Measure Theory
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Lecture 1 - Review of Riemann integration and introduction to sigma algebras

So, we start with a quick introduction to Riemann integration, this is something which you have already learned, but we will just do a quick recap before going further. So in Riemann integration, we deal with bounded functions defined on an interval and we define their Riemann integral by looking at what are known as lower sums and upper sums. So, let us do a recap.

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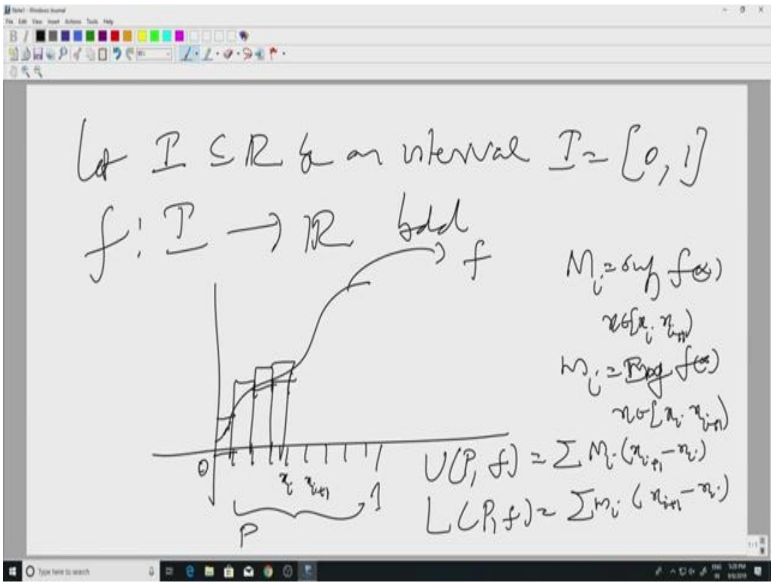

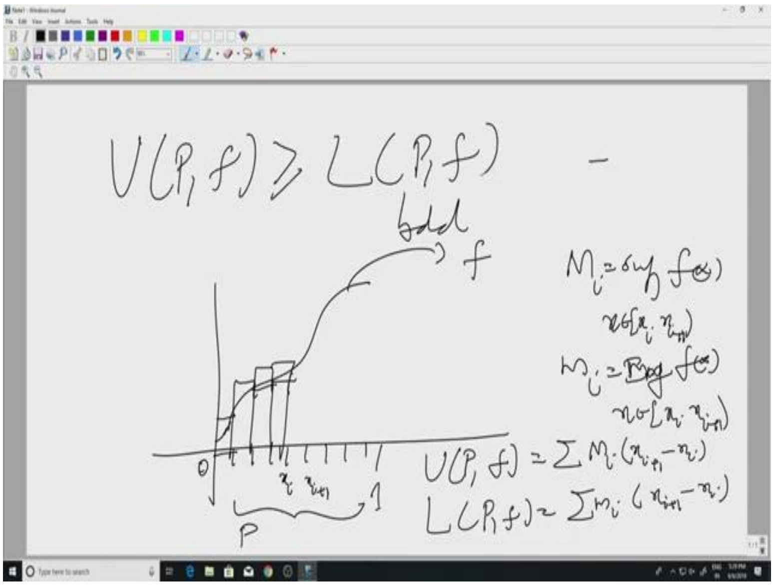

So, let I be an interval contained in R B an interval. For the sake of simplicity we will just choose I to be the closed interval 0, 1. And we have a function f from I to R bounded. So let us say we have 0, 1 here, and I have some function here. So this is the graph of f, we decompose the domain of f into small pieces intervals like this and in each interval, we choose the maximum of f and the minimum of f and so on, so the interval so let us say this is x i and this is x i plus 1.

In the interval let us say capital I is supremum over x in x i, x i plus 1 f of x and small m i is equal to minimum of or infimum of f of x, x in the same interval, then we form the upper sum. So, let us say this partition is called p, then upper sum of f corresponding to this partition is summation capital M i into the length of the interval so, that is x i plus 1 minus x i. Similarly, the lower sum is you choose the infimum of f in these intervals, and multiply again by the length of the intervals.

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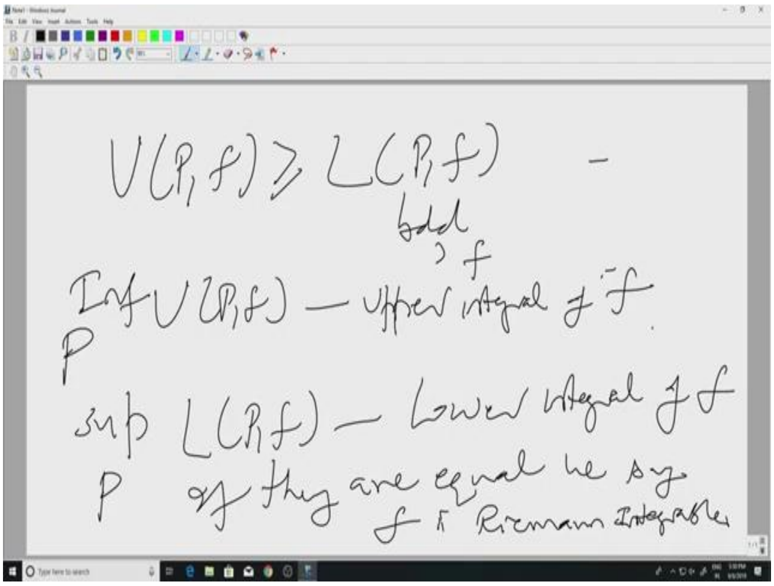

So you get 2 quantities, and it is easy to see that the upper sum is greater than or equal to the lower sum. And you can take the infimum of the upper sums and the supremum of the lower sums over all partitions. So, you look at supremum over P which is a partition of upf and similarly, sorry, not supremum infimum and supremum over P of the lower partitions. So, this is called the upper integral of f and lower integral of f, if they are equal. If they are equal, we say f is Riemann integrable. And the common quantity is called the Riemann integral of f over the interval I, which was 0, 1.

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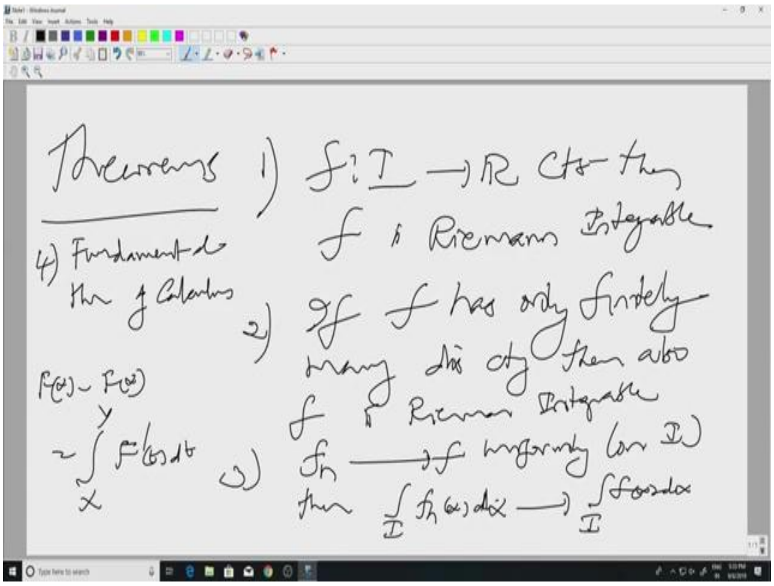

Now, let us recall some of the main results from Riemann integration. So I will just write them as results and then we will go ahead or theorems. One, if I have a continuous function then f is Riemann integrable. So, that is a large class of functions which is Riemann integrable, 2 if f has only finitely many discontinuities then also f is Riemann integrable. So, in particular, monotonic functions which have only finitely many discontinuities in the interval 0, 1 will be Riemann integrable.

3 suppose I have a sequence of functions f n Riemann integrable converging to f uniformly, so on the interval, right I then their integrals also will converge to integral of f s dx. So, notice that we are interchanging the limits and integrals here and the 4^{th} one is the fundamental theorem of calculus which says that I can write f of x minus f of y equal to integral over x to y f prime t dt provided f prime is a continuous function, then the indefinite integral of f prime is a differentiable function and you have this equality which is the fundamental theorem of calculus. Our aim in these lectures would be to replace these theorems in a much more general setup.

So, one of the things you should realize from the Riemann's construction is that you are decomposing the domain of the function into smaller pieces, looking at the supremum and infimum, forming upper and lower integrals and then taking certain limits. If the limits agree then we say the function is Riemann integrable and the integral is the common value.

Now, in Lebesgue approach, this is what the course is about. In Lebesgue approach instead of decomposing the domain of the integral, we decompose the range of the function, not the domain of the function, the range of the function into smaller pieces, then take the inverse image of those smaller pieces, attach certain length to those sets, and then add up and then take an appropriate limit just like what we do with Riemann integral. This gave rise to a theory which was much more general than Riemann integration, and which was also quite useful for the development of analysis in the beginning of 20^{th} century. So let me start with some simple examples to motivate what we will be doing.

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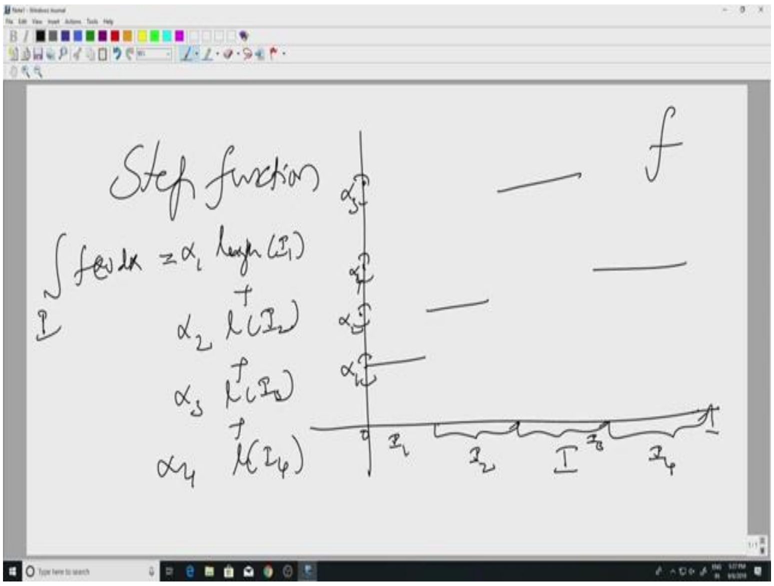

Let us take a step function. So it would be something like this. Let us say it is here and some value here, some value here and some value here and so on, so this is the domain on which it is defined, let us say 0, 1, so these are certain values which we call alpha 1, this is alpha 2, alpha 3, alpha 4 and so on. So f takes the value alpha 1 in this interval, and next interval it takes the value alpha 2 and so on. What is the Riemann integral of this function? So, if I call this f, the integral of so this is the interval i f of x, dx is nothing but so let us call this interval, I1 the next interval I2. And next interval I3 and this is I4.

Then the Riemann integral of this function is simply alpha 1 times the length of I1 plus alpha 2 times length of I2 etcetera. alpha 3 times length of I3 plus alpha 4 times length of I4. Now, as I was saying in Lebesgue approach we will be decomposing the range of the function. So, the range of the function is only these 4 points. So, let us look at small intervals around these values.

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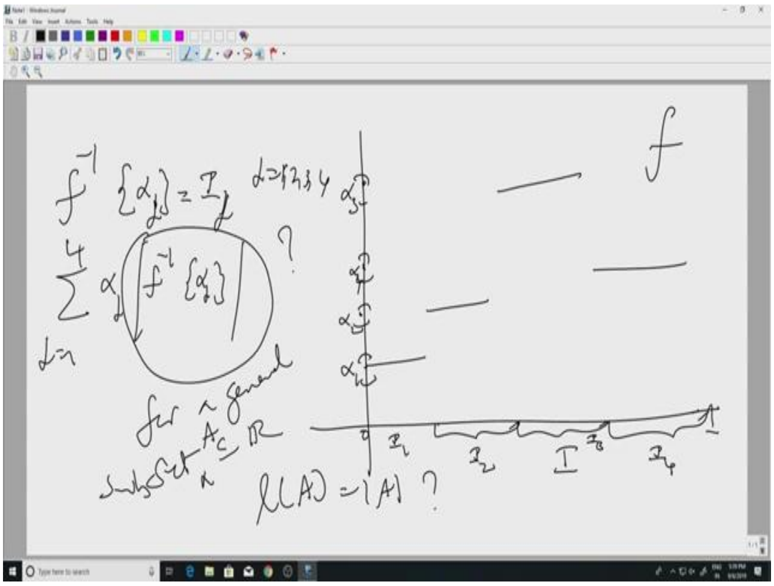

And let us say we are taking the inverse image of the so, what would be the inverse image of these values f inverse of the set or the point alpha 1 is simply the interval I1. This is precisely the set where f takes the value alpha 1 similarly, so f of, I will write alpha j, j equal to 1, 2, 3, 4. Now, the Riemann integral can be written as the sum of the values, some of you multiply the values of f alpha j with the inverse image and then you sum up. This is a set, we are taking the length of that set.

So, this is what Lebesgue approach is, you decompose the range of the function into smaller pieces, look at their inverse image, you need to have a length function associated to the inverse images of these sets. So, when you take the inverse image, you will have different kind of sets, subsets of the real I, in this case the function was a step function. So, when you took inverse images of certain values in the range of f, you got intervals. You may not get intervals when you look at general functions, so one needs to understand what is meant by this for a general set. For a general set, subset of R, what is meant by the length of that set?

So, let us say the set is a, what is the length of a? Or modulus of A, let us say, how do you define this? So, one needs to know two things, one is the collection of sets for which length function can be defined, which will satisfy the properties of the length of the intervals which we know.

For example, if I take 2 disjoint intervals, the length of their union should be the sum of the length of the components. So, similar properties you want to be true and it should also take care of a large class of collection of sets, subsets of the real I, which should allow us to use the procedure we have just seen to define the integral for a large collection of functions. We will have to decide which functions are integrable and then it make sense, right? That is what we will be doing.

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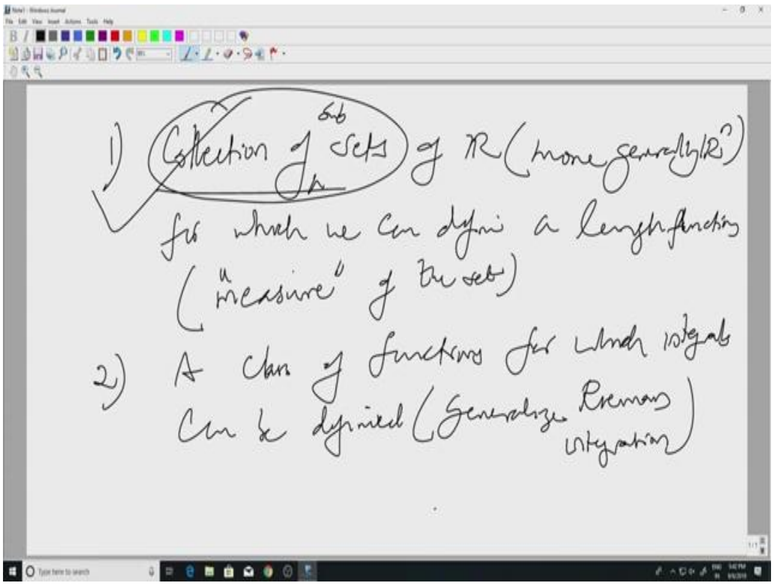

So, we need 2 things, one is collection of sets or subsets of R or more generally R n for which we can define a length function. This is what we will call the measure, measure of the set. Measure of the set should be in the case of the real line, if I take an interval the measure of the interval should be the length of the interval. If I take a square in R2, it should be the area of the square. If I take a ball or a cube in R3 then it should be the volume. So, whatever notion of measure you already have in R, R2 and R3 should continue to hold when we define this in more generally in R n for sets, which are much more general than intervals or squares, rectangles or balls and so on, that is one thing.

So we should, the collection of sets should contain these sets which we already know like the intervals like the balls like the squares and so on, right, then we also need a class of functions. Class of functions for which integrals can be defined, which should generalize Riemann integration. So that would be our first take. So we will start with collection of sets for which some measure will be defined. So, instead of looking at R n initially, we will look at abstract spaces for which this collection of sets we will formalize the properties of the collection of sets we want. And we will call that sigma algebra. So that is the first definition we will be dealing with.

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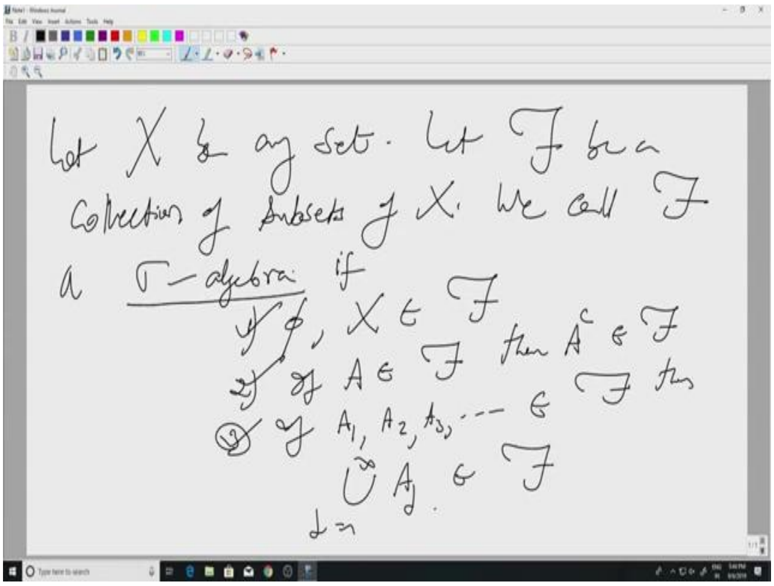

Let X be any set, nonempty set. Let script f be a collection of subsets of x. So x is a non-empty set and I am taking some collection of subsets of x. So, we call script f, a sigma algebra. If three things are satisfied, one, the empty set and the whole space should be in script f. Two, if I have a set A in script f, then the compliment of A should also be in F. Three, if A 1, A 2, A 3, etcetera. So countably many sets, they are all in script f, then their union should be in A j, j equal 1 to infinity should also be in script f. If I have these 3 properties, then we call script f a sigma algebra. So, for any given set x, a sigma algebra is a collection of subsets of x, which satisfies these three properties. So, before we go further, let us look at some easy examples.

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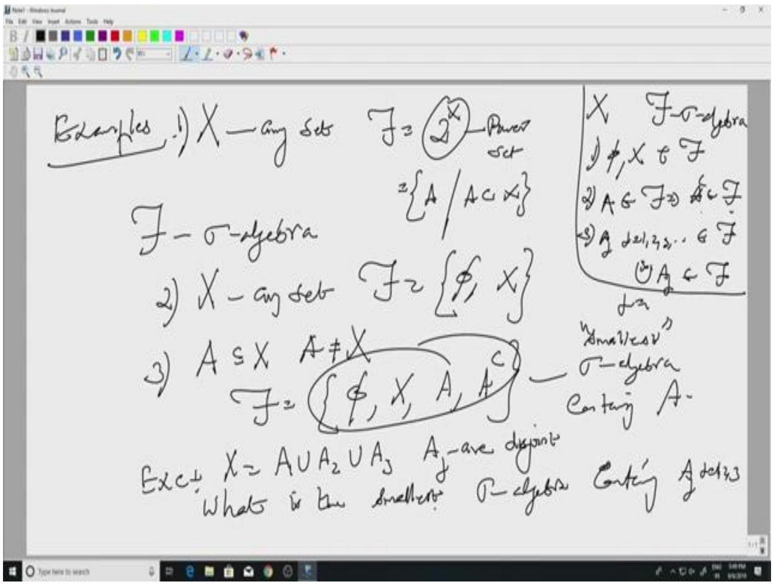

So, let us write down the definition in this corner. So, I have x, I have script f, right, this is the Sigma algebra. Firstly, the whole space and the empty set should be in script f to A in script f implies A compliment is in script f. Three, if I have a j, j equal to 1, 2, 3, etc. countably many, countably infinitely many right, in script f then their union is also in script f. So, to check that a collection of sets as a sigma algebra, we check all these 3 properties. So, examples: let us take x to be any set and script f to be the power set, what is the power set?

So, this is all subsets of x. So, that is generally denoted by 2 to the x. Then clearly script f is a sigma algebra that is trivial because you have the empty set which is a subset, x which is subset of x. So they will be into to the x which is script f. If A is a subset of x, then A compliment is also a subset of x. So, that will be there, if A j j is equal to 1, 2, 3 and so on are subsets of x, then the union is also a subset of x. So these 3 properties are trivial to check. And this is the biggest sigma algebra you can think of because these are any sigma algebra, or subsets of x will contain subsets of x and this is all of them.

So, this is the largest sigma algebra you can think of. Second one is the smallest one, so, let us take, just put whatever is necessary. So, I know that the empty set should be there, I know that the whole space should be there and nothing else, this is obviously sigma algebra because all these three properties are satisfied. Now, let us start with some interesting examples. Let us take some set A contained in X, A not equal to X let us say. I look at script f to be equal to, I know that it should have the empty set, I know that it should have the whole space, let us put A. If A is there, I know the second property tells me that A compliment should be there. Third property tells me that unions should be there.

So, but the union of any of this will give me some set here itself. So, if I look at just this, this is a sigma algebra, this is the smallest sigma algebra containing A because if I take any other sigma algebra containing A that sigma algebra will contain A and A compliment, the empty set and the whole space. So, in particular script f, so, everything in script f will be in any sigma algebra containing A. So, a simple exercise would be to write down a sigma algebra containing more than two sets.

So, let us take X to be equal to A1 union A2 union A3, A j are disjoint, what is the smallest sigma algebra containing A j, j equal to 1, 2, 3. So, we saw a smallest sigma algebra containing 1 set A in this example. Similarly, what will be this? You can write down those explicitly, you know that it should have A1 you know it should have A2, you know it should have A3. Similarly, it should be closed into complementation and so on, so you fill up by taking compliments, union, etcetera. You will get all possible sets and that will give you explicit expression for smaller sigma containing A1, A2 and A3.

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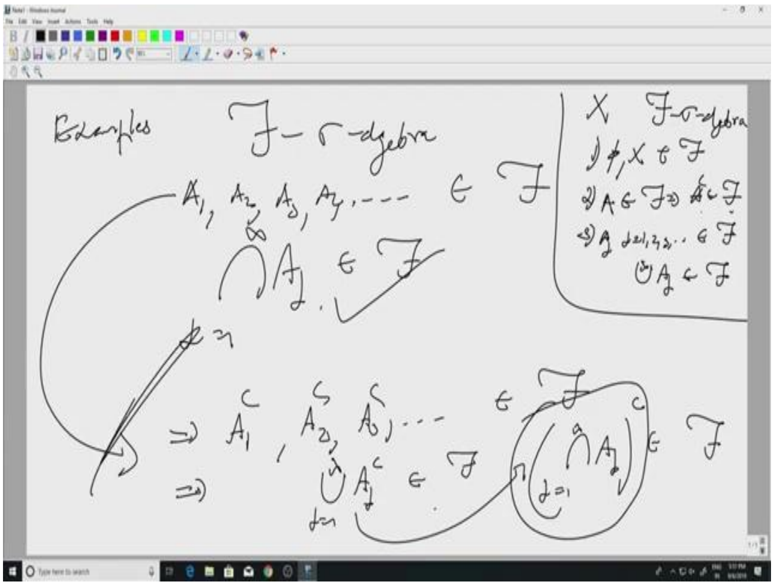

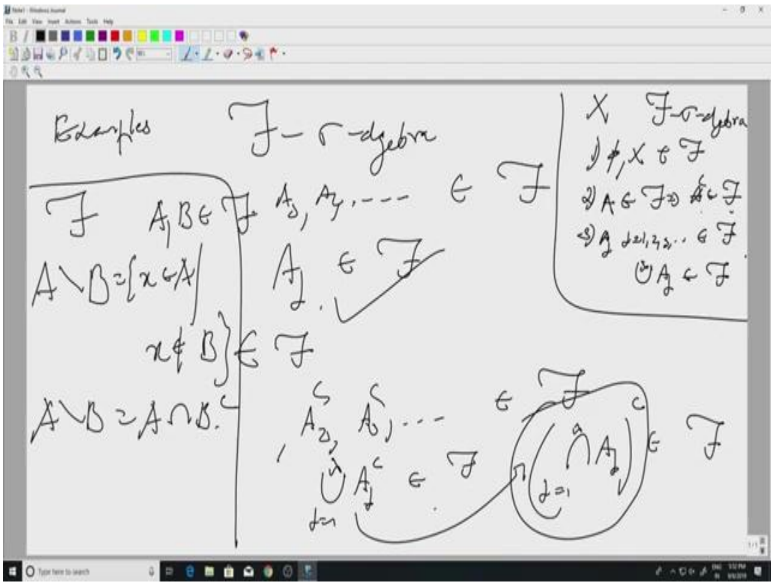

But let us continue with some more examples. The sigma algebra is closed under taking complementation that is a second property. Similarly, the Sigma algebra is closed under taking countable unions, now because it is closed under complementation this also implies that it is closed under intersection. So, what I mean is if script f is a sigma algebra and I have A 1, A 2, A 3, A 4, etcetera, they all belong to script f, then intersection A j, j equal to 1 to infinity also belongs to script f. Why is that?

Because from here, I know that this gives me A1 compliment, A2 complement, A3 complement, etcetera belongs to script f. The third property tells me that union of the complements j equal to 1 to infinity also belongs to script f. But this is nothing but intersection of Aj j equals 1 to infinity whole compliment by Demoivre's theorem. So, this will belong script f. So, the compliment of this set will also be in script f which is this. So, the sigma algebra is closed under unions, complementation, intersections or whatever security cooperation you can generally do.

For example, you can look at, you get two sets A, B in script f then A minus B, what is A minus B? So, these are all points in A which are not in B. This will also be in script f, why is that? A minus B is nothing but A intersection B complement and A is in script f, B is in script. So, B compliment will be in script f. So, their intersection will be in script f, remember the sigma algebra is closed under intersections. And if you have countable intersections, you can also take finite intersection or finite unions by taking other sets to be the empty set, that will do or the whole space depending on the case.

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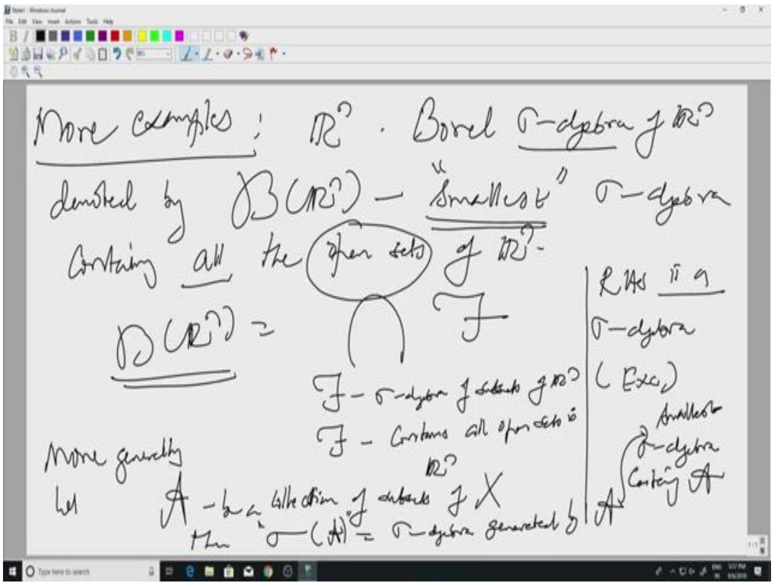

We will look at more examples, so consider R n and we will define the Borel sigma algebra, Borel sigma algebra of R n, we will denote it by B of R n. So I should tell you what is the sigma algebra. This is the smallest. So I will explain what that means, smallest sigma algebra containing all the open sets of R n, so I should tell you, what is mean by the smallest sigma algebra, well, what is the smallest sigma algebra? You can, for any set you can take the power set that is a sigma. But there may be smaller collections than the power set, which is sigma algebra.

So, I am looking at all sigma algebras containing all open sets of R n, for example, the power set itself, the power set itself is a sigma algebra containing all the opens sets, but there can be smaller open sets containing smaller sigma algebras, containing all the open sets. So, you look at all such sigma algebras containing all the opens sets, take the intersection of them, that will give me a sigma algebra which is the smallest. So, let me write it down. So, B of R n by definition is the intersection of all sigma algebras.

So, f is a sigma algebra of subsets of R n and S, so S is a collection of set, I wanted to contain all open subsets of R, open sets in R n. Then this is, so the right hand side RHS is a sigma algebra that is an easy exercise. Well, how will you prove it is a sigma algebra? You need to know if it has all those 3 properties. One is whether the empty set is there, well, empty set is there in all sigma algebra. So, in the intersection also will be there. Similarly, the whole space is there and so on. So, you check all the 3 properties to prove that this is a sigma algebra.

So the smallest sigma algebra containing a collection of sets, in this case it is a collection of open sets will make sense. So take, so more generally or generally let script A be a collection of subsets of let us say some x then sigma of A, this is the sigma algebra generated by script A. In other words, this is the smallest sigma algebra containing A, all the sets from A, that is denoted by sigma of A. So you can say this is sigma of all open sets in R n, we will denote it by B of R n though, that is the standard notation for Borel sigma algebra.

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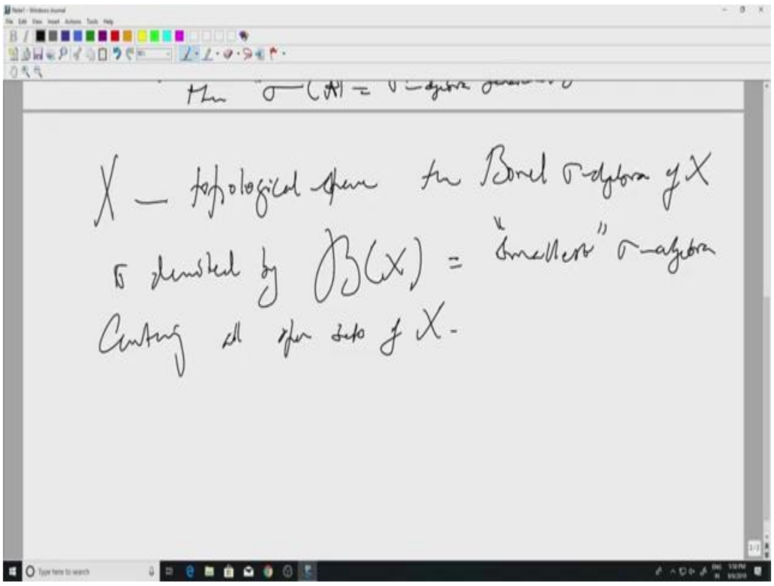

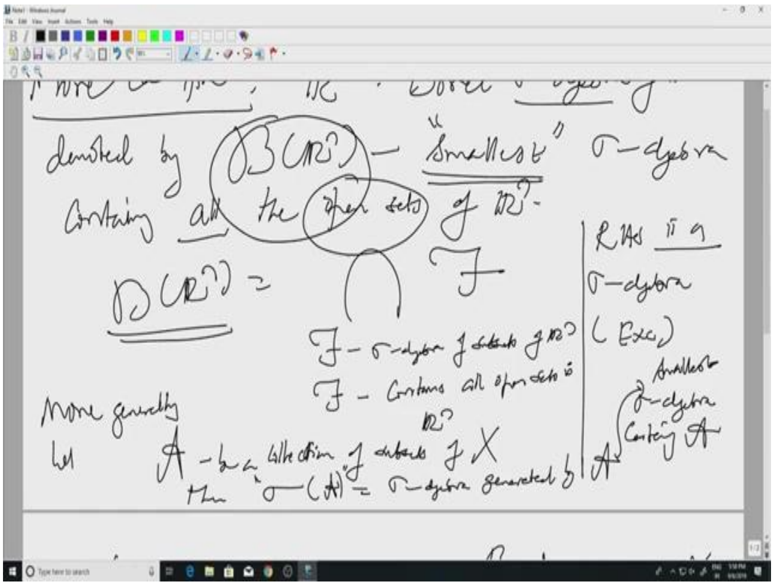

So, one can also, instead of R n if you start with topological space, the Borel sigma algebra of X is denoted by, by B of X. That is the standard notation. So this is of course, the smallest sigma algebra containing all open sets of x. So let us go back to R n. So what are the sets that will be contained in B of R n? So let us, we can look at that a bit more closely.

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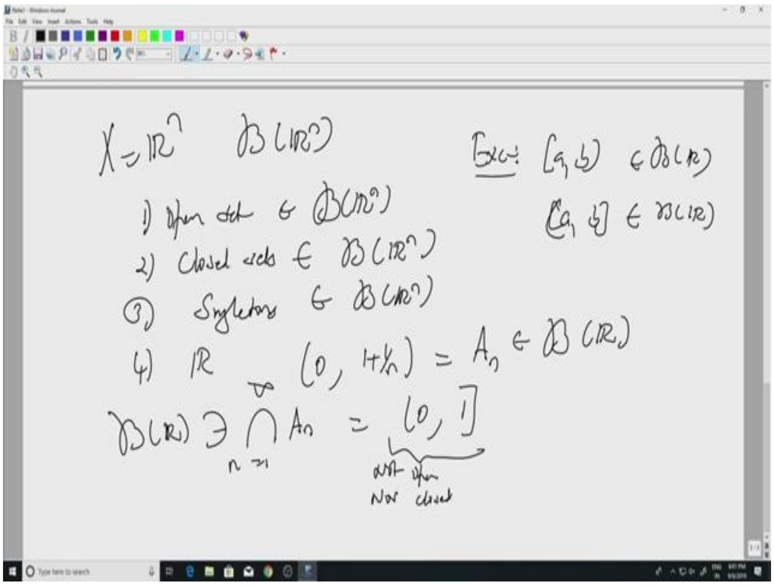

So, if I go back to R n, so, let us say X is R n and I am looking at Borel sigma algebra for R n. So, any open set is an element of B of R n. Open balls and things like that. 2 closed sets right, because closed sets are compliments of open sets and the sigma algebra is supposed to be closed under complementation. So they are also in B of R and, 3 Singleton's for example, that would be also in because Singleton's are closed. Well, the sets need not be open or closed. So remember it is closed under countable intersections, countable unions and so on.

So, let us look at the real line a bit more closely and look at the set open 0, let us say 1 plus, 1 by n. So let us call this A n. This is an open interval, so this of course belongs to Borel sigma algebra of the real R. Now let us look at the intersection, well, what is this? This is open at 0, 1 is there in every set. So, this is going to be closed at 1, you will not get anything more than that. And since A ns are in B of R intersection, A n will also be in B of R, the Boral sigma algebra of the real R, this is not open or closed, right, not open or closed. So, there are sets of all kinds which are in B of R n. So, in particular, so simple exercises would be to prove that sets of the form A, B open at A, closed it B etc, so these are all Borel sets.

So, such sets will be in the Borel sigma algebra, they are neither open nor closed. So, to conclude the first lecture, we have started with a review of Riemann integration and we have seen that generalizing this will take a definition of length function for much more general sets than the intervals. For that, we have defined what is known as sigma algebra. So, sigma algebra is the collection of sets which will have this length function or what we call the measure defined on that. So, we have just reached sigma algebra. We will be doing, we will be defining what is meant by a measure in the next lecture onwards and then properties of the measures, so, that is what we will do.