Properties of Outer Measure on Rn

Professor E. K. Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru

Lecture- 14

Measure Theory

So, next we will look at properties of the outer measure. So, recall that we just defined the outer measure on subsets of Rn, for each subset of Rn we have a positive number associated to it. It may be 0 or a positive number or it could be infinity also and this outer measure is supposed to give us a countably additive measure, when we restricted to an appropriate sigma algebra that will come later. Before that, so towards to move towards such a goal, we look at basic properties of the outer measure we just defined.

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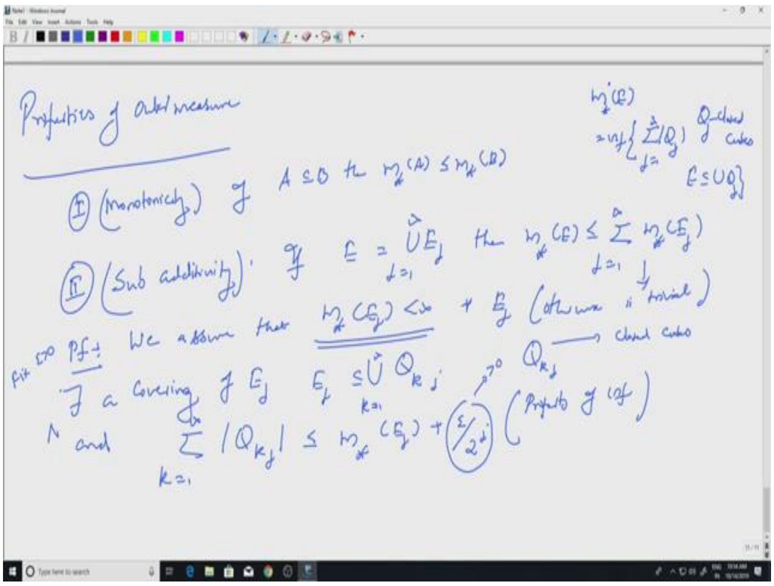

So, let us recall that so properties of outer measure, so will keep this in the side. So, m star of E was defined to be the infimum of certain quantities this is summation mod Qj, j equal to 1 to infinity, Qj were closed cubes and it was a cover of E. So, you cover E with closed cubes form this quantity and take the infimum. So, Properties of outer measure the first one, this is monotonicity we just saw this, so I will be brief here monotonicity, if A is content in B then m star of A is less than or equal to m star of B.

So, this immediately so this we saw earlier so this immediately implies that m star of the whole space Rn is infinity because you can take any take big cubes. Q content in Rn and that will become bigger and bigger because if Q is content in Rn, m star of Q is less than or equal so m star of Q is less than or equal to m star of Rn, but m star of Q is simply the volume of Q and the volume of Q can be made as big as you want, so that is easy. Second one, second property this is called Subadditivity. Subadditivity again a very natural property to have a new look at measures and so on.

So, if E is equal to union Ej, j equal to 1 to infinity, then m star of E is less than or equal summation j equal to 1 to infinity m star of Ej this is called subadditivity. The volume of E so imagine these are all cubes then volume of E will be less than or equal to sum of volumes of Ej because Ej's may overlap. So, that is what is reflected here and it is true for all sets so that is the important point. Let us prove this, so we start with assuming that we assume that m star of Ej is finite for every Ej. Otherwise, the right hand side will have an infinity and then this otherwise this is trivial because the right hand side will have an infinity term and then we have the inequality clearly.

Now, m star of Ej is finite, so there exist the covering so there exist a covering so this is what I had mentioned as soon as the definition of m star was introduced because it is the infimum for every epsilon you have a covering with certain property. So, there exist the covering of Ej so Ej content in Q, well notation has to be clear Qkj, union k equal to 1 to infinity. What are Qkj is? Qkj are these are closed cubes so, for each j I have Qkj's and summation k equal to 1 to infinity modulus of Qkj is less than or equal to m star of Ej plus epsilon by 2 to the j.

So, of course we always start with an epsilon so maybe I should have said that first so let us put this here. Fix epsilon positive, so there exist the covering of Ej such that this happens. This is just the property of infimum, the epsilon by 2 to the j is a positive number so I will have a covering with this property. So, keep that property, keep that inequality in mind because that is something which will you use.

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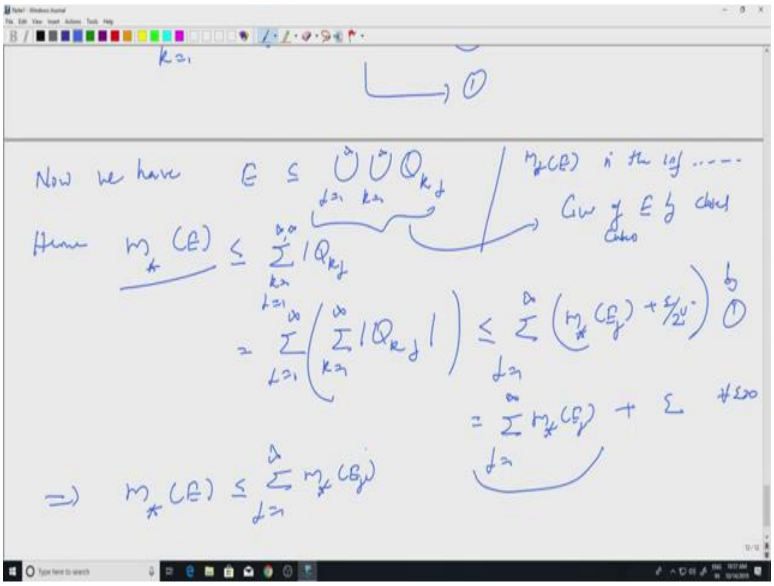

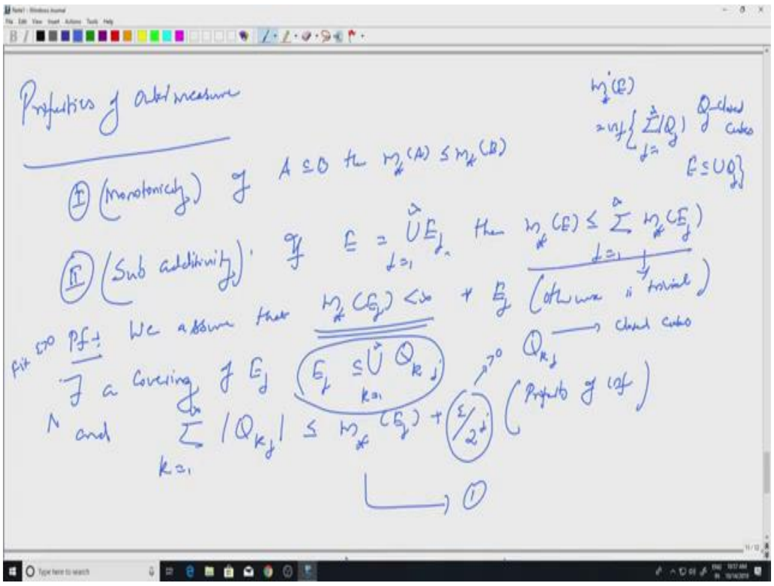

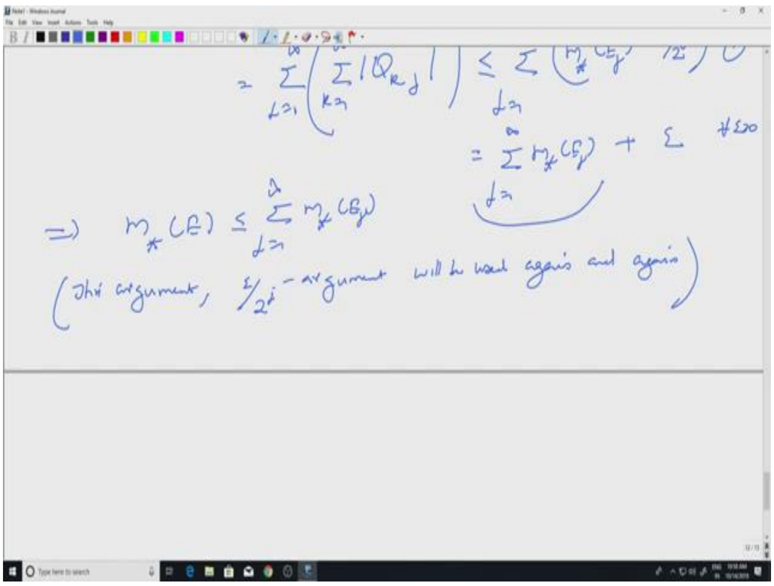

Qkj is cover Ej and Ej is cover E, so by put together I will get so now we have E content in union union Qkj, k equal to 1 to infinity, j equal to 1 to infinity because of this property and E is the simply the union of Ej.

So, hence m star of E remember m star of E is the infimum, m start of E is the infimum of certain quantities and this is a cover of E by closed cubes. So, by sum of mod Qkj's that would be a number in the set where we take the infimum of. So, m star of E will be less than or equal to summation Qkj, k equal to 1 to infinity, j equal to 1 to infinity. But, this is less than or equal to so let me write a one more time so that this is clear, so this is nothing but summation j equal to 1 to infinity, summation k equal to 1 to infinity, mod Qkj, but, this quantity is less than or equal to, so we chose those covers with some property right.

So, this is results with j equal to 1 to infinity, m star of Ej plus epsilon by 2 to the j. so, that is how remember this is we chose it, one so this is by one which is simply summation j equal to 1 to infinity, m star of Ej plus epsilon. So, we have m star of E less than or equal to m star of summation of m star of Ej plus epsilon for every epsilon positive. So, I can let epsilon go to 0 so to get m star of E less than or equal to summation j equal to 1 to infinity m star of Ej which is the subadditivity property.

So, if E is the union of countably many sets we will have this subadditivity property. So, this argument that something we will use again and again, so let us let me give it a name so this argument so will call it, the epsilon by 2 to the j argument will be used again and again so, this is something you should understand very clearly. So, we have two properties, Monotonicity and we have a Subadditivity.

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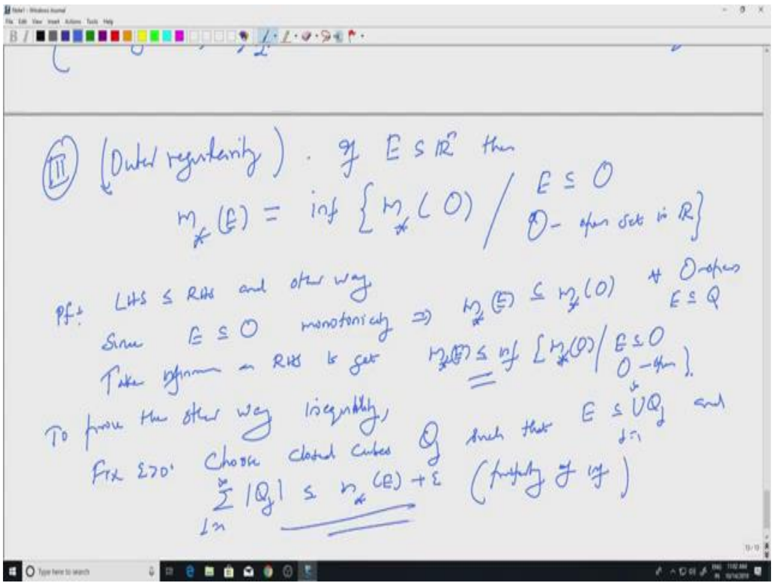

Next one another important property third property, this is called Outer regularity. What does that say? If E is a subset of Rn, then m star of E is equal to infimum of well it is not the definition, it is going to be m star of O, what is O? O is an open set, E is content in O, open set in Rn.

So, you look at all the open sets containing E, look at m star of all those open sets, take the infimum that is what m star of E is. So, remember that this is very surprising but at the same time it is an extremely important property. How you will prove this? So as usual we want to prove two things are equal, so we use the inequalities so will show that, LHS is less than or equal to RHS and RHS is less than or equal to. So, will show that LHS is less than or equal to RHS and vice versa and the other way, so then both of them will be equal.

So, as usual one of the inequalities will be easy so since E is content in O, monotonicity will tell me, monotonicity implies that m star of E is content in m star of O, O open set. But, this is true for every open set containing E, so I can take infimum on the right hand side so, take infimum on the right hand side to get m star of E is less than or equal to infimum of m star of O, such that E is content in O and O open, this is what we want. So, we have one-way inequality, we want to prove the other way. So, to prove the other way, other way inequality choose so you will see the epsilon by 2 to the j argument coming again and again.

So, let us start with fixing epsilon etc. etc. so fix epsilon positive. Choose closed cubes Qj such that E is content in union Qj, j equal one to infinity of course that is possible and summation mod Qj, j equal to 1 to infinity is less than or equal to m star of E plus epsilon. This is possible this is simply the property of infimum. That is the first thing I mentioned after defining the outer measure, property of infimum. So given epsilon I can get a cover with this property satisfied.

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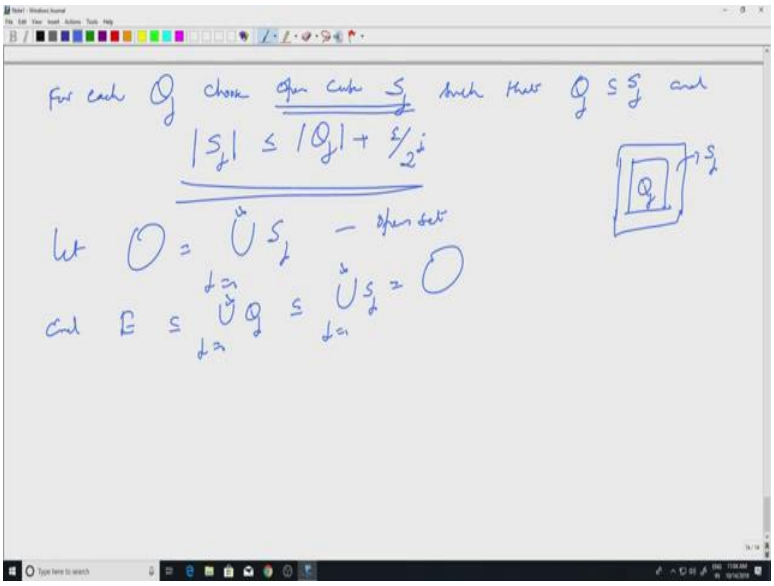

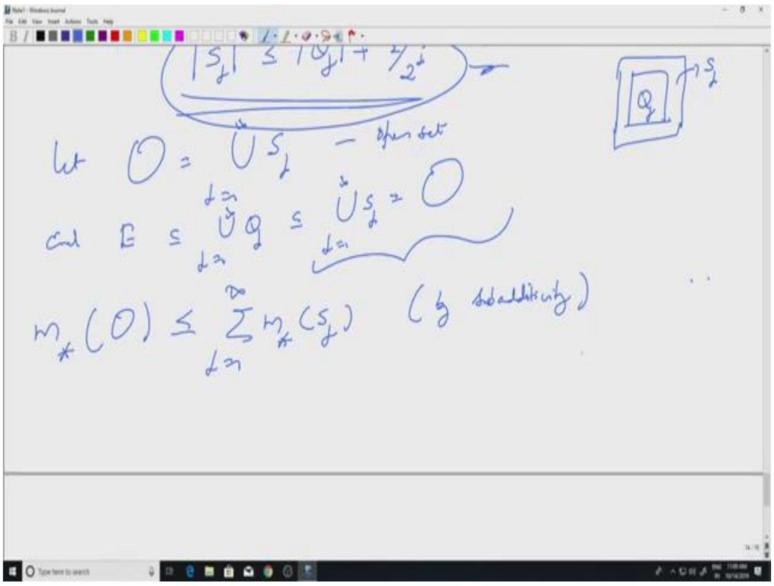

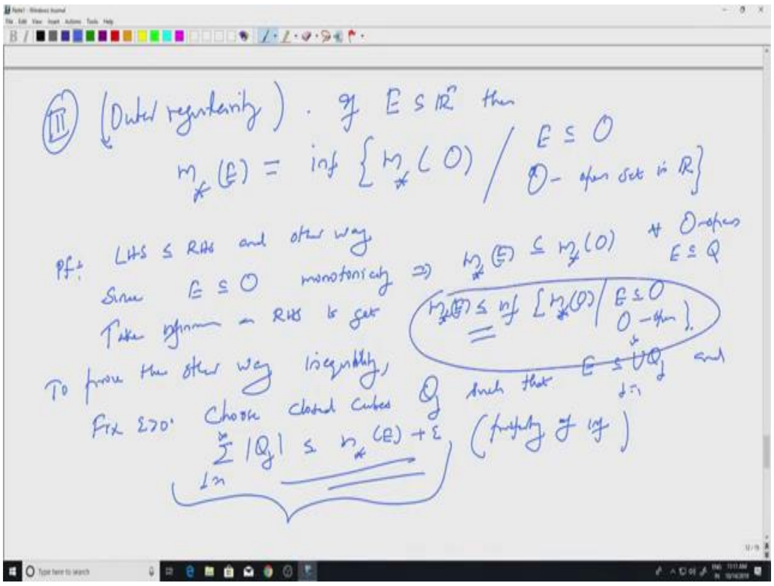

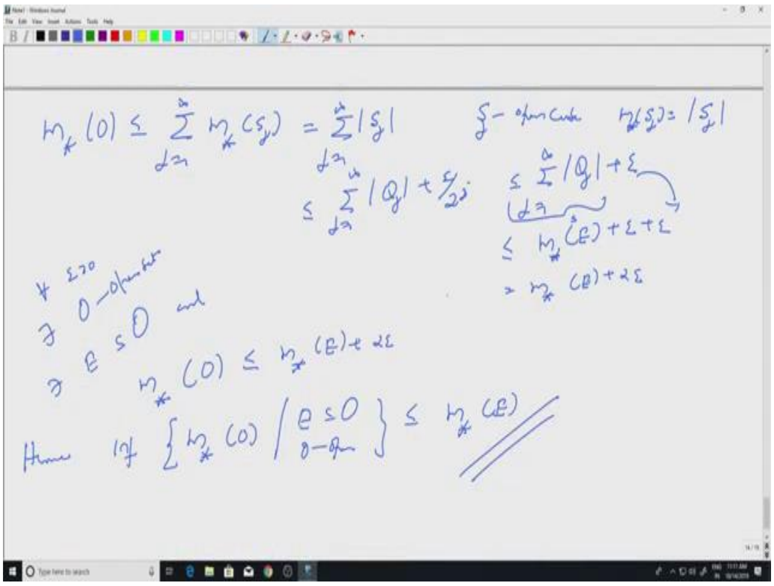

What do we do with this closed cubes? So for each Qj choose an open cube, open cube Sj such that Qj is content in Sj well that is easy.

But, we have the epsilon estimate there so we need to do appropriate, we need to choose appropriate Sj such that the volume of Sj is very close to the volume of Qj. So, mod Qj plus epsilon by 2 to the j, so remember epsilon by 2 to the j is a positive number so all I have to do is to slightly enlarge Qj. If this is my Qj I choose my Sj very closed to it, so that so this is Qj and this would be Sj. Only thing is this inequality should be true so, you enlarge Qj slightly, this is possible, but now these are open cubes so I have an opened set, so let O be equal to union Sj, j equal to 1 to infinity, this is an open set because the union of open cubes and so it is an open set.

And the set E is content in well I know it is content in union Qj, j equal to 1 to infinity but surely content in union Sj, now j equal to 1 to infinity which is my open set O. So, what I have done is getting an open set O which is slightly bigger than the union Qj, so O is the union of things, union of Sj. So, m star of O will be less than or equal to summation j equal to 1 to infinity, m star of Sj this is by subadditivity. So, we have m star of O less than or equal to summation of J equal to 1 to infinity, m star of Sj which is equal to summation j equal to 1 to infinity, mod Sj because Sj is a open cube.

So, m star of Sj will be the volume of Sj which is less than or equal to because the way as j are chosen we have j equal to 1 to infinity, mod Qj plus epsilon by 2 to the j which is less than to summation, j equal to 1 to infinity mod Qj which is less than or equal to m star of E plus epsilon. So, this portion is less than or equal to m star of E plus epsilon and there is another epsilon. So, this is nothing but m star of E plus 2 epsilon, the 2 epsilon does not matter because epsilon is arbitrary epsilon.

So, what we have done is for every epsilon positive there exist O open set such that E is content in O and m star of O is less than or equal to m star of E plus 2 epsilon. This is true for every open set for every epsilon, hence I can take the infimum on the left hand side and let epsilon go to 0. So, infimum of all m star of O where E is content in O, O open this is less than or equal to m star of E, so we have the other way inequality. So, this is one inequality and the first one we started with that is the trivial part this portion.

So, we have the both way inequalities and so we have proved outer regularity, so will stop with one more property.

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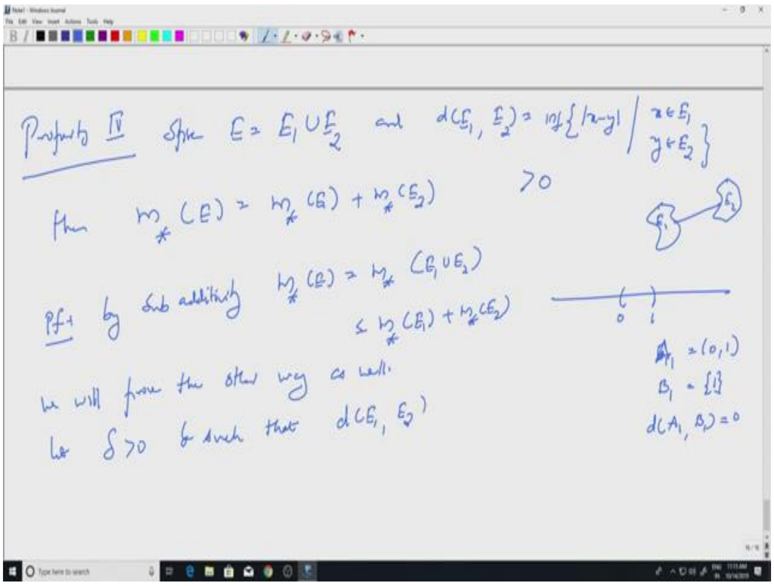

So, property 4 this is some kind of additivity but with some strong conditions. Suppose E is a union of two things, E1 union E2 two sets and distance between E1 and E2, so what is distance between E1 and E2? So, this is the infimum of modulus of x minus y where x belongs to E1 and y belongs to E2 so, that is the distance between the two sets E1 and E2. Suppose this is greater than 0, so they are not just disjoint they are at a distance so this is could be E1 and E2 could be somewhere here. There is a distance between them, this is positive.

Then we have additivity for the outer measure then m star of E equal to m star of E1 plus m star of E2, we want countably additivity so this is a much weaker property that if I have E1 and E2 at a distance then the outer measure adds up. So, let us prove this so E1 and E2 are disjoint but disjoint does not mean that the distance between them is greater than 0. For example, I may have an open interval like (0, 1), E1 could be or let us say A1, A1 is open interval (0, 1) and B1 could be the singleton 1.

Then the distance between A1 and B1 is actually equal to 0 because 1 is in the boundary of A1. So disjoint does not mean distance is positive but, distance positive would surely imply that they are disjoint. So, first of all by subadditivity, we have m star of E is equal to m star of E1 union E2 by subadditivity this is less than or equal to the outer measure of each component. We proved it for infinite 1 but that is the same one for finite unions. So, we have one-way inequality, so we will prove the other way, we will prove the other way inequality as well, so let us prove that.

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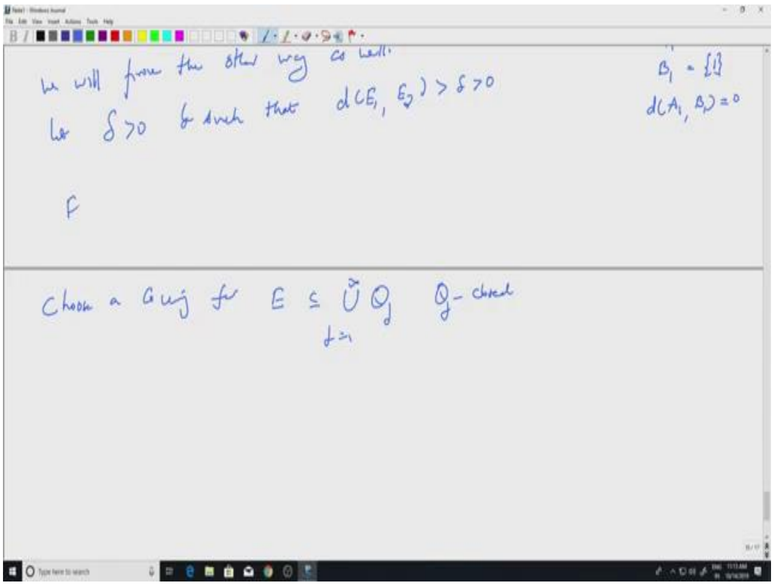

So, let delta positive be such that, the distance between E1 and E2 is greater than delta. So, choose a covering for E, so E is content in union Qj, j equal to 1 to infinity, Qj closed cubes. Then we summed up mod Qj then take the infimum to get the m star of E, such that so again usually there is an epsilon so first fix an epsilon. Fix epsilon positive, choose a covering for E such that, E is content in thus and summation mod Qj, j equal to 1 to infinity is less than or equal to m star of E plus epsilon. So, this we have seen this several times, now this is simply the property of infimum.

So, now if possible so let us perhaps draw picture so that it is sort of clear so I have E1 here, I have E2 here and E is the union of these two. So, I am covering E with cubes like this and so on, but some cubes were intersect both of them, it can be bigger cubes like this. So, what you do is subdivide such ones, so subdivide each Qj if necessary, so that diameter of Qj is less than delta. Then so what is the point recall that E1 and E2 are at a distance delta so I have E1 here, I have E2 here and the distance between them is delta.

So, if I take a cube of diameter less than delta then if intersects E1 it cannot intersect E2, so each Qj then will intersect only one of E1 or E2. Only one of the Ej is it can intersect, it cannot intersect both of them because, the distance between them is delta so if the cube intersects both of them the diameter of the cube will be greater than or equal to delta but, we have chosen, we have subdivided Qj so that it is less than that. So, let us write J1 to be although in this is J, such that Qj intersects E1 and J2 to be although intersects all those in this is K such that Qk intersects E2.

So, J1 and J2 are disjoint because if there is a common index then that particular Qj will have to intersect both E1 and E2 which is not possible.

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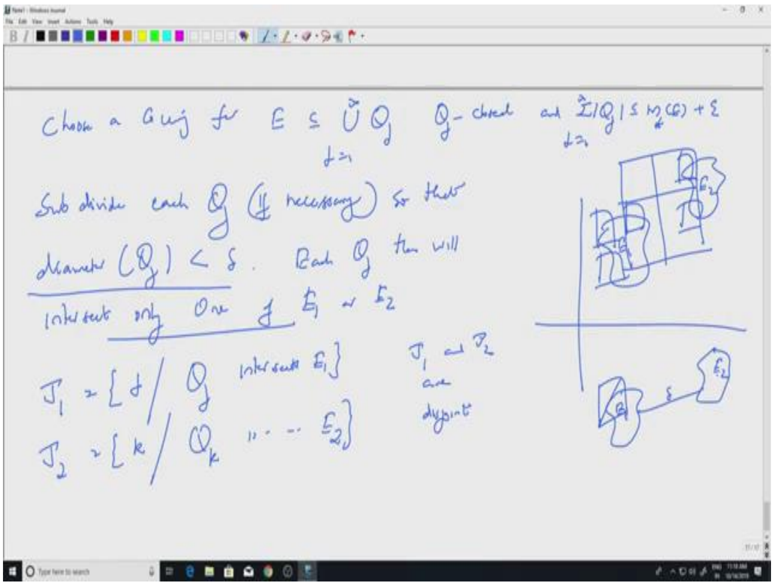

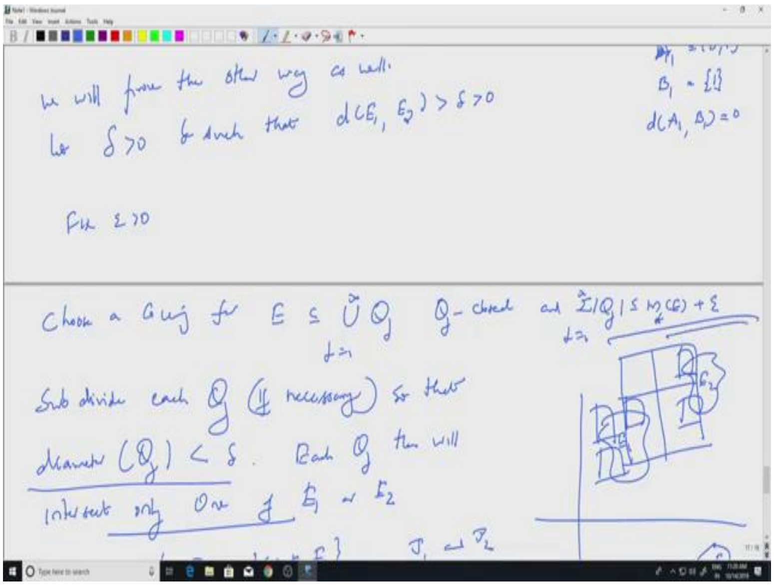

Write $E = E_1 \cup E_2$ $\qquad$ $E_1 \subseteq \bigcup_{j \in J_1} Q_j$ $\qquad$ $E_2 \subseteq \bigcup_{j \in J_2} Q_j$

→ a cover of E_1

Hence $m_*(E_1) + m_*(E_2)$

$$\le \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \le \sum_{j=1}^{\infty} |Q_j|$$

$$\le m_*(E) + \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad m_*(E_1) + m_*(E_2) \le m_*(E)$$

Property IV Suppose $E = E_1 \cup E_2$ and $d(E_1, E_2) = \inf\{|x-y| \mid x \in E_1,\ y \in E_2\} > 0$

Then $m_*(E) = m_*(E_1) + m_*(E_2)$

Pf: by sub additivity $m_*(E) = m_*(E_1 \cup E_2)$
$\le m_*(E_1) + m_*(E_2)$

$A_1 = (0,1)$
$B_1 = \{1\}$
$d(A_1, B_1) = 0$

We will prove the other way as well.

Let $\delta > 0$ be such that $d(E_1, E_2) > \delta > 0$

Fix $\varepsilon > 0$

Choose a cover for $E \subseteq \bigcup_{j=1}^{\infty} Q_j$, Q_j - closed and $\sum |Q_j| \le m_*(E) + \varepsilon$

Hence $m_*(E_1) + m_*(E_2)$

$$\le \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \le \sum_{j=1}^{\infty} |Q_j|$$

$$\le m_*(E) + \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad m_*(E_1) + m_*(E_2) \le m_*(E)$$

Hence

So, I can write E, I will remember this is E1 union E2 to be said E1 is content in union Qj, j in J1 and E2 is content in union Qj, j in J2 because I put together I will cover E, the ones which intersect E1 will cover E1 and the ones which intersect E2 will cover E2, they are disjoint, they are far apart.

So, because of this hence m star of E1 plus m star of E2 is less than or equal to so look at this, this is one covering of E1 so this is one covering of E1. Definition of m star of E1, what do you do? You look at summation mod Qj where j belongs to J1 and then the infimum. So, m star of E1 will be less than or equal to summation j in J1 mod Qj and similarly, m star of E2 will be less than or equal to summation j in J2 mod Qj, but, J1, J2 are disjoint, I can put together so this is less than or equal to summation j equal to 1 to infinity mod Qj, all the Qj's.

But, the Qj's were chosen so that they were at an epsilon distance from m star of E so this is less than or equal to m star of E plus epsilon. Remember that is how the Qj is were chosen so we had chosen Qj is to be with property. So, what did we prove, we proved that m star of E1 plus m star of E2 is less than or equal to m star of E plus epsilon. This is true for every epsilon velocity. So, this will imply that m star of E1 plus m star of E2 is less than or equal to m star of E, so that is one inequality. The other way was easy because of subadditivity. So, by subadditivity we have the other way inequality so this tells me that, they are equal, hence they are equal.

So will stop with this, hence we have m star of E1 union E2 equal to m star of E1 plus m star of E2. So, remember the strong condition we have provided the distance between E1 and E2 is greater than delta or positive so that is enough it is positive, that is an extra condition. So, we will stop with this so what we have seen in this lectures is the definition of outer measure and it is defined for all subsets of Rn. And we saw some properties, monotonicity, subadditivity and outer regularity along with additivity property is the sets are disjoint and at a distance greater than 0.

That is very weaker than countably additivity, so in the next lectures we will restrict the outer measure to a sigma algebra called the Lebesgue sigma algebra and will prove that, it is actually countably additive in that sigma algebra. So, that will complete the construction of the Lebesgue measure on R.