Measure Theory

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Lecture 15

Lebesgue measurable sets and Lebesgue measure on Rn

So, in the last lecture we saw the outer measure. Outer measure is a quantity which is defined on all subsets of Rn. It has some interesting properties, we saw some of them in the last lecture. One was the monotonicity which was which is a trivial property, second was sub-additivity, countable sub-additivity, third was the outer regularity and fourth was an interesting property which said that if two sets are at a distance positive distance, so not just disjoined, they are at a distance, then the outer measure will add up.

So, outer measure is defined on the power set of Rn, it is not going to be a countably additive measure on the power set, we will have to find a collection of subsets of Rn which will be called the Lebesgue sigma algebra on which this outer measure, the restriction of the outer measure to the Lebesgue sigma algebra will be countably additive measure. So, that is our aim in this lecture and the next lecture.

By next lecture, we should be able to complete the construction of the Lebesgue measure. So, we will start with recalling some of the things we did in the last lecture.

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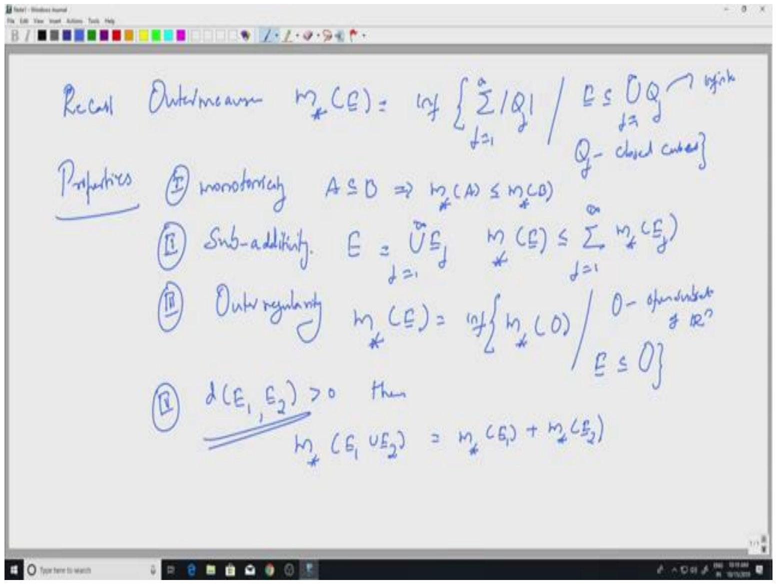

Let us start, so recall outer measure. So, this was denoted by m star and m star of a set E this was equal to the infimum of summation mod Q-j, j equal to 1 to infinity where E was contained in Qj. So, Qj form a cover of E and Qj were closed cubes.

So, let me repeat that one needs the infinite collection here, just taking finite union of cubes will not work that will not give us a genuine countably additive measure. So, this was the outer measure and we had some properties. So, let me recall those properties, so properties which we proved in the last class so one was monotonicity; monotonicity, what was that? If A is contained in B that implies the outer measure of a is less than or equal to outer measure of B.

So, that is that was easy, so, outer measure you should think of as the size of the set. So, the set is bigger the size should be bigger, that is what was proved yesterday. Second one, this was countable sub additivity, sub additivity. So, this said that, if I write E as union Ej, j equal to 1 to infinity, then the outer measure of E is less than or equal to the sum of the outer measure of individual Ej's.

So, when you take the union well, because the Ej's may not be disjoined, even of if Ej's are disjoined, we do not know if they add up, but sub additivity is something which we can prove. Third one was outer regularity, this is an important property out of regularity, this was m star of E so the outer measure of any set E is the infimum of m star of O open sets. So, O was an open set, open subset of Rn and more importantly, E was contained in O, so, you look at all open sets bigger than E and look at the size of them using m star and take the infimum that will give me m star of E, and this is again to for all subsets E.

The fourth one we proved was, if the distance between two sets E1 and E2 is greater than 0, then so that means they are at a distance, so they are not just disjoined, they are at a distance, then the outer measure adds up. So, m star of E1 union E2 this is equal to m star of E1 plus m star of E2. So, we would like countable additivity, this is not countable additivity, this is only finite additivity for sets with this condition.

So that is a much weaker than whatever we want, but this is true for every E1 and E2 which is a subset of Rn. So, this is the property is on a larger, rather large class of sets. So, our aim would be to restrict.

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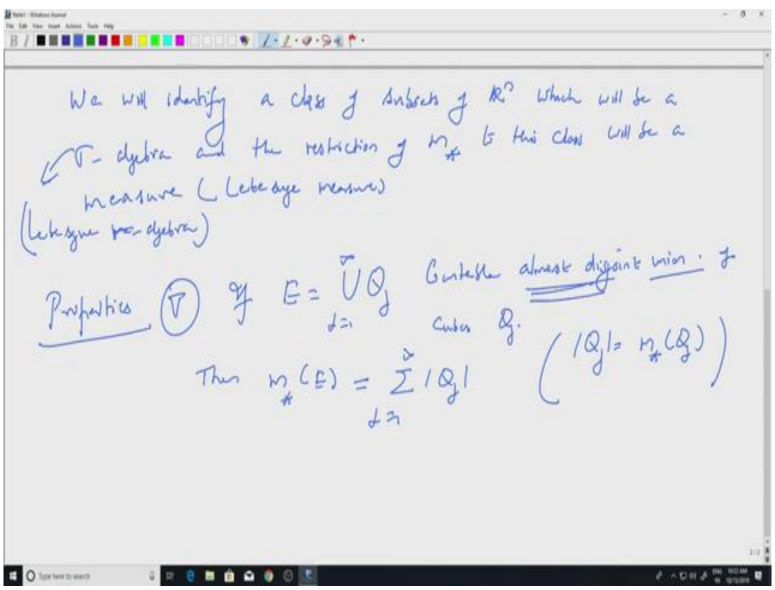

So, aim will be to restrict the outer measure. So, let me see, so, we will, we will identify a class of subsets, of Rn which will be a sigma algebra, and the restriction, restriction of m star to this class will be a measure, countably additive measure.

So, this is called the Lebesgue measure, Lebesgue measure and sigma algebra will be called the Lebesgue sigma algebra. This is going to be a rather rich class of sets, we will see soon, but before that let us continue with the property. So, we had four properties, we will look at the fifth one, which is more or less like countable additivity.

So, property we are continuing with outer measure, we are looking at the fifth property of the outer measure. So, we write E has a countable union of cubes. So, countable, almost disjoint almost. So, this is of close cubes almost disjoint union of cubes, cubes Qj. Then we have additivity, so, then m star of E so remember this is going to be the Lebesgue measure of E, later on if E is a nice enough set which we will identify, but in this particular case when E is actually accountable disjoint union of cubes then the outer measure of E is summation mod Qj.

So, remember the four cubes we know that this is the outer measure. So, we are saying our measure of union of Q days is the sum if they are almost disjoint. So, let us let us try to prove that, so let us keep this statement here.

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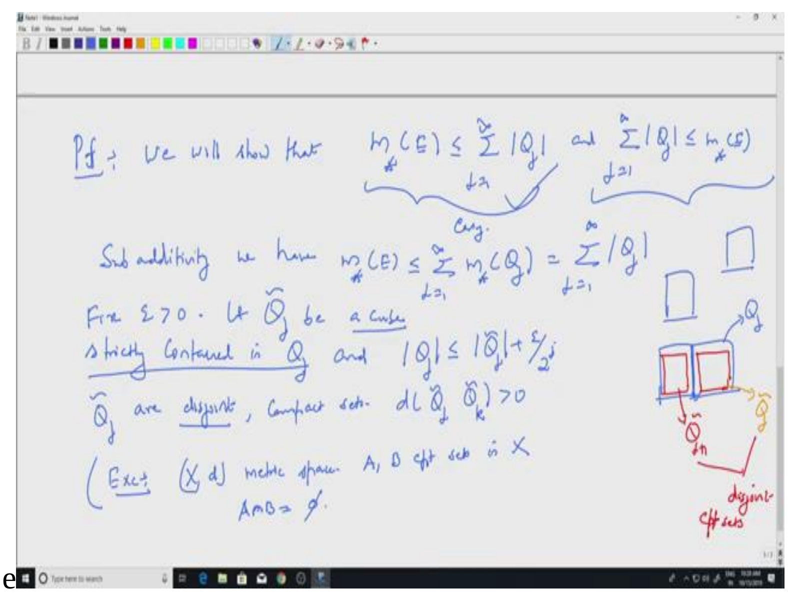
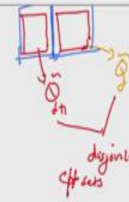
e 

strictly contained in Q_j and $|Q_j| \le |\tilde{Q}_j| + \varepsilon/2^j$

$\tilde{Q}_j$ are disjoint, compact sets. $d(\tilde{Q}_j, \tilde{Q}_k) > 0$

(Ex: (X, d) metric space. A, B cpt sets in X, $A \cap B = \phi$. Then $d(A, B) > 0$

$\inf\{d(x, y) \mid x \in A, y \in B\}$



$$m_*(\tilde{Q}_1 \cup \tilde{Q}_2) = m_*(\tilde{Q}_1) + m_*(\tilde{Q}_2)$$

$$m_*(\tilde{Q}_1 \cup \tilde{Q}_2 \cup \tilde{Q}_3) = m_*(\tilde{Q}_1 \cup \tilde{Q}_2) + m_*(\tilde{Q}_3)$$

$$= m_*(\tilde{Q}_1) + m_*(\tilde{Q}_2) + m_*(\tilde{Q}_3)$$

Property IV of m_*: $d(E_1, E_2) > 0$, $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$

Sub additivity we have $m_*(E) \le \sum_{j=1}^{\infty} m_*(Q_j) = \sum_{j=1}^{\infty} |Q_j|$ (Cary.)

Fix $\varepsilon > 0$. Let $\tilde{Q}_j$ be a cube strictly contained in Q_j and $|Q_j| \le |\tilde{Q}_j| + \varepsilon/2^j$

$\tilde{Q}_j$ are disjoint, compact sets. $d(\tilde{Q}_j, \tilde{Q}_k) > 0$

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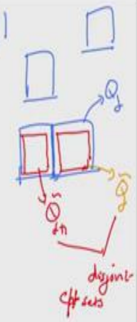
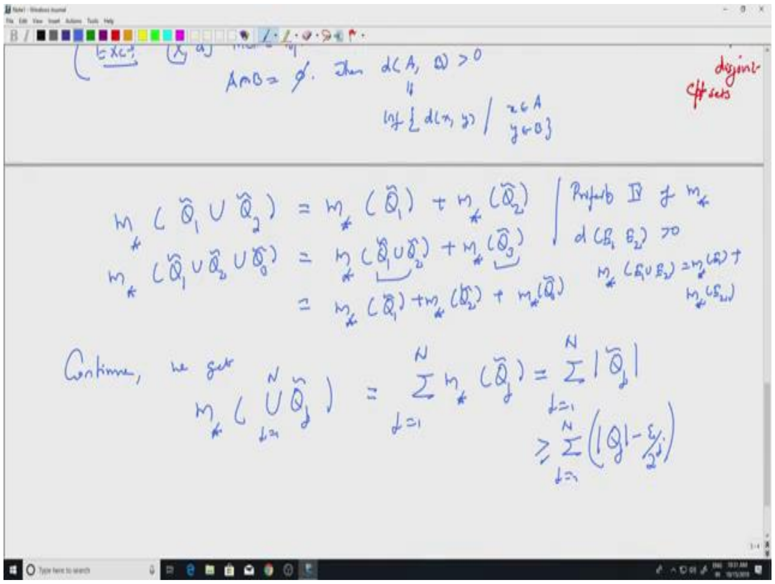


$$m_*(\tilde{Q}_1 \cup \tilde{Q}_2) = m_*(\tilde{Q}_1) + m_*(\tilde{Q}_2)$$

$$m_*(\tilde{Q}_1 \cup \tilde{Q}_2 \cup \tilde{Q}_3) = m_*(\tilde{Q}_1 \cup \tilde{Q}_2) + m_*(\tilde{Q}_3)$$

Property IV of m_*: $d(E_1, E_2) > 0$, $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$



So, we will let start with the proof. So, proof as usual, one will prove inequalities, so, we will prove that so we will show that m star of E, E remember is a is accountable almost disjoint union of cubes is less than or equal to summation mod kj, mod Qj and summation mod Qj is less than or equal to the outer measure of E.

So, once we have both these inequalities, they will have to be equal. So, that is the usual strategy of proof in, in this topics. So, let us start so, so first of all first of all this part is easy, that is simply countable additivity, by sub-additivity we have m star of E to be less than or equal to summation j equal to 1 to infinity m star of Ej sorry Qj, Qj but m start Qj we know that it is the volume of the cubes.

So, for cubes we already know this, so, this we have, so that is one inequality, so this this is done. So, we have to show this one, this is the non trivial part. So as usual fx and epsilon, so by now, you must have seen this epsilon by 2 to the J argument several times so we use something very similar, but we also use some of the properties of m star which we proved earlier.

So, let x epsilon be positive, so let Qj tilde be a cube, strictly contained in, Qj. So maybe we will draw some pictures to see what it is, so remember the Qj are almost is disjoint. So, either they can be disjoint so, you may have a cube here, you may have a cube here, or you may have a cube here and you have a cube here. So, they may be intersecting at the boundary, but that is okay the interiors are going to be disjoint.

So, Qj tilde is a cube strictly contained in Qj. So, let us see, so Qj tilde will be something like this, we will, specify the condition with respect to which Qj tilde should be chosen. So, the outside ones are Qj and let us denote this, who these ones are Qj and the inside ones are Qj tilde. Well how do we choose them? So contained in Qj and mod of Qj tilda or Qj is less than or equal to mod of Qj tilde plus epsilon by 2 to the Qj.

So, this is easy because the Qj tilde as you make them bigger and bigger, they become Qj. So, the volume of Qj tildes will converge to the volume of Qj and this we can do. So, we have seen this several times in the previous arguments. So, notice that Qj are almost disjoint, so Qj tilde being strictly contained in inside Qj are disjoint. So, that is that is one thing, so if you look at this as, so let us say this is Qj and this is Qj plus 1 tilde.

So, these two are disjoint and they are those joint compact sets, because they are bounded closed and they are disjoint compact sets. So, they have to be at a distance, so hence to let me write this statement, Qj tilde are disjoint compact sets. So, if I take any such any two such sets, the distance between so, let us say Qj tilde and Qk tilde will have to be greater than 0, so, well if you have not seen this, this is an easy exercise.

So, this is true in any metric space, so let us say x d is a metric space and if I take two disjoint compact sets, let us take A, B compact sets in x and A intersection B is empty. So, they are disjoint, then the distance between them is positive distance between A and B has to be strictly positive. So, remember this is the infimum of this is equally to the infimum of the distance between x and y where you take x and A and y and B.

So, that is a strictly positive quantity. So, so going back to our proof, we have Qj tildes which are disjoint, so they are disjoint compact sets. So because of that the distance will have to be positive between any two of them. So, that tells me that, if I take n of them, so let us take m star of m star of Q1 tilde union Q2 tilde, well this would be m star of Q1 tilde plus m star of Q2 tilde.

Because so this was property, property four, of the of m star, if the distance between two sets E1 and the E2 is positive, then the m star of E1 union E2 is the sum of the outer measure of E1 and outer measure of E2. So, we know this property already. So, we applied this and I can continue, so I can look at m star of Q1 tilde union Q2 tilde union Q3 tilde, this would be m star of Q1 tilde union Q2 tilde plus m star of Q2 tilde, Q3 tilde, because this set and this set are at a distance they both are compact sets disjoint.

So, they are the disjoint and this will add up again because of the same argument. So, I have Q1 tilde plus m star Q2 tilde plus m star Q2 tilde etc. So, I can do this for and finitely many. So, continue if you continue, we get m star of union to Qj tilde j equal to 1 to capital N let us say, this is summation j equal to 1 to n star of Qj tilde, because they are all compact sets disjoint, so they are at a distance and you can apply the property four again and again.

Well, we know this is j equal to 1 to capital N, mod Qj tilde, because Qj tilde has a cube so its outer measure is the whole N which we know is greater than or equal to j equal to 1 to n mod Qj minus epsilon. So, let us let us go back, sorry epsilon by 2 to the J. So, we know this because we have chosen Qj, Qj such that this is true Qj tilde such that this is true. So, remember the inequality is this way.

So, this is what we have, which of course, we can since you have epsilon by 2 to the J's we can replace it with epsilon.

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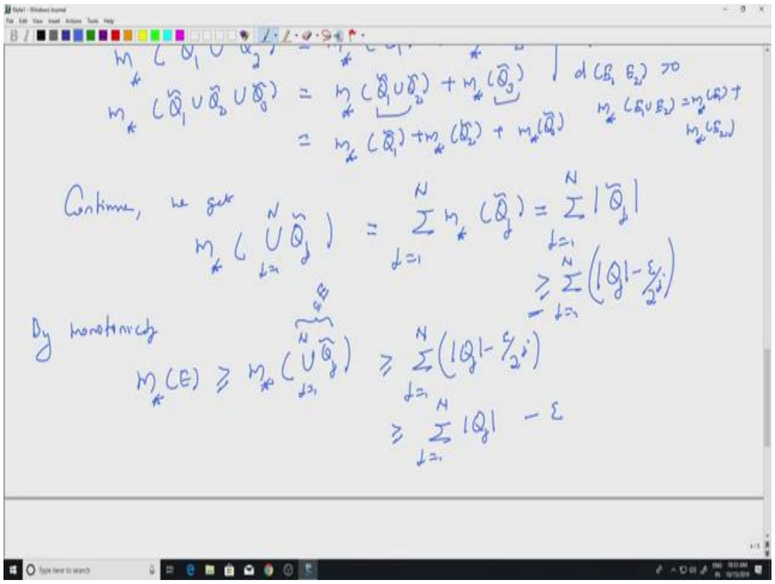

So, by monotonicity, we have m star of E is greater than or equal to m star of j equal to 1 to n Qj tilde because Qj tilde is simply smaller cubes, so Qj tildes are the smaller cubes and union of Qj's is what makes up E and Q-j tilde are contained in E.

So, this union is contained in E, so this set is a subset of E, so by monotonicity we have that this by whatever we have just proved is greater than or equal to summation j equal to 1 to n mod Qj minus epsilon by 2 to the J which of course, is greater than or equal to summation j equal to 1 to n mod Qj minus epsilon, because epsilon by 2 to the J's if you add you will get at most epsilon.

So, I am subtracting a bigger quantity and so, the inequality goes in this direction. So, what do we have?

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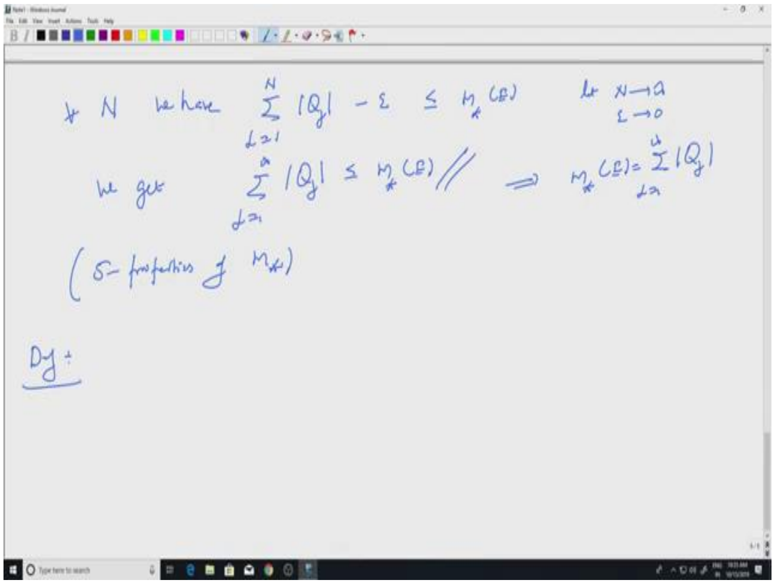

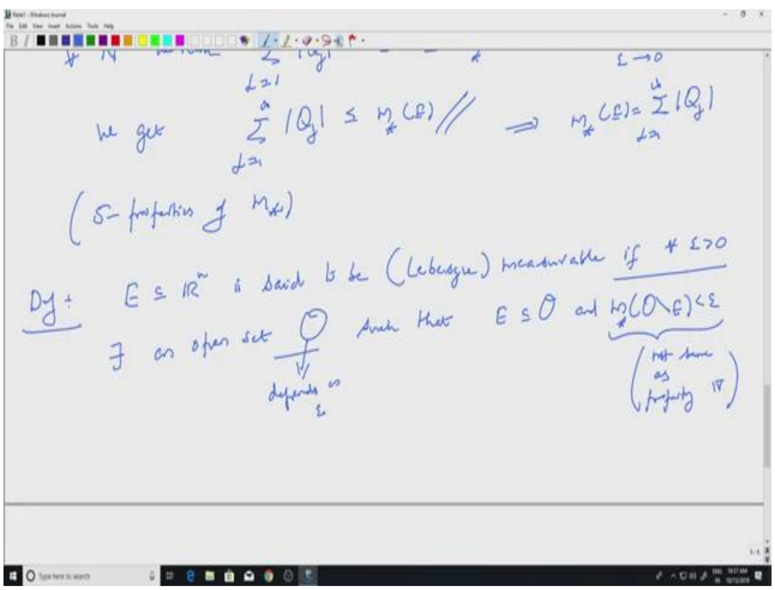

So, let us rewrite it, so what we have is for every capital N, we have summation j equal to 1 to capital n mod Qj minus epsilon is less than or equal to m star of E but this is true for every n and the right hand side has nothing to do with N. So, you can let n go to and go to infinity and epsilon go to 0.

So, let n go to infinity and epsilon goes to 0 we get we get summation j equal to 1 to infinity mod Qj is less than or equal to m star of E. So, that is the other inequality. So, let us go back to the starting of the proof, we wanted to show two inequalities, this one was trivial, this is what we were trying to prove and what we have just established is that summation mod Qj equals 1 to infinity Qj is this.

So, all these will imply that m start of E is actually equal to summation j equals 1 to infinity mod Qj. So, we have countable additivity in a very restricted sense in the sense that if E is a set which is a countable almost disjoint union of cubes, we have additivity. So, we have five properties, so five properties of m star is what we have, so, now let us define.

So, definition now we are ready to define the nice collection of sets on which m star is going to be a countably additive measure define. So, you take a set which is in Rn, is said to be is said to be measurable, so we will give it a name. So, this is called Lebesgue measurable, so we will not keep saying Lebesgue measurable, we will simply say measurable by measurable, we mean Lebesgue measurable, if we are talking about sets of subsets of Rn Lebesgue measurable if for every epsilon positive.

So, this statement is important for every epsilon positive, there exist an open set, open set O, such that first of all E is contained in O and the measure of O, the outer measure of O minus E is less than epsilon. So, we are saying E can be covered by an open set O, at a size less than epsilon, so, this is important and this you should convince yourself that this is not same as the outer regularity not same as property four.

So, that is different, so there is there is a this is a slightly stronger condition than the outer regularity we had, outer regularity was true for all sets remember that all subsets of Rn had the outer regularity property, but both condition that m star of O minus E is less than epsilon will be true only for certain sets. So, as epsilon changes the set O will change. So, the set all depends on epsilon. So, epsilon is something else epsilon becomes smaller and smaller, the set O will also have to be smaller and smaller and very close to E. So, will denote the collection of measurable sets by script L of Rn.

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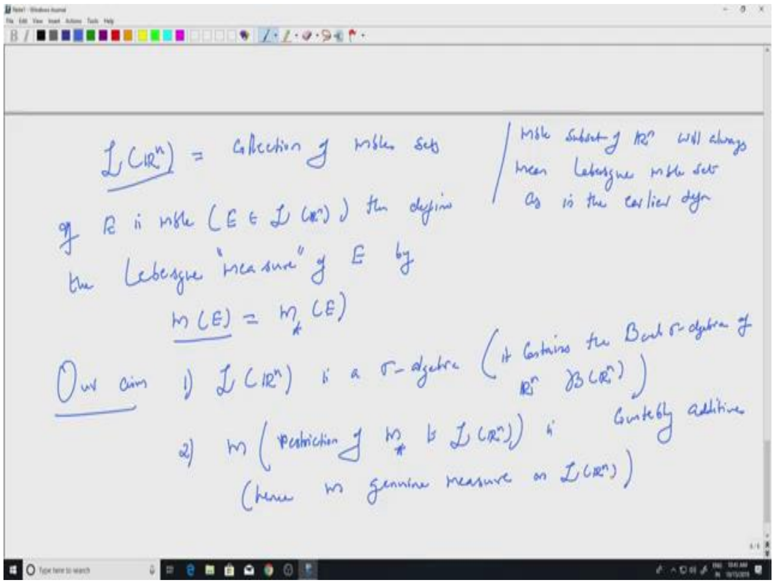

So we let us give it a notation. So, script L of Rn this is the collection of measurable set, so I will keep writing MBLE or measurable, measurable will always mean Lebesgue measurable sets.

So, measurable subset of Rn will always mean Lebesgue measurable set. So, as the earlier definition, So, this is the collection of measurable sets. So, if E is measurable, that means, E is a member L of Rn then define the Lebesgue measure of E by m of E. So, m will denote the Lebesgue measurable of E to be equal to m star of E.

So, remember m star of E is defined for all subsets. When we say Lebesgue measure of some set, that set will have to be an element in the L of Rn in the collection L of Rn. So, our aim now we have two steps to aim. So, we have defined L of Rn and we have defined Lebesgue measure. So, when we say Lebesgue measure, so measure means it is a countably additive measure so, we have to show that.

So, one is to show that the collection of Lebesgue measureable sets is a sigma algebra and it contains the Borel sigma algebra of Rn, remember it was denoted by script B of Rn, this is the smallest sigma algebra generated by open subsets of Rn to m. So, what is the m? m is simply the restriction of m star.

So, this is the restriction of m star to the Lebesgue sets or Lebesgue measureable set is that is accountability additive measure is so E is countably additive that will do, so E is countably additive. So, that it is a genuine measure on, so hence m will be a genuine measure on L of Rn. So, these are the aims.

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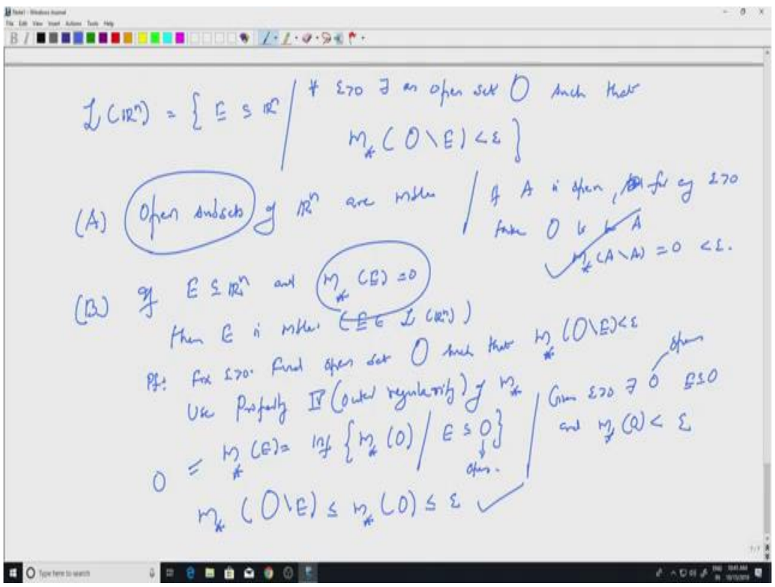

So we will start with the, we will start with the first, first one. So look at L of Rn, so remember what is this?

This is the collection of all those subsets of E which has a certain property, what is that property? For every epsilon positive there is exists an open set, open set O. So, O depends on epsilon remember that such that, the outer measure of O minus E is less than epsilon, this is the property, this collection. Well what do we know? We know that, so from the definition some of the things are very trivial.

One is open sets are measurable, open subsets of Rn are measurable. Well, why is that? If I take an open set O, if A is open say instead of O is open take O to be A for any epsilon positive take O to be A itself, so you will have m star of A minus, A minus A is empty set to 0 and that is less than epsilon. So, this is a trivial, trivial statement, so all the open sets are already in already measureable so, that is a big collection to start with.

Second if E is a subset of Rn and m star of E is 0, so size of the set is 0, then E is measurable, then E is measurable that means E is an element of L of Rn. Well, why is that? So, let us fix epsilon, so prove fix epsilon positive, I want to find an open set such that M star of O mines E is less than epsilon, find open set O such that m star of O minus E is less than epsilon.

So, use property four, so this was outer regularity of m star. So what does it say? It tells me that m star of E equal to infimum of, infimum of m star of O where E is contained in O, but the left hand side is 0, so given epsilon, so, given epsilon positive, there exists O such that E is contained in O and m star of O is less than epsilon, because it has become 0.

So, it has to be less than epsilon for some open set O, this is open. Now, m star of if you take that particular open set, m start of O minus E I know is less than or equal to m star of O by monotonicity and it will be less than or equal to epsilon. So, that is O so, any set which has size 0 is measurable, open sets are measurable. So, we will stop with this right now, what we have what they have done is to restrict the outer measure to a class of sets called the Lebesgue measurable sets.

This class of sets we have to prove that it is a sigma algebra and the outer measure restricted to this is accountably additive measure. So, first we are trying to prove that the Lebesgue sets form a sigma algebra, so we get a rich class of sets, there open sets are there, we have just proved that sets with size 0 are also there. So, next step will be to prove that it is a sigma algebra.