Measure Theory
Professor E.K Narayanan
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 11
Consequences of MCT, Fatou's Lemma and DCT

So far we have seen abstract integration theory, in the last lecture we actually proved three important results. So, let me recall that, first one was the Lebesgue monotone convergence theorem, which concerned increasing sequence of positive measurable functions. The second one was an inequality called the Fatou's lemma again concerning non-negative measurable function.

Third one was the dominated convergence theorem, where you had a sequence of complex measurable valued functions converging to something but dominated by a function in L1, then we were able to interchange the integral and the limit. These three are extremely powerful theorems, which contributed to the growth of analysis in the beginning of 20th century.

We will see many applications of these, these theorems as we go along. Today, we will, we will start with some of the consequences of these theorems. Again, continuing with abstract integration, in the first session, we will actually finish whatever we wanted to do with abstract integration. And in the second session, we will actually start with construction of the Lebesgue measure on Rn.

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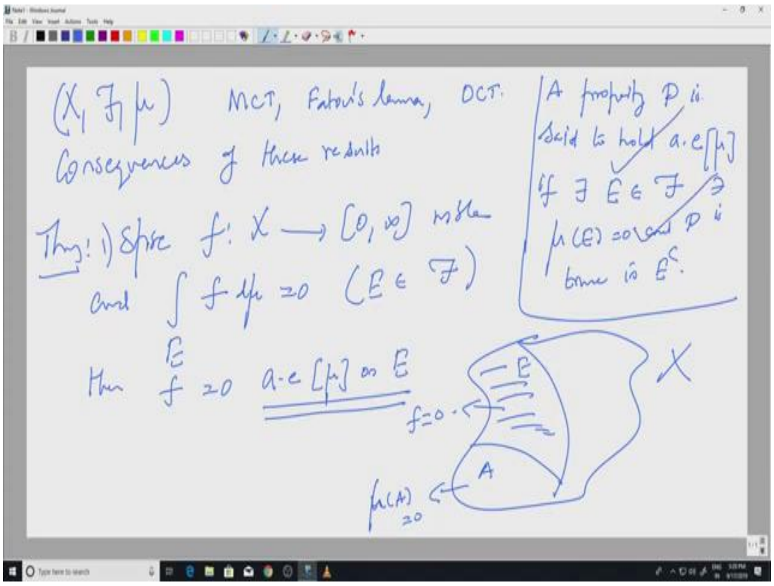

So, let us recall things we have done before. So we started with space sigma algebra and the measure as always, we had three theorems, one was the monotone convergence theorem, another one was for Fatou’s lemma and we had the dominated convergence theorem, okay. So, we will look at consequences of this consequences of these theorems of these results. And also understand the property P which holds almost everywhere.

This is what we introduced in the last lecture. So, a property let me recall that property P, said is set to hold almost everywhere is set to hold almost everywhere. So, remember we abbreviated it as a dot e. This is with respect to a measure always so, almost everywhere with respect to mu, if there exist a set E in script F such that mu of E is 0 and P is true, the property P is true in E compliment.

So, outside a set of measure 0 the property holds, then we say the property P is holding almost everywhere. So, we saw some examples of this in the last lecture, so we will continue with that. So, let me state this as a theorem suppose, f is a measurable function, non-negative measurable function and integral over E f d mu is equal to 0. So E of course will have to be a set in the sigma algebra we started with.

Then, let us call this one, then f equal to 0 almost everywhere with respect to the measure mu on E, what does that mean? That means so if this is my space X, and let us say this is E something is said to hold almost everywhere on E, what does that mean? That means there is some set like this let us call that A which has measure 0 mu of A is 0 A is sitting inside E, and outside A the property holds that means f is 0 here. So it is like considering E as the whole space and applying this definition whatever we have earlier defined.

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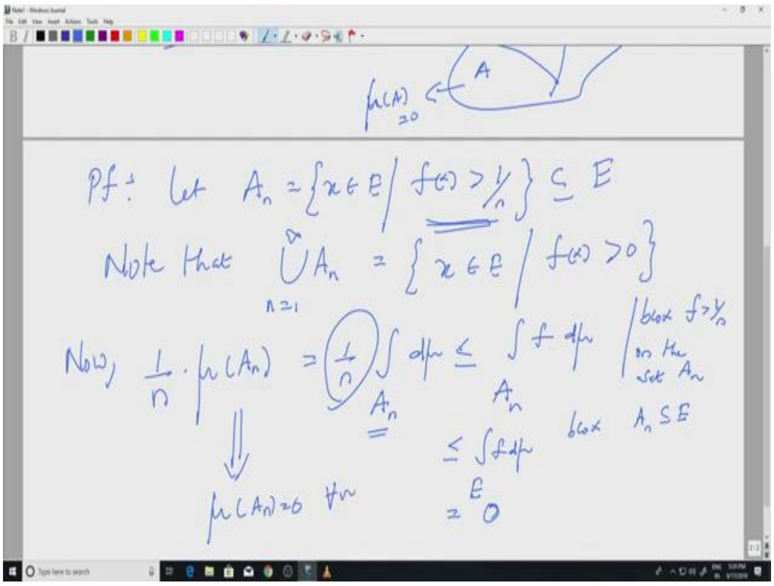

So, let us start with the proof of this. So, these are these are easy consequences of whatever we have done so far, but it is important to know that most of these things hold only almost everywhere, the sets of measure 0 are negligible, but most of the conclusions are always almost everywhere on the space we are interested in, so remember that.

So, let us try to prove this. So, let An be the set of all those points x in E such that f of x is greater than 1 by n. Well, why are we looking at such a set, we are trying to prove that f is 0 almost everywhere on E. So, if f is not 0, remember, it is a positive measurable function.

So, f will have to be greater than 1 by n for some n, so, such points will be in A. So, let us so notice that note that union of A n, n equal to 1 to infinity is precisely the set where f is strictly greater than 0. Now, start with A n, A n is the set where f of x is greater than 1 by n. So, if I look at 1 by n times mu of An.

Well, this I know is 1 by n times integral over An d mu, that is the definition of the integral if you integrate the indicator, then you will get the measure of the set, this is less than or equal to on the set A n I have f of x to be greater than 1 by n. So, I can replace this 1 by n by f of x. So, I can write integral over An f d mu because f is greater than 1 by n.

So, because f is greater than 1 by n on the set An and monotonicity of the integral, if these are all positive functions, which is of course less than or equal to integral over E f d mu, because An is a subset of E, the way we have defined, this is a subset of E but integral over E f is 0 so this is zero. So, what does that imply? Implies that 1 by n times mu An is 0, so mu of An has to be 0 for every n.

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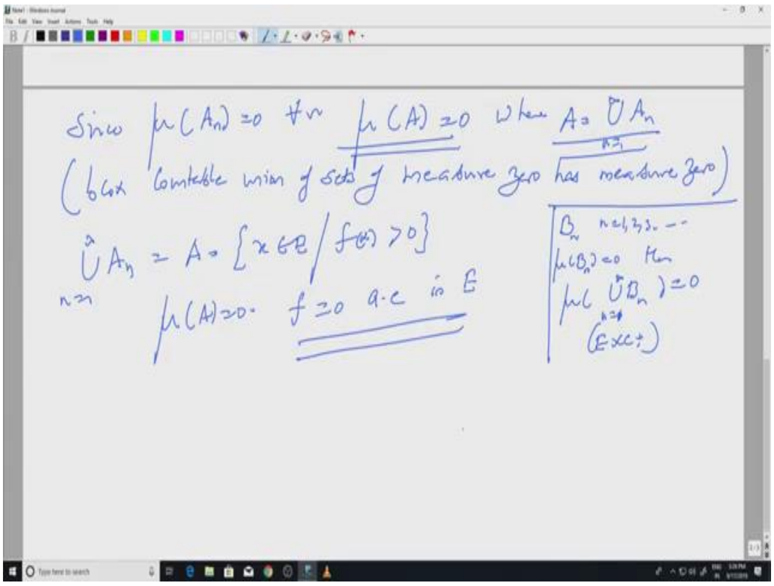

So, mu of A n is 0 for every n, mu of A equal to 0, where A is the union of An, well, why is that? Because countable union of sets of measure 0 sets of measures 0 has measure 0. We use it in the last class, but if you are not comfortable, you may prove this as an exercise this is very easy so let us give this in the corner here.

So, take Bn n equal to 1, 2, 3, etc measurable mu of Bn is equal to 0 then mu of union Bn is also equal to 0 n equal to 1 to infinity, this is also 0 exercise. This is this is pretty easy you can (())(9:34) them if you want, or you subadditivity or whatever else you have, you have seen before.

So, because of that, mu of A will be 0 where A is the union, but union of An is the set where small f is positive. So union An, n equal to 1 to infinity equal to A is precisely the set where x is where x in E such that f of x is strictly positive, and we are saying mu of A is 0. So that means, wherever f is strictly positive is 0, but f is a non negative function, which means that f is equal to 0 almost everywhere.

Of course, everything is happening in E. So E, that is E, that is the conclusion. So that is not very surprising if I have a positive or non negative function whose integral is 0, then it has to be 0. So it is like in the integral, if you if you have a non negative (())(10:43) function whose integral is 0, that means the area under the curve is 0, which is same as saying the function is actually on the x axis, which means that f is zero.

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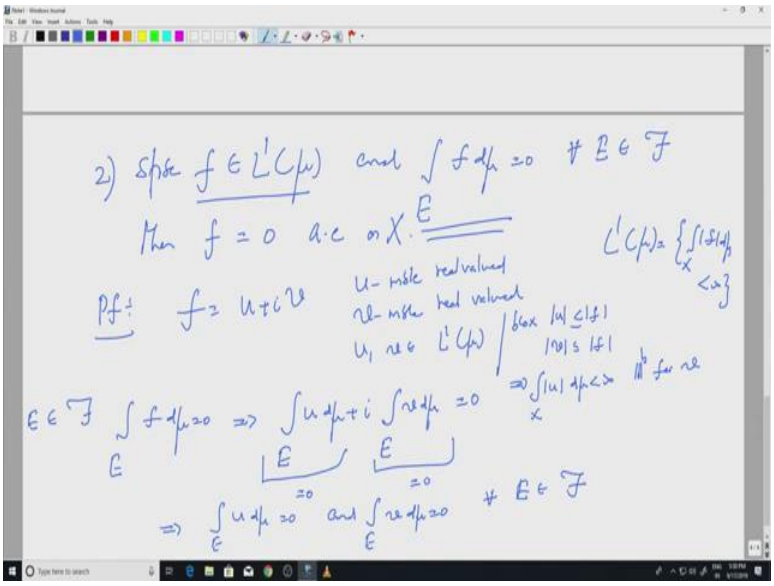

So, that is one consequence. So let us start with the next one. Second one suppose f belongs to L 1 of mu and integral over E f d mu equal to 0 for every E in script F, then f equal to 0 almost everywhere on x, of course with respect to the measure mu, mu is not changing, but f is an entirely f is fn f vanishes everywhere almost everywhere on x with respect to the measure mu, if I know that all the integrals of f vanish.

So, let us prove this, again, not too difficult and it also follows from whatever we have done in the in the last class. So, recall f is, f is in L1 mu, L1 mu is the collection of all complex valued measurable functions whose integral is finite. So, we are looking at mod f d mu to be finite this is the collection of functions we are looking at these are complex valued measurable functions.

So, I can write f as u plus iv as we have done earlier, u is measurable real valued, , v is measurable real valued, and both u and v are also in L1, why? This we have seen earlier because mod u is less than or equal to mod f mod v is also less than or equal to mod f and so the integrals of u and v would be finite, this would be finite and similarly for v.

So, they are all integrable functions also the positive part is integrable, the negative part is integrable etc. so there is no infinity minus infinity situation here. So, now let us apply the condition that integral of f is 0 over all sets E, well what does that mean? So if I take the set, E whatever set so take an arbitrary set E in a script F, sI know this is 0 by the definition of the integral, this tells me that this is integral of u plus i times integral of v d mu this is 0, that means the real part is 0 the imaginary part is 0.

So, the same conclusion holds for u and v, so what we get is integral over E d mu is 0 and integral over E v d mu is 0 for every E in script F.

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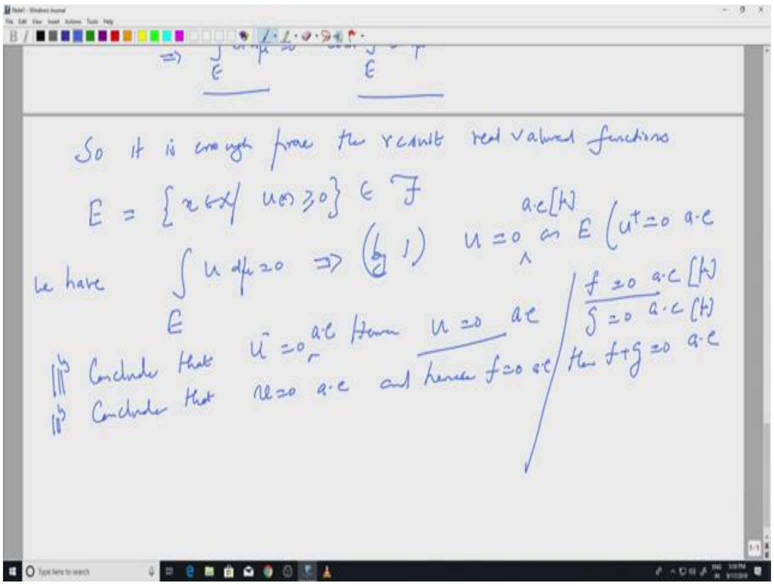

So, it is enough to so it is enough to prove the result for real valued functions real valued functions because we got both u and v having the same property and they are real valued functions.

So, if you show that this implies u and v are 0, then f is also 0 almost everywhere. So, let us see that, so let us see what happens to u. So, let us let us take the set E to be this is the set where the function u is positive, this is of course a measurable set, because u is measurable and so, by the condition we have integral over E u d mu is 0 implies by 1, the first result. So, what do we what do we prove in the beginning if I have a positive function whose if I have a positive measurable function whose integral is 0 over some set E then the function is 0 almost everywhere on E, so we use that.

So, this tells me by 1 u equal to 0 on E, this is same as saying u plus is 0 well u equal to 0 almost everywhere remember that part, this is same as saying u plus equal to 0 almost everywhere, wherever u is positive, u will be u plus wherever u is negative minus u is u minus that is a negative part of u.

So this tells me u plus is 0. Similarly, conclude that u minus 0, hence again, 0 almost everywhere, remember that, hence u is 0 almost everywhere. So, here we are using that if f is 0, almost everywhere g equal to 0 almost everywhere, then f plus g equal to 0 almost everywhere.

So, this is a simple exercise, remember there is a for f equal to 0 almost wherever we get a set whose measure is 0 and outside that f is 0. Similarly, for g we get another set, so for f plus g, you take the union of those two sets whose measure is 0, okay so that will prove then f plus g is 0 almost everywhere.

So, this tells me u is 0 similarly, conclude that v is 0 almost everywhere and hence f is 0 almost everywhere. So, if I have the end all integrals of f to be 0 over sets, then f is 0, almost everywhere. So, let us look at another consequence which is good enough to be written as a as a theorem.

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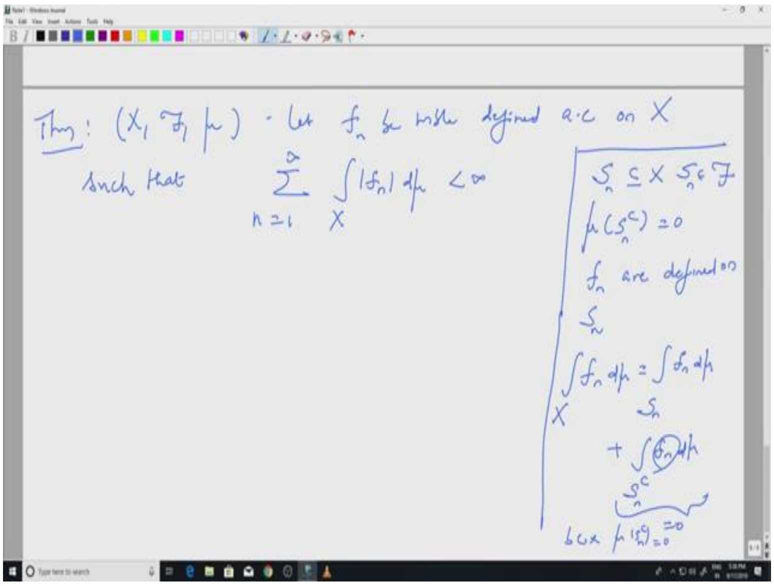

So, I have the triple x F mu let fn be measurable defined almost everywhere on x, that means for each fn there is a set Sn whose measure is whose compliment has measure 0 and fn's are defined on Sn. So, this can be complex valued defined almost everywhere on x, such that the following is true. Well, what is it? Summation n equal to 1 to infinity integral over x, mod fn d mu is finite.

So, when I write integral over x you should notice that fn's are only defined on some set. So, let us say there is Sn contained in x Sn are measurable and mu of Sn complement is 0 and fn are defined on Sn. As n changes, the set changes, the fn's are defined on some set Sn.

Then, what is integral over x fn d mu, because whatever you define on Sn compliment, it does not actually contribute anything. So, I can write this plus integral over Sn compliment fn d mu, but this is 0 because the measure of Sn is Sn compliment is 0. So it does not matter what is fn on Sn compliment, as far as the integral is concern. So you take it to be 0 if you want or some constant if you want,s, so that it is a measurable function.

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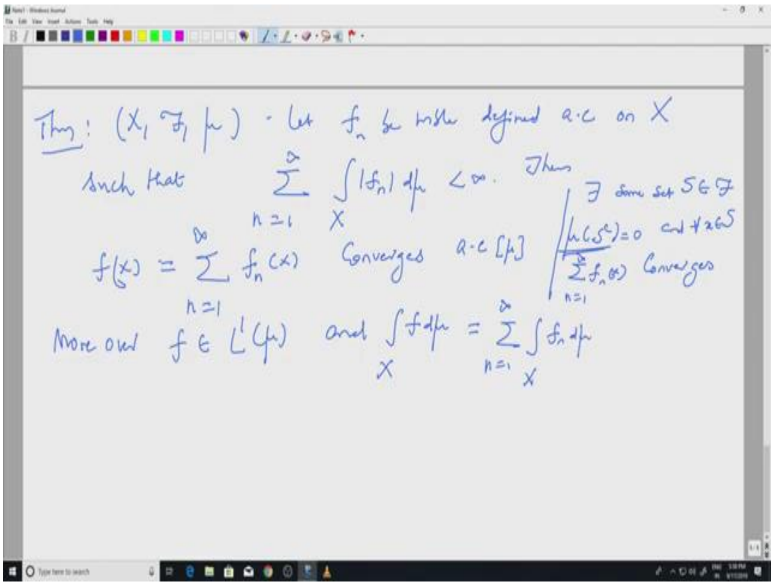

So we will come back to this when we look at the proof. So, assume that fn's are defined almost everywhere on x and we have this inequality that summation n equal to 1 to n, integral over x mod fn d mu is finite, then so conclusion, f of x equal to summation n equal to 1 to infinity fn x.

So, for each x fn x is a complex number, and I am adding them, this converges almost everywhere, what does that mean? That means, there exists some set let us say S in script F with mu of S compliment 0 and for every x in S summation fn x convergences.

So, convergence takes place on a set whose complement has measure 0, in other words, there is a set whose measure is 0 such that outside that set convergence takes place. Well, it converges and not just that, so moreover, the function f belongs to L1 of mu, that means if it is measurable of course you are adding measurable functions.

So, you will get a measurable function, it actually belongs to L1 of mu that means the integral of f mod f is finite and whatever you expect to happen will happen. So, integral of f d mu remember f is simply the sum of fn. So, you want it to be the sum of the integrals. So, this is sum n equal to 1 to infinity integral over x fn d mu this is true.

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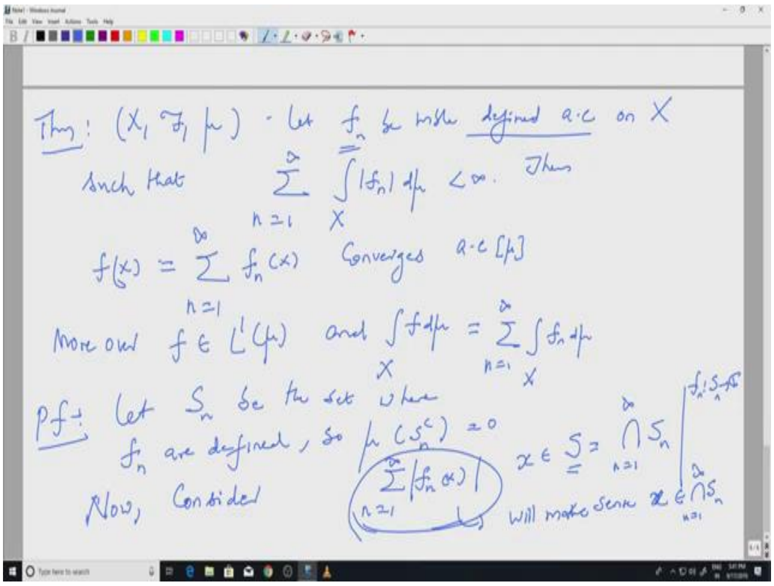

So, that is the full statement, let me read it again. You have a sequence of measurable functions complex valued measurable function defined almost everywhere on x, such that the summation n equal to 1 to infinity integral of mod fn’s converge then, the sum of fn they themselves converge almost everywhere of course, and you have the limiting function f to be an L1 function whose integral is the sum of the integral of fn.

So, let us prove this, let Sn be the set where fn are defined. So, remember fn are defined almost everywhere which means measure of wherever fn’s are not defined, that is Sn compliment, that will have measure 0. Now consider, f of x equal to summation fn x, sorry so let us before we go to f of x, consider the modulus of fn x n equal to 1 to infinity, of course this can be infinite you are adding positive numbers, so it is either finite or infinite, but consider this for x belonging to S, what is S? S is the intersection of Sn n equal to 1 to infinity, why am I taking the intersection? Recall that fn is defined on Sn.

So, if I want to add all these, so, this will make sense for x in all Sn, if x is in Sn for every and, f n of x makes sense for every and, and so I can add them I may get infinity but I will I this quantity makes sense this I can, u as a function defined on S.

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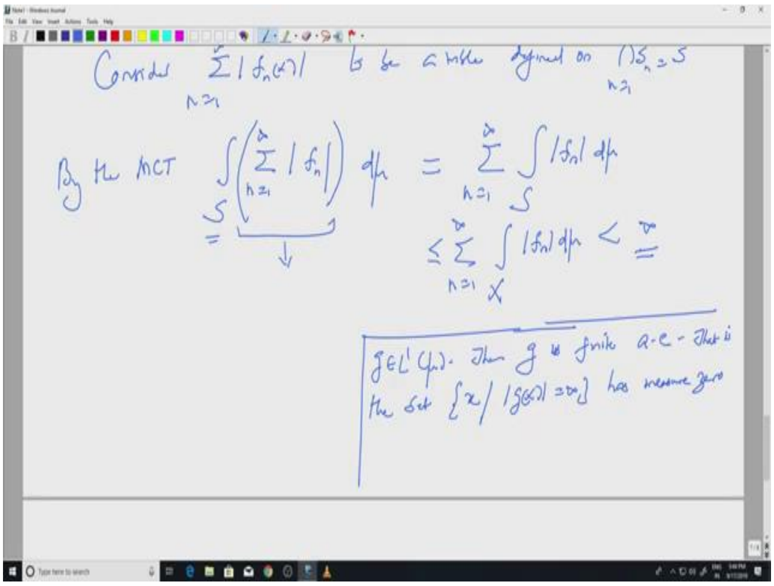

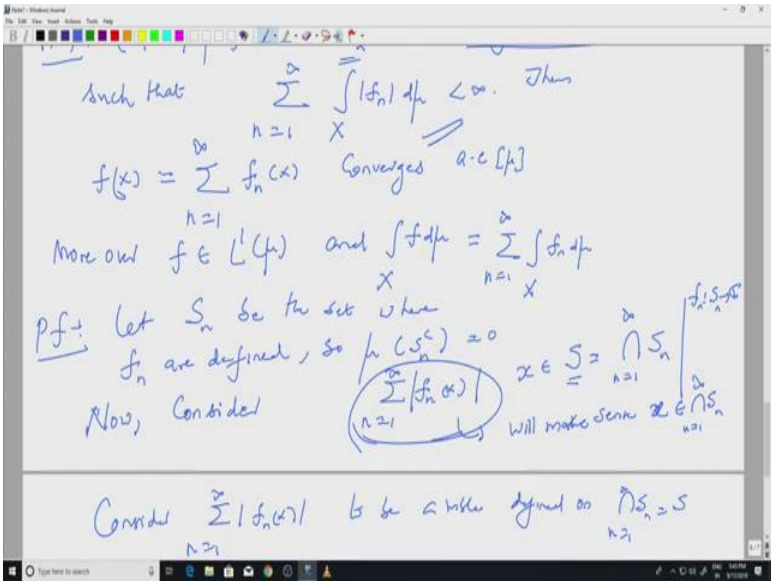

So, consider summation mod fn x, n equal to 1 to infinity to be a measurable function of or adding measurable function, so it does not matter, defined on intersection Sn, which is what we called S. Now, by the monotone convergence theorem, this we have seen earlier, if I integrate over S, where everything is well defined, sum of positive functions, I am adding mod fn's d mu I know that this summation and the integral can be interchanged by MCT because I am adding positive function.

So this is same as n equal to 1 to infinity integral over S mod fn d mu, which of course, is less than or equal to summation n equal to 1 to infinity integral over x mod fn d mu by our assumption this is finite. So, let us recall that so our assumption here says that if I add the integrals of fn over x, I am going to get a finite quantity..

So, if I look at this function summation mod fn as a function that has integral over S finite because of this. So, this will have to be finite almost everywhere. Why is that? So let us see, let us do that as a different result itself, I should have done this in the beginning perhaps. So, let us say g is in L1 of mu, then g is finite, almost everywhere. What does that mean? if I look at if I look at the set, so that is that is the set where mod g is infinite has measure 0.

So, this set has measure 0 and outside this set it is of course less than infinity. Well why is that? So let us let us give a one line or two line proof for this.

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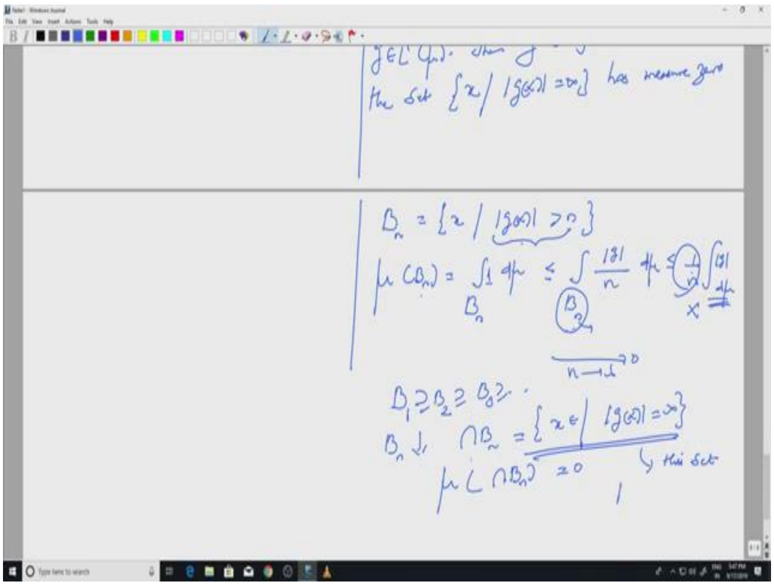

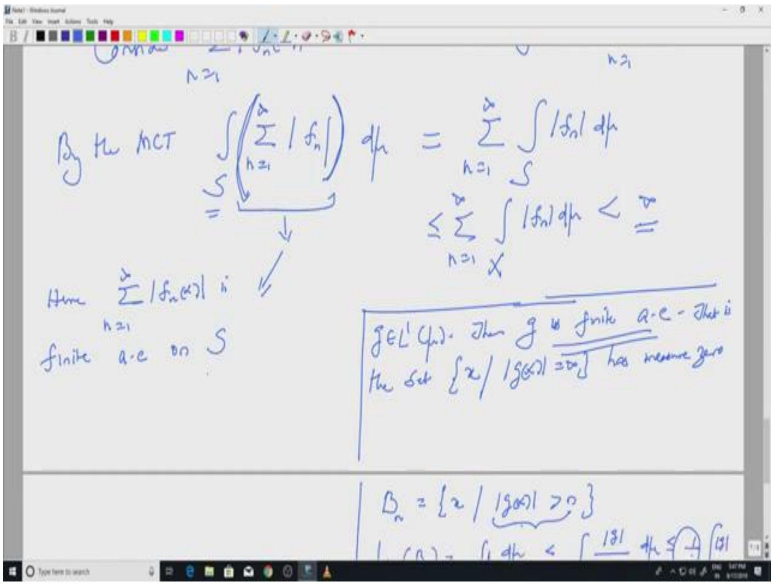

So, look at the set Bn where mod gx is greater than n, then measure of Bn, equal to integral over Bn d mu by definition, but on Bn mod g is greater than n, so I can replace or dominate this by integral over Bn mod g divided by n, so mod g divided by n on Bn is greater than 1.

So I am replacing something here one by something bigger than one. So by monotonicity of the integral, we have an inequality d mu, which is of course less than or equal to 1 by n, that is a constant which comes out, I am integrating over Bn I replace it by x. So, I will have integral over x, mod g d mu, which goes to 0 as n goes to infinity, because of the 1 by and, integral over mod g d mu is finite because g is in L1.

So, this goes to 0, that means mu of Bn goes to 0, so what is mu of bn? So what can you say about the Bn? Well, it is a decreasing set B1 is bigger than or equal to B2 is a superset of B3 and so on. So Bn's are decreasing intersection Bn, well what is intersection Bn? Intersection Bn is precisely all those points where mod g is greater than n for every n.

So that is where g is infinite, but Bn's go to the intersection Bn each of them is finite mu of B1 is less than or equal to 1 times integral over x mod g d mu, which is finite so this will go to 0. So, mu of intersection Bn is 0. So this set where g is infinite has measure 0. So, this set has measure 0.

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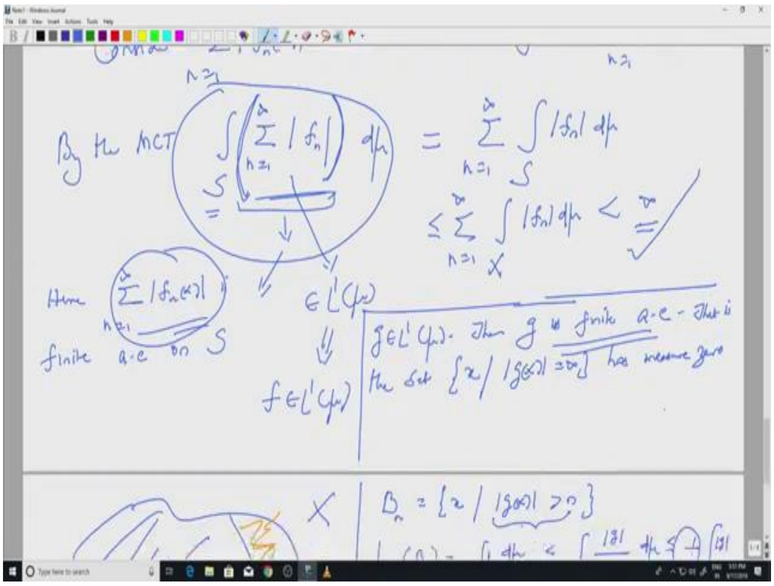

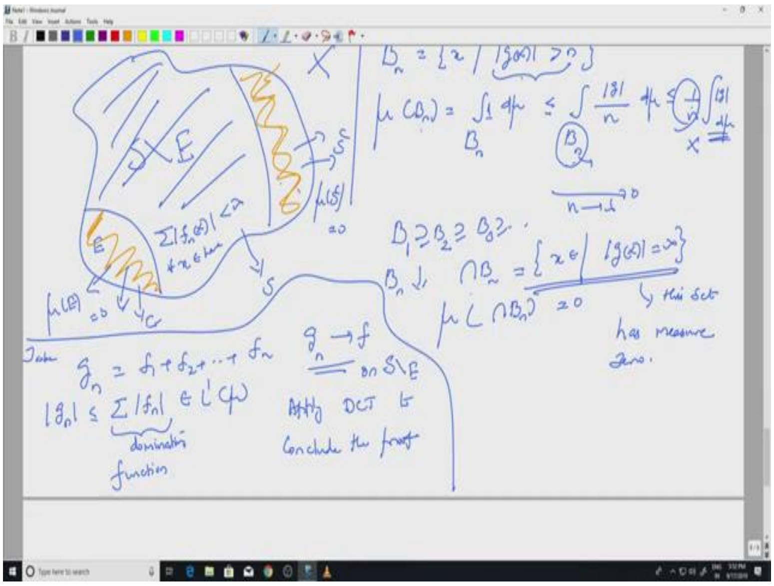

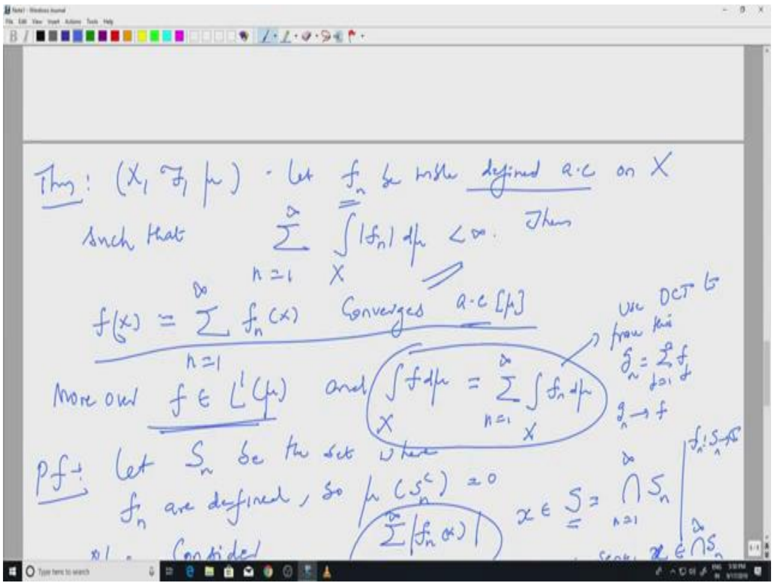

So, that is why g is finite almost everywhere. If I have L1 function, it is finite almost everywhere. Now, we are in the same situation, I have a function which is, so if I look at this function this is a positive function whose integral is finite, hence summation n equal to 1 to infinity mod fn x is finite almost everywhere on S.

So, what do you have? So we have let us say this is my set x. This is my set S, this is S compliment, S compliment has measure 0, so let us forget that guy. Now on S summation mod fn x is finite almost everywhere on S. That means there is some set here, let us call that E such set measure of is also 0. And in, in this area summation mod fn x, is finite for every x here in this set.

So, this is a set of measure 0, this is a set of measure 0. So the union of these two sets the set and the set will be of measure 0. And on the compliment of that summation mod fn x converges, so that is precisely our conclusion. If this happens, then this converges almost everywhere.

So there is a set of measures 0 and outside that set of measure 0, I have absolute convergence. I have absolute convergence and absolute convergence, of course implies the usual convergence. Now, it also tells us that f is in L1 of mu L, why is that? Because summation mod fn x is integrable.

That is what this says if I integrate summation mod fn over S, S compliment has measured 0, so it does not matter. So over x S it has finite integral, so that means this function is in L1, but this function of course, is bigger than f, mod f will be less than or equal to sum of mod fn's, so f is also in L1, so this is an L1 from this we conclude that f is also in L1.

Now, the interchanging the summation and the integral, this is a easy exercise. So, I can leave this to you. So use DCT to prove this to prove this, how will you do that? You take gn to be summation j equal 1 to n fj, then gn's will converge to f gn's converge to f, you just need a dominating function. So, mod gn will be bounded by summation mod fn which is in L1 so, you can apply DCT.

So, that proves the that proves the theorem we want.. So, maybe I can read it here. So, take gn to be f1 plus f2 plus etc etc etc up to fn, then mod gn is of course less than or equal to summation mod fn, which I know is in L1. So, this function is my dominating function.

And I know that gn's will converge to f almost everywhere then I know that well I can, okay instead of almost everywhere I will say this happens on S on S minus the set E. Then apply, apply the DCT to conclude the proof, this is a set E and on the set S minus E, I have this convergence and integrals on S minus E will converge via dominating convergence theorem integral over E and integral over S complement has measure 0, these sets are measures 0 so integrals over them are 0.

So, what we have seen so far is the abstract theory of integration. We started with sigma algebras and measures on sigma algebras. Then we defined integration with respect to integration of positive functions with respect to this measures and then we extended it to complex valued functions.

We saw that it was linear and three major theorems, one is the monotone convergence theorem. The second one is the Fatou's lemma and, third one is the dominated convergence theorem. And then, we saw what is meant buy a property to be true almost everywhere depending on the measure and so on.

And we saw how the theorems can be rewritten with respect to this concept of almost everywhere. So, the convergence can be always on a set which is of full measure in the sense that the compliment has measure 0, then one can still conclude that appropriate convergence take takes place under the hypothesis of monotone convergence theorem or dominated convergence theorem.

In particular, if I know that there is a sequence of functions converging almost everywhere to 0, and the sequence of functions is dominated by an integrable function, then the dominated convergence theorem will tell me that the integrals will go to 0. Even if the convergence is not everywhere, one can still conclude that conclude using dominated convergence theorem or monotone convergence theorem, what will happen to the limit of the integrals because the integrals over sets of measures 0 is 0 and can be discard. In the next session, we will start with the construction of Lebesgue measure.