Measure Theory
Lebesgue Sigma Algebra

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Lecture 16
Lebesgue sigma algebra

So, we will continue looking at Lebesgue sets. So just recall that Lebesgue sets are, Lebesgue sets or Measureable set or Lebesgue Measureable sets these three terms will mean the same thing. The collection of Lebesgue sets we want to proof is a sigma algebra. So right now have all the open sets to be Lebesgue sets have sets whose outer measure is 0 also to be Lebesgue sets. We still have to proof that the collection of Lebesgue sets or the Measureable sets is closed under countable unions, compliments and well the empty set and the whole space that is the trivial condition that they are also there. So, that is our aim now.

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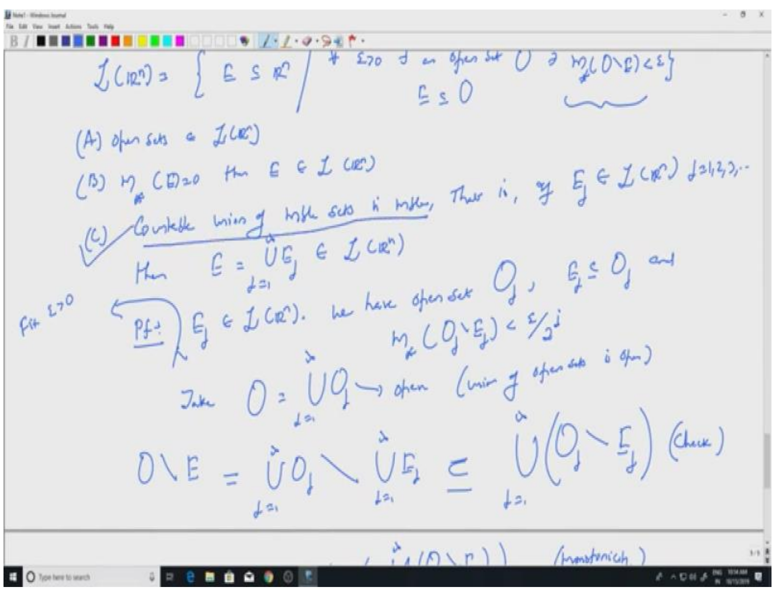

So let us start. So, two things we have proofed, so let us recall L of Rn. So, L of Rn is the collection of sets E contained in Rn such that for every epsilon positive there exist an open set, open set O such that the measure of O minus E is less than epsilon. This is the condition we have.

So, what we have just shown is that open sets are open sets belong to L of Rn that is trivial, B we have shown that if M star of E is 0, then E is a Measureable set. So, let us go to the non-trivial properties, countable union, countable union of Measureable sets is measureable. So, that is the, that one of the properties of the sigma algebra that the collections of the sets should be closed under countable union.

So, what did we just say that is, so let us complete the sentence that is, if Ej they are the sets in L of Rn that is EjR measurable J equal to 1, 2, 3, etcetera. Then the union of Ej, so let us call that E, E equal to union Ej j equal to 1 to infinity. That is also a Measureable set, so it belongs to L of Rn. So let us proof this, so this is not very difficult it follows from whatever sub additivity property we have for the outer measure.

So, each Ej is measurable, so let us fix an epsilon first. So, we start with that, fix an epsilon, fix epsilon positive, so remember to show that E is measureable what we do fix epsilon positive and then construct an open set so that this is true. So, let us start with Ej I have fixed an epsilon, so we have, we have open set, open set let us call that Oj because that is one works for Ej, such that first of all Ej should be contained in Oj.

So, that is condition which you should always remember the set should be contained in the open set and the outer measure of the difference. So you approximate the set nicely by open set. That is what measurability, qualitatively means, that is nicely approximated by open sets from above. So, this would be less than or equal to epsilon by 2 to the j. So, you have seen this argument before epsilon by 2 to the j. So instead of taking epsilon, I take epsilon by 2 to the j as a positive number and I have an open set for this.

So, now it should be obvious to you what to do because I have Oj for each Ej. So, for E I should take union of Ej, so the corresponding open set should be union of Oj. So, take O to be union of Oj j equal to 1 to n 1 to infinity this is of course open because each Ej each Oj is open and I am taking the union of open sets, union of open sets is open.

So, that is union of open sets is open. So because of that so now we want to approximate the set E by O so take, so look at O minus E and I want to look at the outer measure of those and say that it is small. Well what is this, this union Oj j equal to infinity minus union Ej j equal to 1 to infinity. So, this I will leave it to you to check that this is actually contained in it is not equal to it is contained in union j equal to 1 to infinity Oj minus Ej. Check this, this is a trivial set theoretic exercise.

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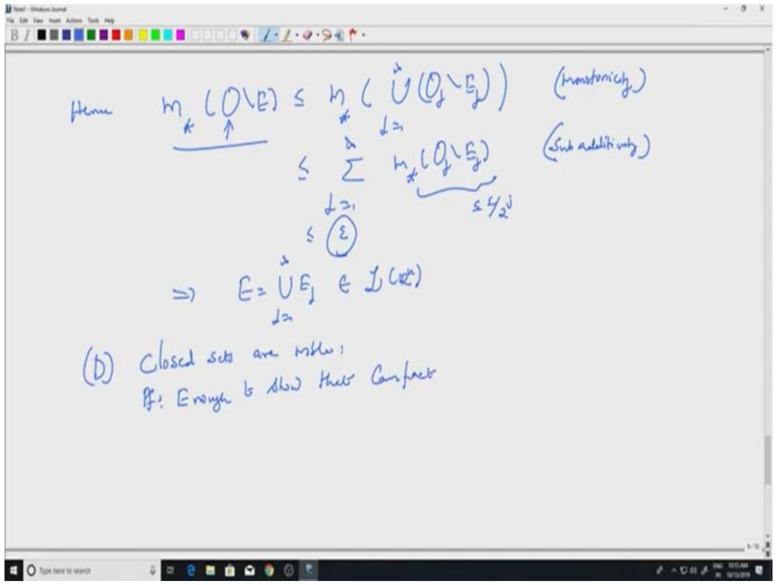

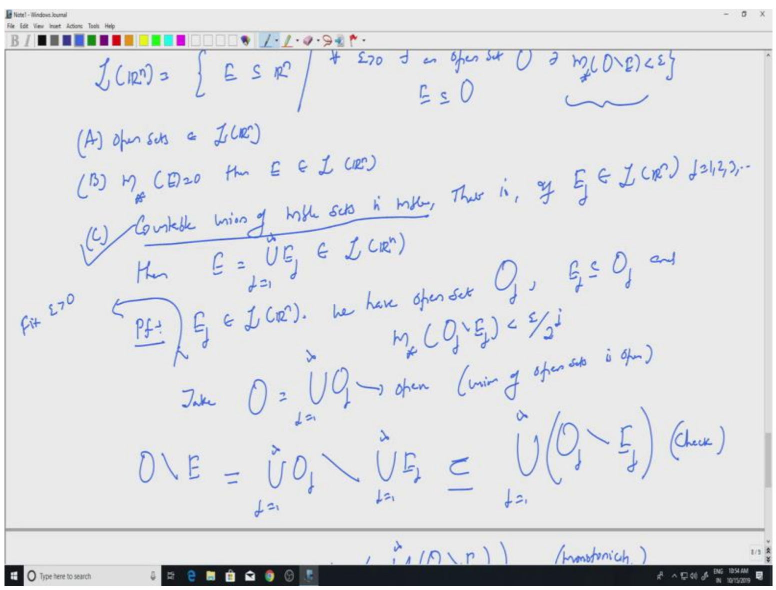

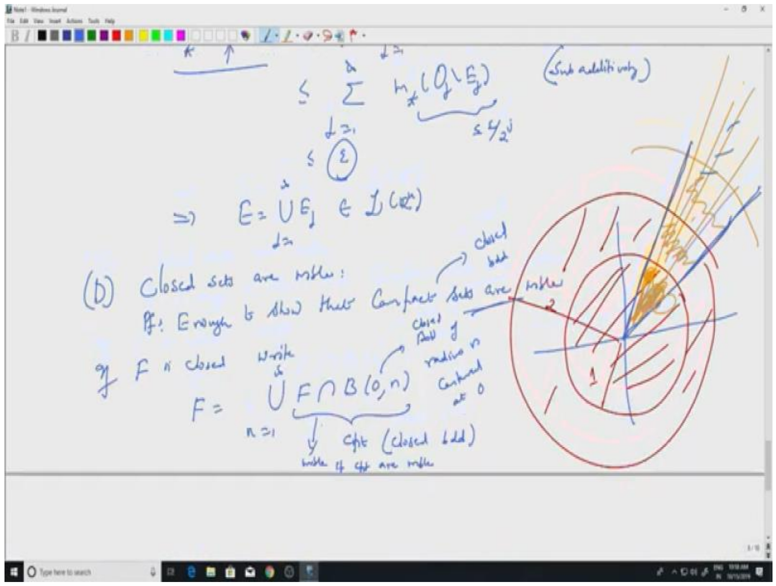

Because of this, so hence I can apply subadditivity, monotonicity. So, M star of O minus E less than or equal to M star of union j equal to 1 to infinity Oj minus Ej which is less than or equal to by… So, this is just monotonicity and this will be less than or equal to summation j equal to 1 to infinity M star of each of them this is simply sub additivity. By sub additivity and of course less than or equal to epsilon because each of them is less than or equal to epsilon by 2 to the j and I am adding them, so I will get epsilon and that is all I need.

So, this implies, so for every epsilon I have found an open set O such that this is less than or equal to epsilon. So, this implies that the set E this is the union of Ej j equal to 1 to infinity is measurable. That is precisely what we wanted to proof. So, we have proved C. So, this is one of the properties for L of Rn to become a sigma algebra. We are still two steps away from proving that it is a sigma algebra. So, let us do that, so that is the next one let us call that D close sets are measurable, close sets are measurable.

So go back the first one is that open sets are measurable but that was trivial, now we are saying close sets are measurable. So, that is like taking the compliment of course we do not know if we take the arbitrary set in L of Rn and take the compliment we still land in L of Rn, we do not know yet. But this is the first step close sets are measurable. So let us, let us proof this, so first thing to notice is that enough to, enough to show that compact sets are measurable, compact sets are measurable. So why is that?

So let me draw some picture for you, the close sets, so compact sets are closed and bounded, it is closed and bounded. So close sets need not be bounded it can be an unbounded set. So let us say we have an unbounded close set so something like this, so let us take this sector so this including the boundary lines. So that it will be a close set everything whatever is inside, what you do is you take, you take a ball of radius 1 and then you take a ball of radius 2 and so on this is radius 2. So, this portion is this the intersection of ball of radius 1 with the close set that is compact.

Similarly this portion that is compact and so on. So, what we can do is if you intersect with balls of radius n and take the union I am going to get well, so when you intersect you will, when you intersect you get only this portion. Whatever is inside, so this would be compact and then you will get this portion which is compact then you will this portion which is compact and so on.

So, you can write the close set as a union of countable union of compact sets. So, that is the reason, so let us write that, so if F is closed you can write F to be union of F intersected with ball of radius n centered at 0 n equal to 1 to infinity. So, this is ball closed ball of radius n centered at, centered at 0 and each of them is compact this is compact because it is closed and bounded, closed and bounded and it is a countable union.

So, if compact sets are measurable then these are measurable if compact sets are measurable and the earlier property we have proved is that countable union of Measureable sets is measureable. The property C which we countable union of Measureable sets is measureable. So, it is enough to show that each of these intersections is measurable then the, because it the countable union we are done. So that is why it is enough to show that compact sets are measurable.

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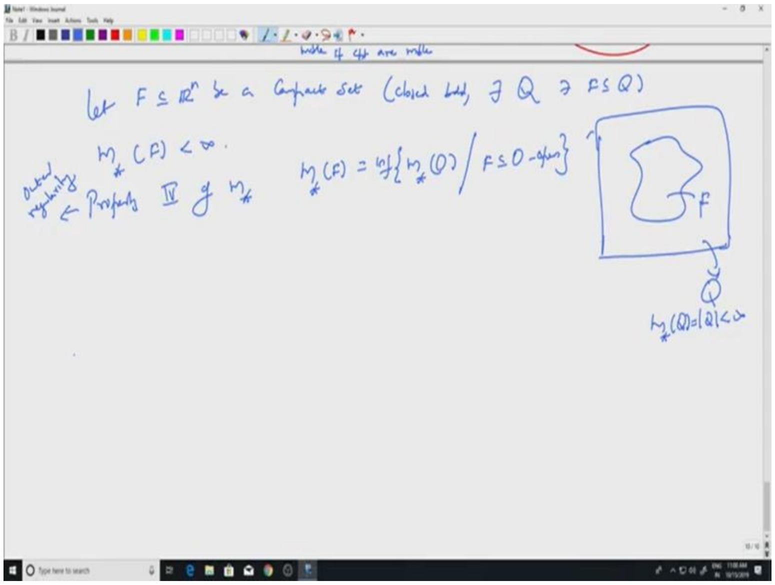

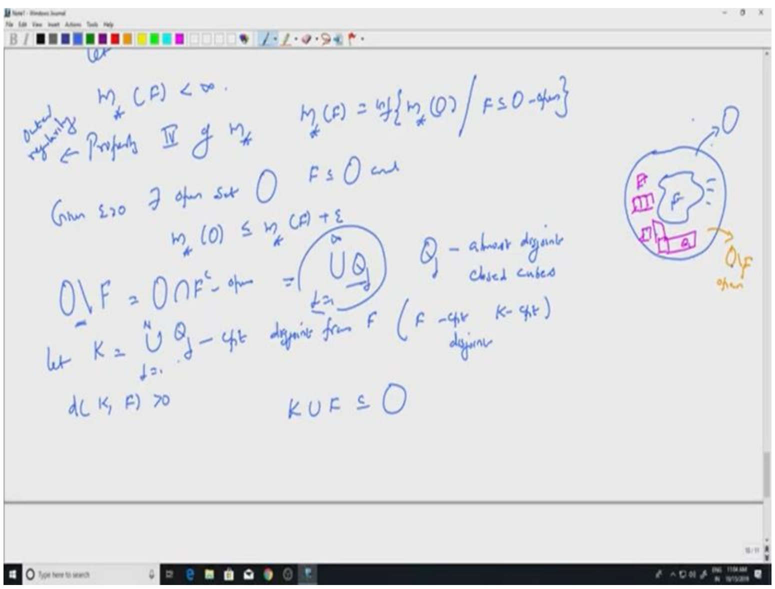

So we will do that, so let f contained in Rn be a compact set, be a compact set. We want to show that F is Measureable, since it is a compact set it is close and bounded. So, it will be closed and bounded, so there exist some big cube Q such that F is contained in Q, So, because of that M star of F the outer measure of F will be, will be finite.

Because the compact set will be something like this close and bounded and you have a big cube surrounding it because it is close and bounded you can put it inside a big cube and M star of Q is the volume of Q. So, this is finite, it is a product of the side length, so with that is finite and so compact sets will have outer measure finite.

So, now use property 4 of M star. We know that M star of F is equal to infimum of M star of O with f contained in O and O open. That is the outer regularity so property 4 is the outer regularity which is true of all subsets of Rn. So, we have this property we use this so now may be some picture is needed, so let us let us take this picture so I have F and I have so let us draw this picture again, so I have I have F which is a closed and bounded set this is my F and I am putting it so there is an open set around it so let us draw an open set around it.

I am drawing a bigger open set just to, so that the picture is clear, so it will be a much smaller open set. So, given epsilon I can so given epsilon positive there exist open set O such that F is contained in O and M star of O is less than or equal to M star of F plus epsilon, that is the property of infima. You look at M star of F plus epsilon there would be an element from that set which is less than or equal to that, that is M star of O.

Now you look at O minus F. So, what is O minus F? So that is this portion so this is O minus F. this will be open why is that? Well O minus F by definition is O intersected with F compliment. So, you are looking at O which is an open set and F compliment which is an open set and you are in the set T. So, this is open, because it is open you can write this as, so this is one of the properties we proved when we started with rectangles you can write this as union of Qj, j equal to 1 to infinity Qj close cubes almost disjoint close cubes. Almost disjoint closed cubes.

This was one of the properties we proved. So let me, let me try to show this in the picture so we can write the remaining of F inside O as, it need not be of same size, some may be bigger some may be smaller and things like that. So, this part is written as disjoint union of almost disjoint union of closed cubes. So, now you take finitely many of them, so let K be equal to union j equal to 1 to n Qj. So, I am taking only finitely many, so you take may be this one Q1 may be this one, may be this one and this one and so on.

So you take finitely many. Then this is also compact because finite union of this is compact, disjoint from F remember both are compact sets. So, F compact K compact disjoint because the Qjs are inside the compliment of F, it is inside O minus F, you are writing O minus F as union of

Qjs and you are taking Qjs. So they are disjoint, so I have two compact sets which are disjoint, so the distance between them will be greater than 0, because they are 2 disjoint compact sets.

So, this was the earlier exercise I gave if you have two compact sets, which are disjoint then their distance is positive and also K union F is contained in O. So let us look at the picture again.

So, I am taking the compliment of F, so O minus F that is this portion that is written as union of Qjs, I take finitely many Qjs and F. So, F is this. So that will be inside O, I am going outside the open set I started with.

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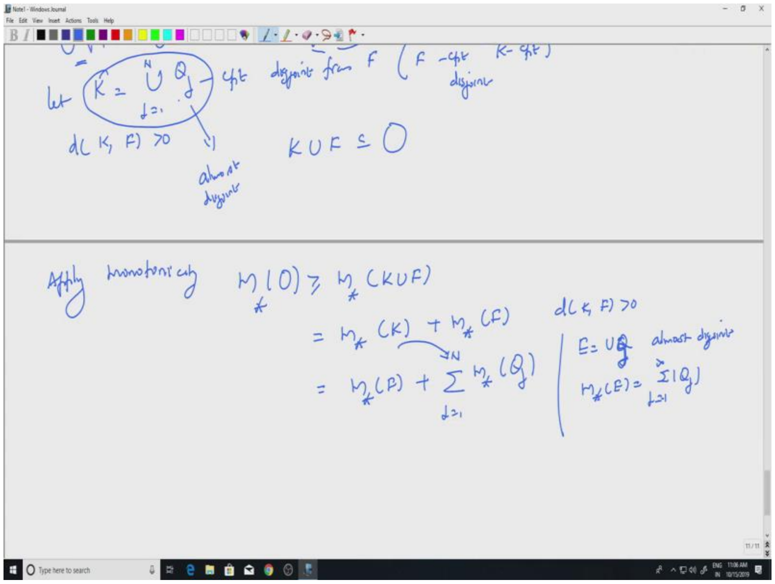

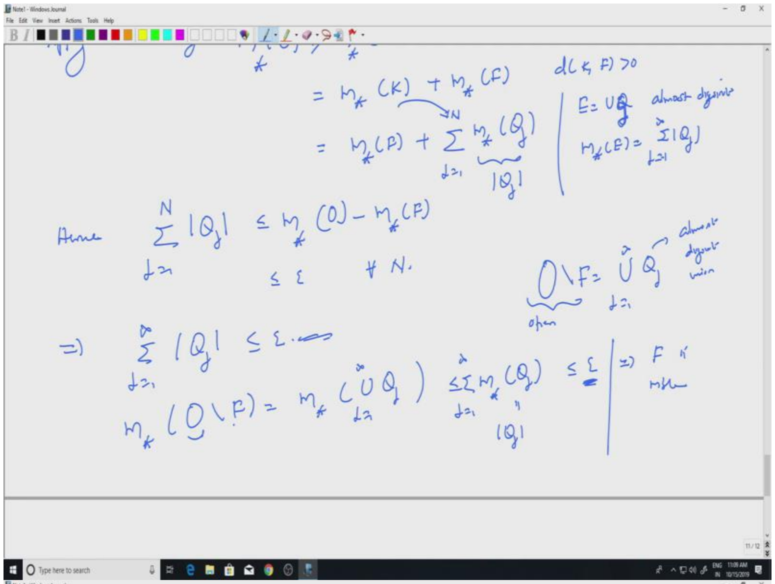

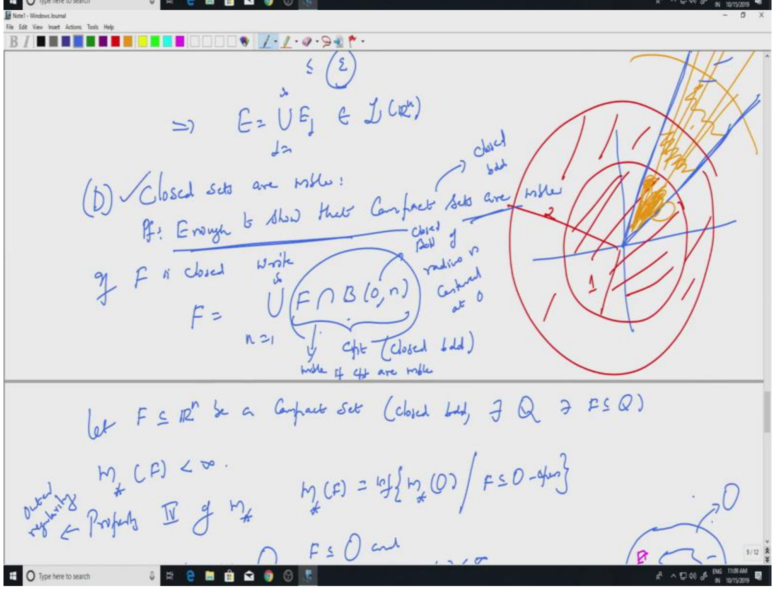

So, because of so apply monotonicity, monotonicity so you will get M star of O to be greater than or equal to M star of K union F, because K union F is a subset of this which is equal to M star of K plus M star of F, why is that? Because the distance between K and F is positive and we know that M star add sub when you have disjoint sets at a distance at a positive distance, which is equal to M star of F. So, that stays as it is plus well what is K? K is the union K is the union of Qjs where are almost disjoint, almost disjoint.

So, that was the property we proved earlier if you have almost disjoint union of sets then M star will add up. So, this is simply summation, so M star of K would be summation J equal to 1 2 capital N, M star of Qj. This was one of the property remember that if you write E as union Ej almost disjoint Qj almost disjoint, then M star of E was equal to summation mod Qj. Countability but a very restricted countability . So this is true, hence so now we apply whatever we know as a, so this remember is simply the volume of Qj. So, if you look at summation j equal to 1 to capital N volume of Qj this would be less than or equal to M star of O minus M star of F. Remember everything is finite M star of F is finite, M star of O is also finite.

So this is less than or equal to epsilon. Because we started with the set O so that so that this is 2. So because of that M star of so remember this implies that M star of O is also finite. Because M star of F is finite, F is a compact set and is less than or equal to epsilon. But this is true for every epsilon for every capital N. So, I can let N go to infinity, so this will immediately imply j equal to 1 to infinity mod Qj is less than or equal to epsilon.

Well what does that say what is Qjs? So, remember the open set O minus F, as union Qj, j equal to 1 to infinity. This was this was the open set and this was almost disjoint unit, almost disjoint unit. So, if you look at the outer measure of the open set O minus F this is nothing but well M star of union Qj. But remember this is a countable almost disjoint union of cube, so we know that it by well you can apply sub additivity. Here if you want, j equal to 1 to infinity, M star of Qj, and I know this is simply mod Qj but we know that this is less than equal to epsilon. So, apply this less than or equal to epsilon.

So that is all. So we, this implies that F is Measureable, F is Measureable, why? Because we have found an open set O such that O minus F has size less than or equal to epsilon. Whatever epsilon you start with, so that tells me that the compact set E is measurable. So what that gives me is because any closed set is an accountable union of compact sets and compact sets are measurable closed sets are measurable.

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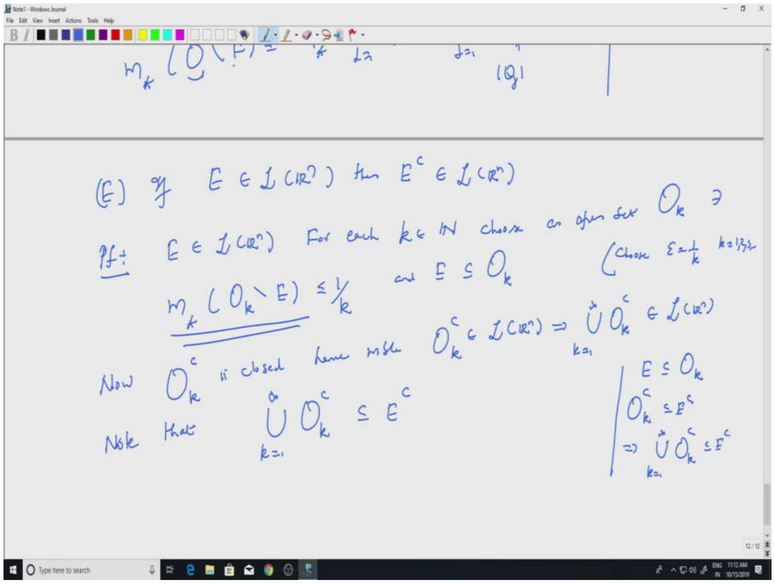

Let us go to the next property we are still far away from proving that it is a sigma algebra. So the next property will tell me it is a sigma algebra. So if E belongs to L of Rn then E compliment belongs to L of R, so we already proved that countable union of measurable sets is measurable. Now we are proving that it is closed under complementation, so that will tell me that the collection of measurable sets form a sigma algebra.

So, let us prove this, so if I take E in L of Rn what do I have for each, so let me write down the statement and then explain, for each k in natural numbers choose an open set we will call that Ok, such that the size of Ok minus E this is less than or equal to 1 by k and of course the set E should be contained in Ok.

This is true, because, I can do this for every epsilon, so choose epsilon to be 1 by k. So, this is simply choosing epsilon to be 1 by k, where k is 1, 2, 3, etcetera. So, for each epsilon I have an open set. So, for epsilon equal to 1 by 2, 1 by 3, 1 by 4 etcetera we get open sets O1, O2, O3, O4, etcetera with this property. Now Ok compliment if I look at the compliment of Ok this is closed this is closed. Hence, measurable we just proved that closed sets are measurable.

So Ok compliment is a element of L of Rn. So this would imply union of Ok compliment k equal to 1 infinity is also in L of Rn. We proved that countable union of measurable sets is a measurable set. So I am taking Ok compliments and taking the union. So that is also a measurable. So note that note that union k equal to 1 to infinity Ok compliment is contained in E compliment. Well why is that? Because each of them is contained in E compliment, so the union is also contained in E compliment. Remember, so this is simply because we looked at Ok which is like this, so this tells me that Ok compliment is contained in E compliment.

So, this would of course imply the union is also contained in E compliment. So, we are in good shape. So we continue, now some set theoretic inclusions are needed. So this is a slightly it will look messy but it is actually pretty trivial.

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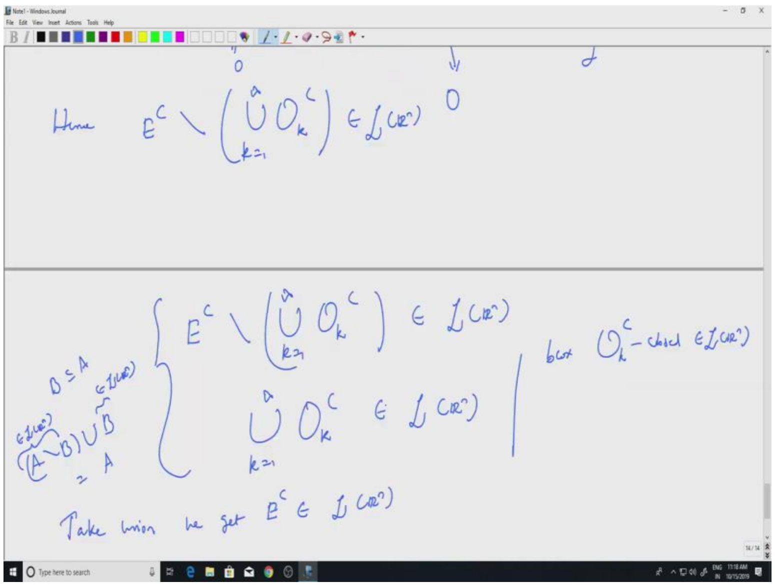

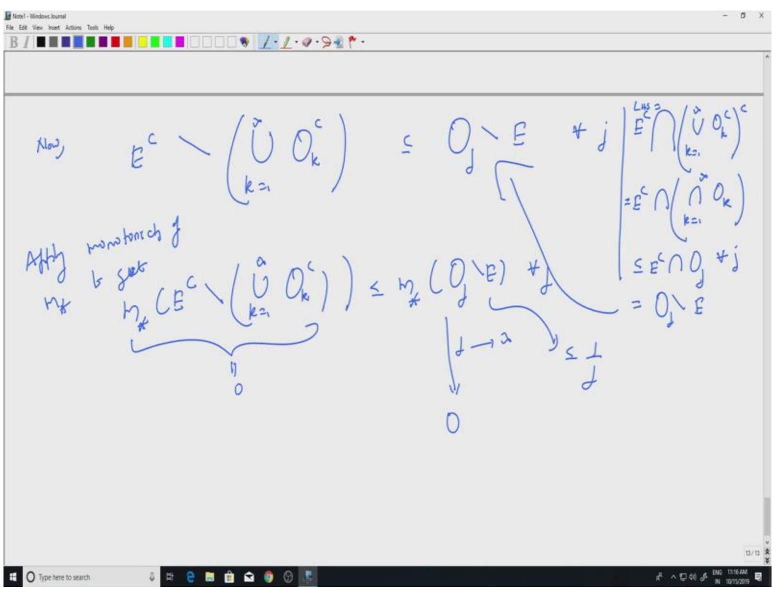

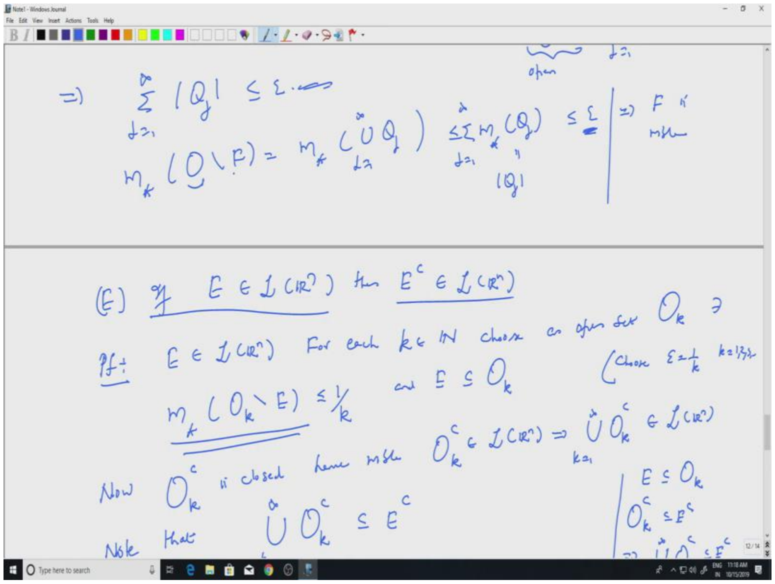

So let me write down that the first. So E compliment, so now what we do is check this E compliment minus union k equal to 1 to infinity Ok compliment, so I will take this, this is contained in Oj minus E for every j. So, this looks a bit messy but let us see why?

So let us prove this so what is the LHS? So, LHS is E compliment intersected with the compliment of whatever is there A minus B is the intersection of A with B compliment. So, LHS is equal to E compliment intersected with the compliment of this. So let, I will write one more step. So, that is this clear, k equal to 1 to infinity Ok compliment whole compliment. So, it looks messy as I said but it is exactly pretty trivial.

Which is E compliment intersected with so if I take the compliment of the union I will get the intersection of the compliment. So, that is intersection of k equal to 1 to infinity Ok. Which is of course contained in E compliment intersect any of them Oj for every j. But E compliment intersection Oj is simply Oj minus E. So, that is all is the here. So, proof though even though it looks messy the proof is rather set theoretic inclusions and its sort of straight forward.

So, now apply M star applies monotonicity of M star to get M star of E compliment minus union k equal to 1 to infinity Ok compliment is less than or equal to M star of Oj minus E for every j. So, the right hand side has j but the left hand side have is independent of j.

So I can put j, I can let j go to infinity and this will go to 0 because this thing is less than or equal to 1 by j. That is how we have chosen the sets Ok remember that it is less than equal to 1 by k. So, here Oj will be less than equal to 1 by j and like j go to infinity I will get 0. So, what does that mean? That means this side is equal to 0, so I have a set whose outer measure is 0, so that belongs to the Lebesgue sigma algebra.

So hence, let us write it here hence E compliment minus union k equal to 1 to infinity Ok compliment belongs to Lebesgue sigma algebra because it has size 0. Any set which has M star outer measure 0 is a measurable set. So, let us let me write down two things, so that the we can stop, so this would be k equal to 1 to infinity Ok compliment, this I know now belongs to L of Rn. I also know that union k equal to 1 to infinity Ok compliment also belongs to L of Rn. Why is that? Because Ok compliment is closed and so belongs to L of Rn and countable union of Measureable sets is measureable.

So, I have two things belonging to L of Rn. So, if I have two sets belonging to L of Rn their union will also belong. So, take union, take union, so union is simply, so I am subtract, so it is like, it is like A minus B union B, this is simply A because B contained in A. So, remember B is contained in it. So, when you take the union you will get, we get E compliment and of course if this belongs to L of Rn, so one of them belong to L of Rn and the second set also belongs to L of Rn then the union is in L of Rn.

So that is all we need. So, we proved that if E belongs to L of Rn measurable then E compliment also belongs to L of Rn. So, we can stop with this, what we have proved is that the Lebesgue sets the Lebesgue Measureable sets defined using M star of O minus E to be less than epsilon or in other words for every epsilon there is an open set O such that M star of O minus E is less than epsilon.

That collection of sets E with this property called the Lebesgue sets form a sigma algebra. We proved that countable union of Measureable sets is measureable and we just proved that if E is measureable set its compliment is also a measureable set. Now the outer measure restricted to the sigma algebra, we have to prove that it is countably additive. So, that will complete the construction of the Lebesgue measure. So that is what we will do in the coming lecture.