishes the proof.

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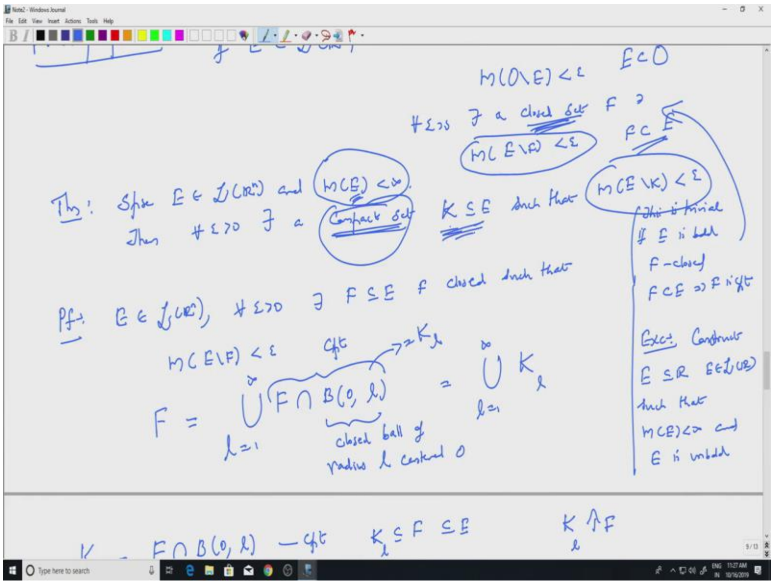

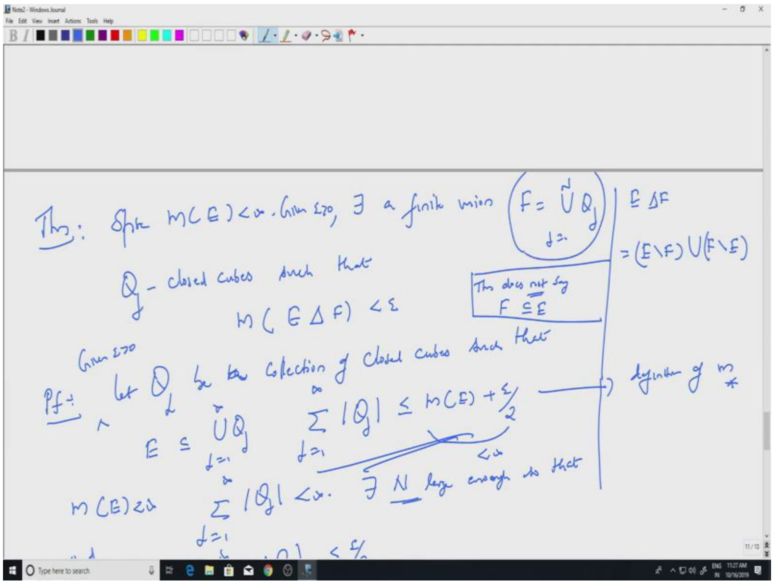

So just to repeat, these two theorems are slightly finer properties of the measurable sets. So, here we have a compact set, which is both for sets of finite measure. So, here we have a compact set and we have an inclusion.

Here we have, we do not have a compact we have a compact set, but the compact set is much more explicit in the sense that it is actually union of closed cubes, but then we do not have this portion that F, F need not be contained in E unlike here K contained in E. So, that is that is the point you should understand.

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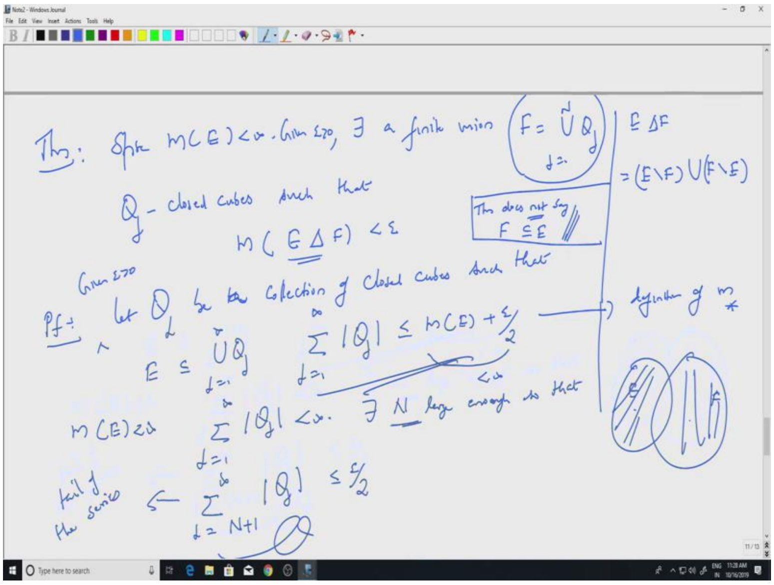

But then the conclusion is that conclusion is that if you if you take the symmetric difference right. So, E is like this f is like F can be like this. So, if E is this and F is this, then the symmetric difference is this part that is very small which means that F and E are very close to each other.

So, E can be approximated by union of intervals. So, that is what we have. So, this is a good point to stop. So, what we have done so far is complete the construction of the Lebesgue measure and look at certain finer properties of the measurable sets. So, the measurable sets in Rn Lebesgue measurable sets in Rn can be approximated from inside by compact sets if those sets are sets of finite measure, and if and also you can change the compact set by a union of close cubes, finite union of close cubes.

And this would be very close to the original set original Lebesgue measure set provided the set has a finite measure. So, sets of finite measure can be approximated by sets, the kind of sets you want from what you see from topology and so on. So, we will go ahead with those in the next session as well and see that they do not differ too much from Borel set.

So, if you look at compact sets, they are Borel sets they are closed sets, so they are Borel sets. Similarly the union of closed cubes you have seen there also Borel set. So, we are trying to say that Lebesgue set is very close to a bottle sets and that is how we prove that the Lebesgue sigma algebra is the completion of the Borel sigma algebra. So, we will take up that in the next sessions.