Measure Theory

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Lecture 12

Rectangles in Rn And Some Properties

So, we have seen abstract integration so far, we had a space, sigma algebra and the measure and we know how to integrate with respect to that measure and we saw three major theorems. So, that is the end of abstract integration theory. The only example of a measure strictly speaking, we know is the counting measure.

We can of course, change the measure of each singleton and then try to get a slightly more general measure, but other than that, we have not seen any concrete example of measures. So, from now onwards, our aim will be to construct, the so called Lebesgue measure on Rn which will be a concrete example of a genuine measure which will generalize the concepts which you already know like, the length of the interval area of the rectangle and volume of cubes and things like that in R3. But we will do this in general on Rn and it will have the usual properties you want. S

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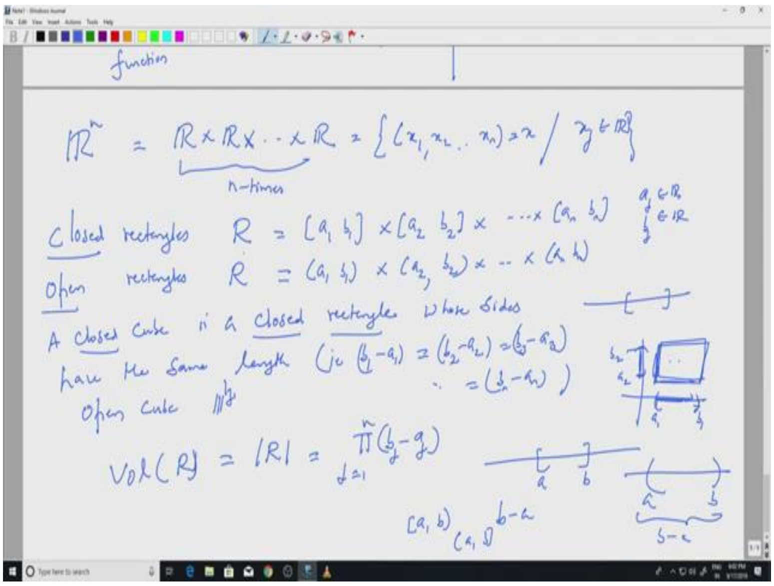

So let us start, so we start with so let us look at Rn. So Rn remember is the cartesian product of R the real line with itself n times, n times and we denote this by tuples xn. So, any point x would be an n tuple of real numbers. So, some easy definitions, closed rectangles so, we will denote rectangles by R, so, this R is different from the real line R. This is simply product of closed intervals.

So, in dimension one, a closed rectangle is a closed interval. So, a2 b2 cross etc, etc times an bn, where aj and bj are real numbers. So, in the real line this would be closed interval in R2 this would be a box like this, you take a 1 here b1 here and a2 here b2 here and so on, these are basic object so closed rectangles.

When I say open rectangles, we will still use R for that but it will be specified instead of taking close intervals, we will take open intervals etc, etc times an bn.. So, we will be looking at this and this, so the product will not have the outside line, so only the interior part.

A cube a closed cube is a closed rectangle whose sides have the same length, well what does that mean? That is the length are b1 minus a1, that is one one length and same as this length. So, that is b2 minus a2, of course in other dimensions you have more coordinates. So, all the sides should have length same, then we say it is a cube, and it is a closed cube, if we are looking at a closed rectangle with same sides, open cube similarly, it would be an open set that is the only difference, but the sides are same.

So, in R2 a cube would be a square because both sides are same, and volume of a rectangle closed or open this is denoted by modulus R, this is the product of this length of the sides, this is what you would want even if it is so, let us look at the dimension one case. Suppose, I have a closed interval, what is the length of it? The length of this is b minus a.

If I have an open interval ab, even then the length is the same, because the one point does not add to the length, the length of this has also b minus a, the length of an interval like this, that is also b minus a, length of open a at closed b also b minus a. So, any such interval has length b minus a. And when I take volume of a rectangle, whether it is closed, open or maybe closed at some point, some part and open it the other part and etc, you will still get the same volume, it is not going to change. So, it is simply the product of length of the sides, that is the definition. So that is the basic object we will be looking at.

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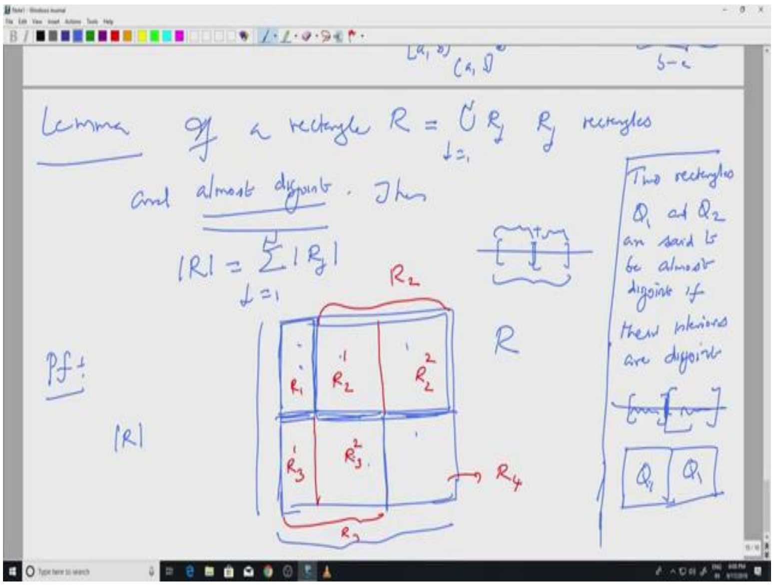

And we will use that to define length of various other sets. So, some simple lemmas, if a rectangle R can be written as a finite union, j equal to 1 to n finite union, where Rj are also rectangles and almost disjoint, what does that mean? Two rectangles let us say Q1 and Q2 are said to be almost disjoint if their interiors are disjoint.

So, in the real line, if I take two sets like this, they are almost disjoint because their interiors are disjoint they will intersect only at the boundary. So, some set like this, and another set like this, these are these two are in almost disjoint, but sets of this form are not because there is an intersection like this, it is not the boundary.

So, that is the concept of almost disjoint union, then the volume of the rectangle is actually the sum of the volume of each of them. Well, that is clear if you have if you have two, two close intervals like this. If I take the union the length is this which is the sum of length of each of them, that is all is saying but this is true in all dimensions.

The proof of this is rather simple, so I will give a sketch of it. So, let us say we have a big rectangle R like this, this is R and it is the disjoint almost disjoint union of finitely many other rectangles. So, let us say something like this, so I will draw one or two. So I have R1 here, I have R2 here, I have R3 here, and I have R4 here. So I have four rectangles, so each of them I can look at the closed rectangles like this.

So, they intersect only at the boundary, so, they are almost disjoint. And from this picture, it is clear that the area of this big rectangle is same as the sum of the area of the smaller rectangles, but you know we are trying to justify that in all dimensions. So, what do we do?

Well, it is what you do is simply, simply extend the sides so that you have a grid, so from here, you extend this like this, this you extend like this.

So now, you have more rectangles, so maybe I should write this as R1, this is R2 so the whole thing is R2, this whole thing was R3, and this was R4. So now, we have more rectangles R1 is the same, R2 is union of two almost disjoint rectangles, R4 is the same, R3 is also union of two disjoint rectangles almost disjoint rectangles.

But what is the advantage now? The volume of R, volume of R is the length times the breadth, that is the area, that now you can say is actually the sum of these smaller rectangles. That is because the sides add up, the side of this plus this plus this will be the same as will be times the sides added up on the left side that will give me the area of the bigger rectangle.

And you apply the same logic for each of these rectangles. So, you look at this rectangle for example, you will see that the area of the, this bigger rectangle is the sum of these two because again the sides add up, sides adding up is what helps here.

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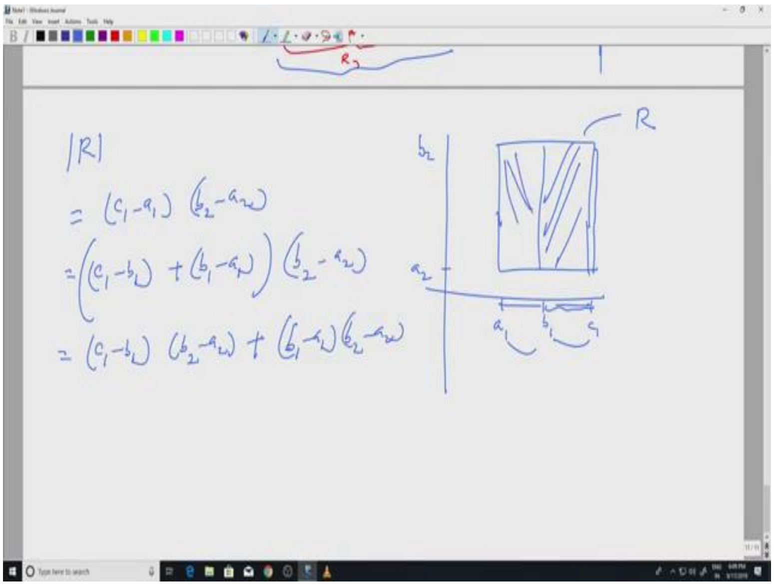

So, let me let me elaborate on that a little bit. So, if I have R2 like this and I have a bigger, bigger rectangle like this, let us say, it is bifurcated into two so I have let us say, a1 here, I have b1 here, and c1 here. Here, a2 and b2, the area of the bigger rectangle, so bigger rectangle is called R. So area of the bigger rectangle is nothing but this is c1 minus a1 that is the length times the breadth, so that is b2 minus a2. But c1 minus a1 is c1 minus b1 plus b1 minus a1.

So this is what I meant by sides add up, times b2 minus a2, and this I can distribute, so this I will get c1 minus b1 into b2 minus a2, which is so c1 minus b1 times b2 minus c2. So, that gives me this area plus b1 minus a1 into b2 minus a2, so that gives me this area. So, that is the area of the bigger rectangle, so whenever sides add up, you can do this.

So that is what is done by extending the sides, so this portion and this portion essentially allows us to make a grid where all the sides will add up and so you can you can write the volume of the bigger rectangle as sum of the volume of smaller rectangles and sum of these smaller rectangles add up to the original rectangles we started with and so they are equal.

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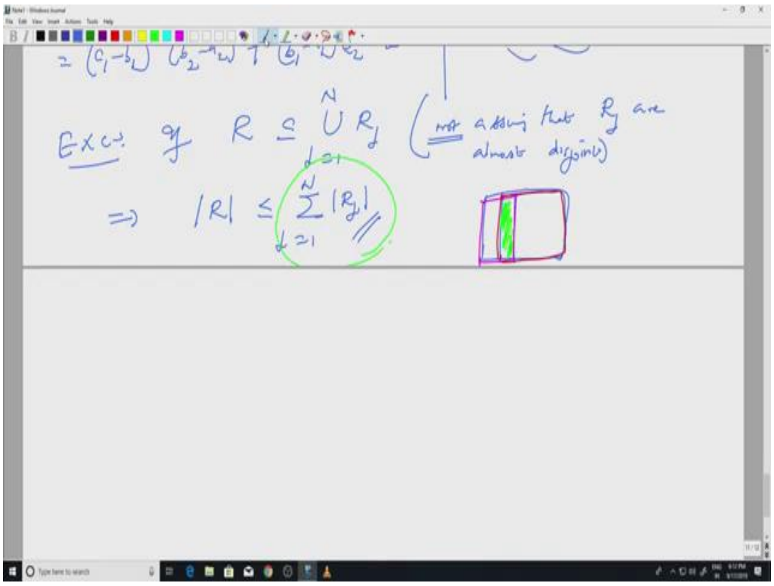

So, it is it geometrically is very very clear, but in since we have different dimensions, we need to prove this. So, let me let me let me give a simple exercise here, so if R is contained in union Rj, instead of being equal, I am not assuming them to be not assuming that Rj are almost disjoint, so not assuming this implies that the volume of the rectangle R is less than or equal to sum of this model.

So, you can use the same proof, then difference that when you when you make a grid like this, the rectangles you get need not be disjoint almost disjoint like this, they may be overlapping. So, when you add you will be getting bigger quantities so what I mean is you may have something like this, and then you may have something like this. So, let me use a different color, so, I may have one rectangle like this and I may have another rectangle like this.

So, the area in between, area in between is added twice, so you will get a bigger quantity here because of this, that is an easy exercise.

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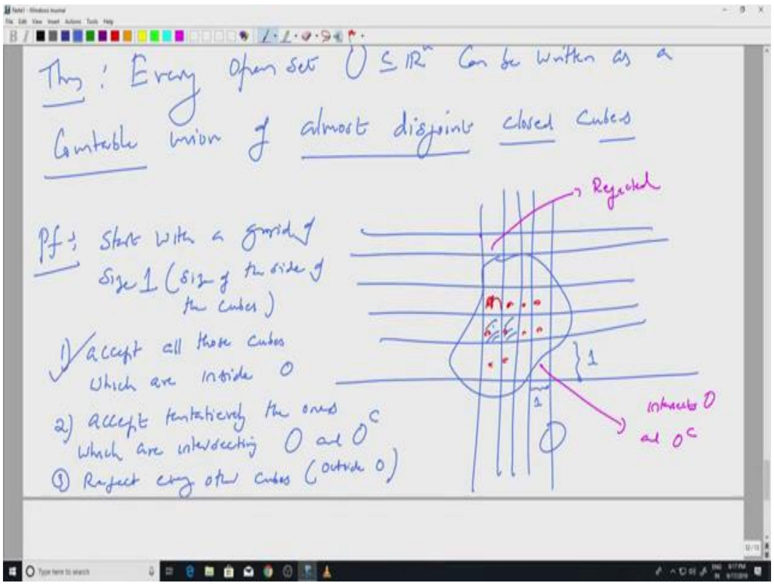

So, let us state another theorem again, there is no measure theory here this is more topological result, but this is something which we will use. So, every open set O in Rn can be written as a countable union countable is very important countable union of almost disjoint, almost disjoint closed cubes. So, let us proof this so what you do is, again one needs to draw something to get an idea about the proof. So let us let us draw this. So let us say this is my open set O.

So let me draw the slightly smaller,, this is my open set O, what do I do? I draw a grid, let say this is my axis. So I draw a grid of length one or size one like this. So, each side is of length 1, that is how I start. So, well in the picture it may not look like that, but these are all squares.

Now, you select cubes which are entirely inside O, so you look at cubes which are entirely inside so this one this one this one this one, this one this one this one and so on, which are entirely inside. So, this is this is not entirely inside so, let us. So, that is one selection, so prove first you start with the start with grid of size 1.

So, this is the size of the side size of this side of the cubes. (())(19:19) in the higher dimensions these are cubes, in R2 you see squares, . Then you do three things, one accept all those cubes which are inside O, so these are almost disjoint union, so, these two this cube and this cube are almost disjoint. So, whatever we have accepted which are entirely inside O are almost disjoint to each other, but it does not give you all of the set O we have to still choose.

Accept tentatively the ones which are intersecting O and O compliment. So, there would be cubes which intersect both O and O compliment, for example, this one, it intersects O and O compliment, you choose that as well, reject everyone else, reject every other cubes.

So, these are outside, outside O, so this for example something which is coming here can be completely rejected, this one this is rejected. So, we have accepted those cubes which are entirely inside O, and accepted tentatively, so only for the time being the ones which are intersecting O and O compliment.

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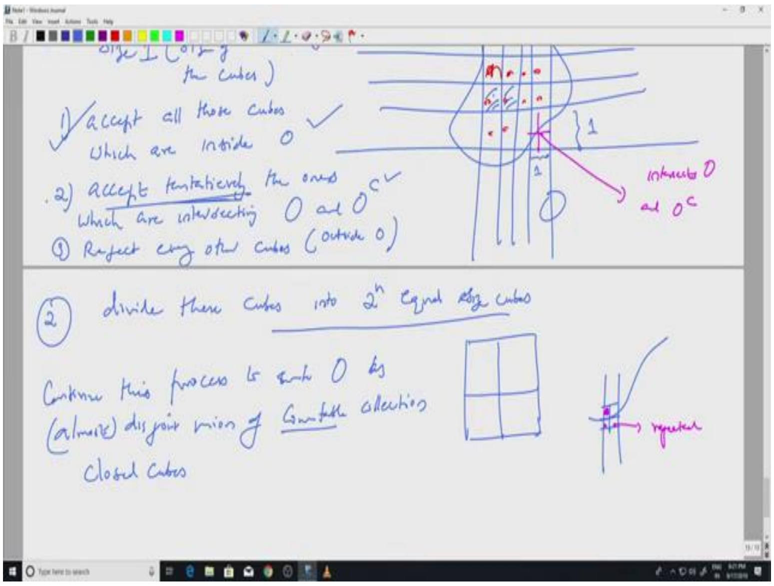

Now we continue this by bifurcating this into smaller size grid. So the ones so the second collection, you can divide these cubes into 2 to the n equal size cubes, so maybe I did not write it properly what I mean is, so if I if I have one square like this, I can divide them into four equal parts.

So, this is in R2, in R3 there would be eight such and in Rn, there will be 2 to the n equal size cubes. So, this happens to the ones which are intersecting both O and O complement. So, this one for example, I can I will divide this further into four equal ones, and then see which ones are entirely inside O, and intersecting both O and O compliment, and continue the process.

So, in each step, you started with cubes of size 1, you accepted some cubes. So, that is the first step which are inside O, tentatively accepted ones which are intersecting O and O compliment, reject ones which are outside, the ones which intersect both O and O compliment you again divided into 2 to the n equals size cubes, and then continue the process which means accept the ones which are completely inside O reject the ones which are completely outside O.

Tentatively accept which intersect O and O compliment, you continue this. So, you can see what is happening. So whenever you are close to the boundary of O and let us say you have you have some cube like this. Then, you will be decomposing this into four equal cubes like this, and then choose whatever is completely inside. This, this and this will be rejected, and you continue this obviously you will write O as union of continue this process to write O as a disjoint union of almost disjoint union of disjoint union of countable collection of close cubes, countable is important closed cubes.

Well, why are they accountable? Because the interiors are disjoint and so, they are disjoint open sets and you cannot have more than countable, you cannot have uncountable collection of disjoint open sets in R2 because R2 is separable.

So, maybe we will, will stop here. So, we have just started with Rn. And we have looked at basic objects called the cubes for which what is meant by volume of the cube is understood, those are the building blocks from the definition of the volume for cubes and rectangles, we want to go to more general step, more general sets for like the example, open sets or open balls and so on.

For which we should be able to define a concept of volume, which should of course, coincide with whatever we know in the case of dimension 1, 2, and 3, which will follow from various properties which we will prove for whatever is known as the Lebesgue measure which we will define in the next, next two three lectures.

We will start with what is known as the outer measure that is assigning a sort of a measure to any subset of Rn, but the countable additivity property we want will not be true on the power set of Rn, we will have to restrict ourselves to what is known as the Lebesgue sigma algebra. So, that will be the content of the next few lectures.