An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science Bengaluru
Orientation on Manifolds 3
Lecture 68

Hello, and welcome to the last lecture in this series. So, let us, this will be all about orientation. So last time I sketched an argument that having a non vanishing, nowhere vanishing n form on a smooth manifold gives rise to a system, a collection of charts an atlas in fact, such that with the property that the transition functions have positive determinants of their derivatives. And such an atlas is called an oriented atlas.

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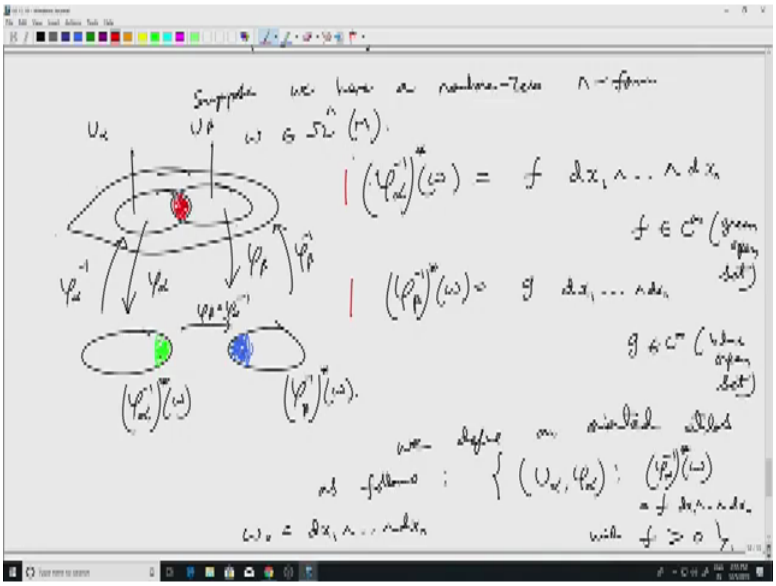

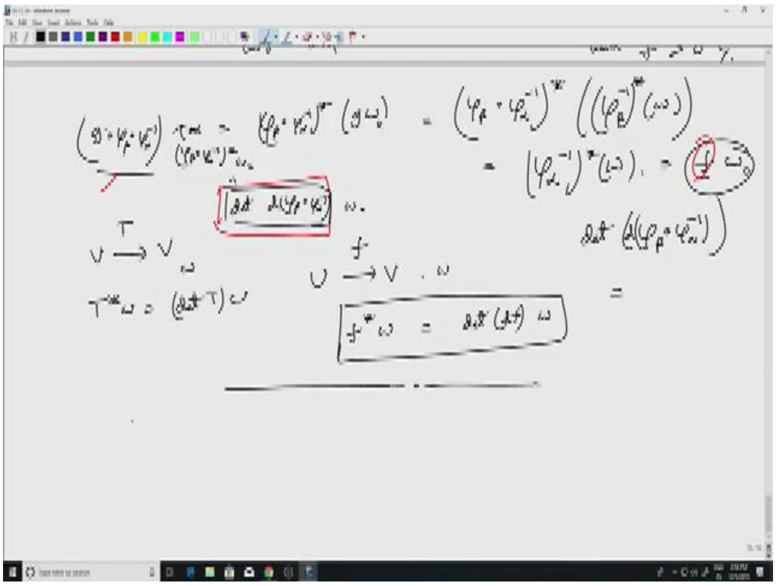

And the way we picked out this atlas is quite simple, we just pull back the form to an open set in Rn. And once we pull it back, it will be some function times the standard n form and we just, we demand that the function be positive. Well, the moment you pick out those charts which have this property, then we showed that the transition functions have the required property, namely this thing here. This is the object we are concerned with, so, and this turned out to be positive because this is positive as well as well, the right side this is positive. Now, two things. Yeah, so what we have to check is that, so what we have shown is that we can get, we can define a system of charts this way. But it is not clear that, well to begin with, are there any charts of this form and are there enough of them to cover M? Well, both questions are easily answered by noting that we can always get such charts.

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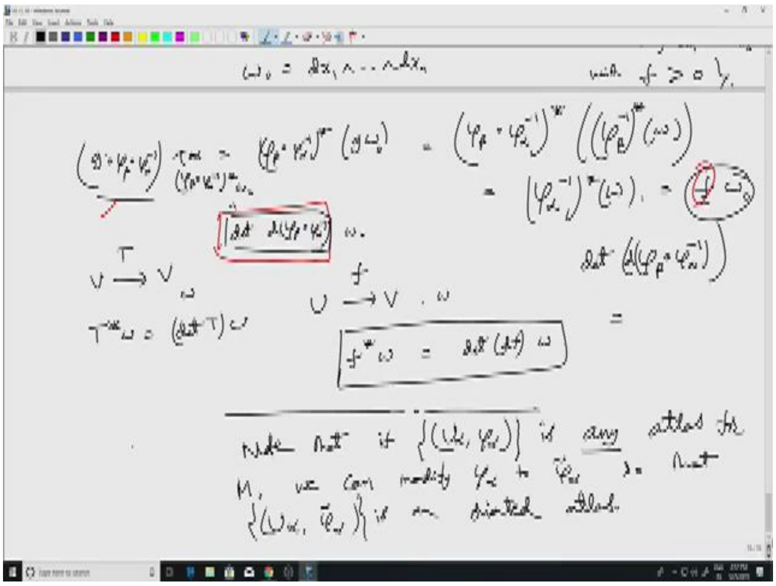

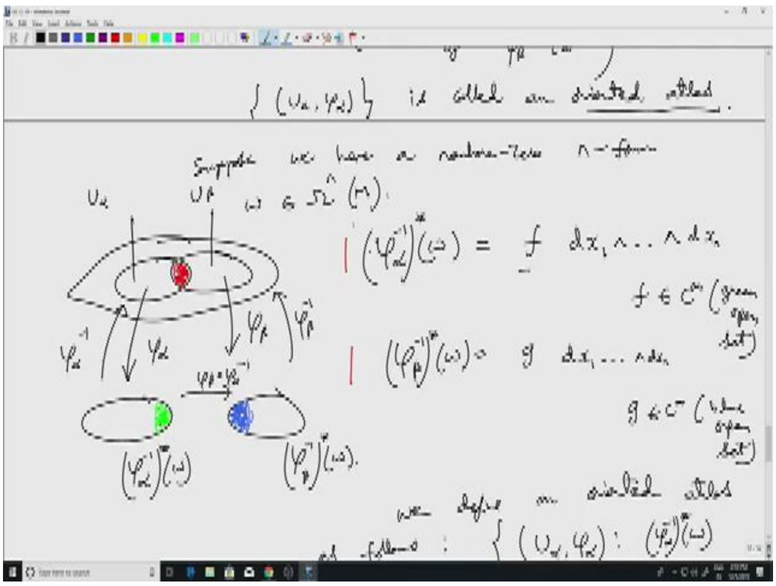

So note that if U alpha phi alpha is any atlas for M not necessarily oriented, we can modify it, modify the chart diffeomorphisms, let us call it phi alpha tilde, so that U alpha, phi alpha tilde is, this is an oriented atlas. And since the original collection of open sets U alpha covered M, and this will, well we are not really changing the U alpha at all. So this fact that this cover M is as follows from the original property. Then the only thing we are changing are these maps phi alpha and the way we modify them is, so recall that what we need actually to conclude to get an oriented atlas is that this, when I pull back the form to Rn, I should get a positive function here, this f should be positive.

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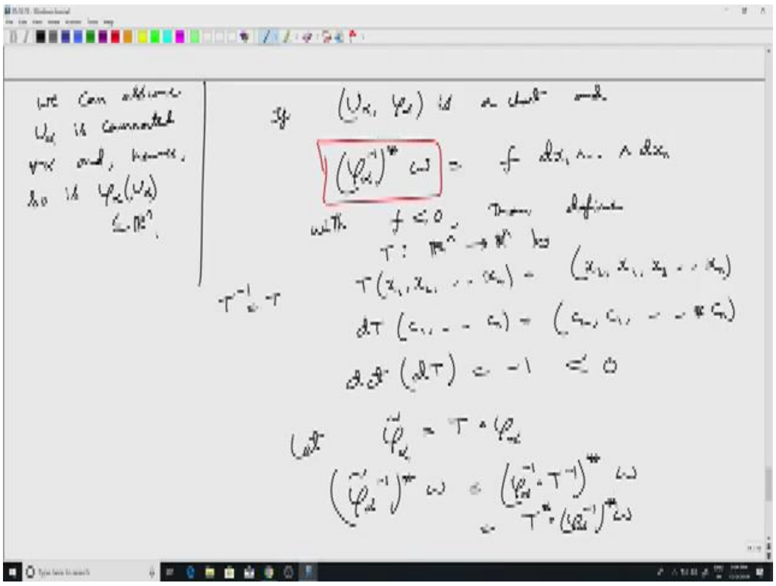

So let us start with the chart and pull back U alpha phi alpha is a chart and phi alpha inverse star omega equal to f dx1, dxn with. Here I make the observation that by always taking slightly smaller charts if necessary smaller open sets U alpha, we can assume U alpha is connected for all alpha and hence, so is phi alpha U alpha, these open sets in Rn that we get we can assume are connected. Well once it is connected and I write it like this, then I know that this is a nowhere-zero form this side, the left hand side is nowhere-zero. Therefore, this function is nowhere-zero. Hence it is either strictly positive or strictly negative.

So let us, if it is strictly positive, we are in good shape. So if it is strictly negative, then define, I will just, what I will do is just interchange, use a reflection in the first two coordinates in Rn, then define T from Rn to Rn by T of x1, x2, etcetera, xn to be just x2, x1, x3, the other stuff, other coordinates will be kept as they are. So I just interchange, this is a linear map and therefore its derivative dT is just itself, dT of c1, c2, cn is c2, c1, cn. And the main property of this that we need is the, if you write down the matrix, it is just similar to what we had done earlier. The determinant of the derivative of this will be is actually equal to minus 1, but what we need that it is negative.

So, then let phi tilde, the new chart map be phi tilde of phi tilde alpha be, first let us do phi alpha and then compose with this T. Well, so now let us see what the pullback form looks like. When I pull back this, I will get this, actually when I do inverse already the order gets switched, phi alpha inverse would be, first I will have to do T inverse then phi alpha inverse and then I have to do a star, in which case order again gets flipped, T inverse. Also, yeah, let us, the way T has been defined actually T inverse is equal to T. If I apply it in T twice, I will just get back to the original one. So, I need not actually continue to write T inverse, but let us just, T composed with, T upper star composed with phi alpha inverse upper star omega.

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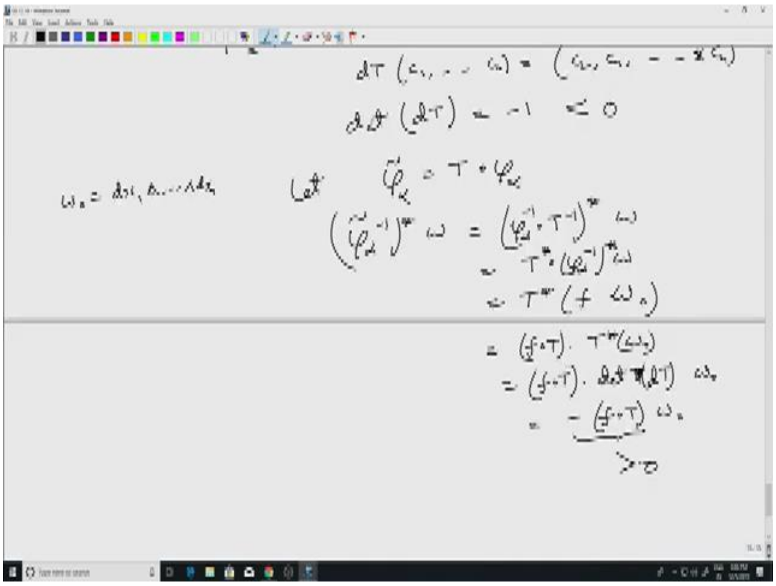

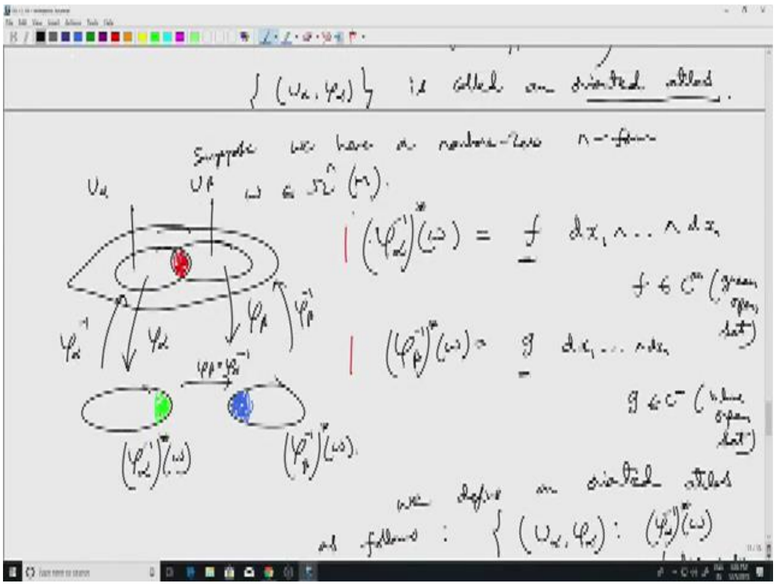

Equal to, this was what we are called f, T star of f of well dx1, so which is omega naught, using the notation omega naught for the standard n form. So this is equal to this and this we know is the same thing as f composed with T and here is the, actually at this point is the, we know that for a linear transformation, this will be determinant of T and then, here I did determinant of derivative of T but in this case, I really do not, yeah, okay, maybe I can use that, it will be the same in this case.

Determinant of dT times omega naught, which is minus f composed with T of omega naught. So in short for this new chart map, I have got a function here, which is strictly positive, because f itself was negative. So just by modifying this, by entertaining the first 2 coordinates, I can rectify so that I have this required property that all of these functions here are positive. And the moment I do that, we have seen that we will get an oriented atlas.

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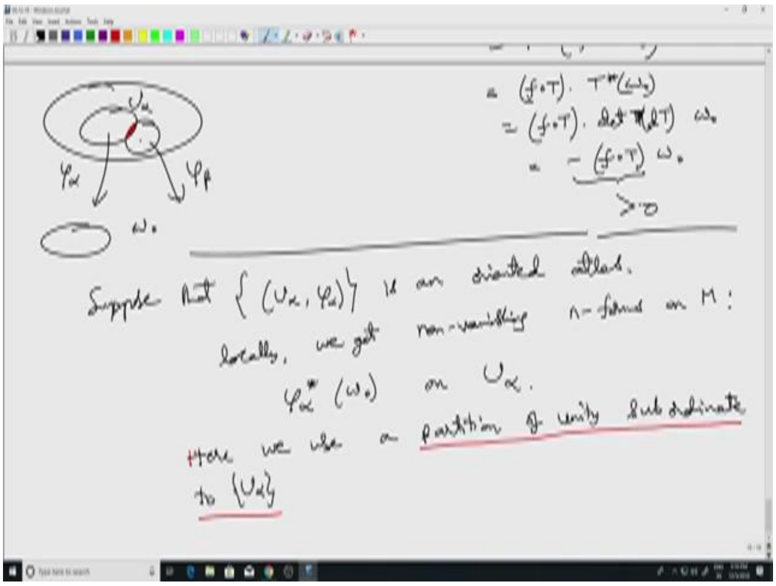

And conversely, if this is an oriented atlas phi alpha, U alpha, phi alpha is an oriented atlas. Suppose that U alpha, phi alpha is an oriented atlas, we would like to construct, come up with nowhere-zero n form on the manifold. Well we can certainly do that, so on locally, in fact, we do not even need oriented atlas. Let, like in the previous construction we had an n form on the manifold and we use charts to get n forms on open subsets of Rn, here we go the other way. So, take a chart, here you have a standard wall, n form omega naught and this is the chart map phi alpha, U alpha. So, I just pull it back to U alpha, phi alpha upper star omega naught and this will be defined on U alpha.

Well there, now the question arises as to how these, how to patch these up. So, these are on every open set, there will be about an n form and we cannot ensure that for instance that suppose I have a phi beta and on this intersection the form phi alpha star omega naught and phi beta star omega naught that they actually agree on the intersection is something that we cannot insure. So, what here we have to use a device which I did not actually have occasion to use throughout the course, but which is actually one of the most useful things in the context of starting with locally defined objects and getting a global object on a manifold, namely this thing called, here we use a partition, what is called a partition of unity subordinate to U alpha, the open cover U alpha.

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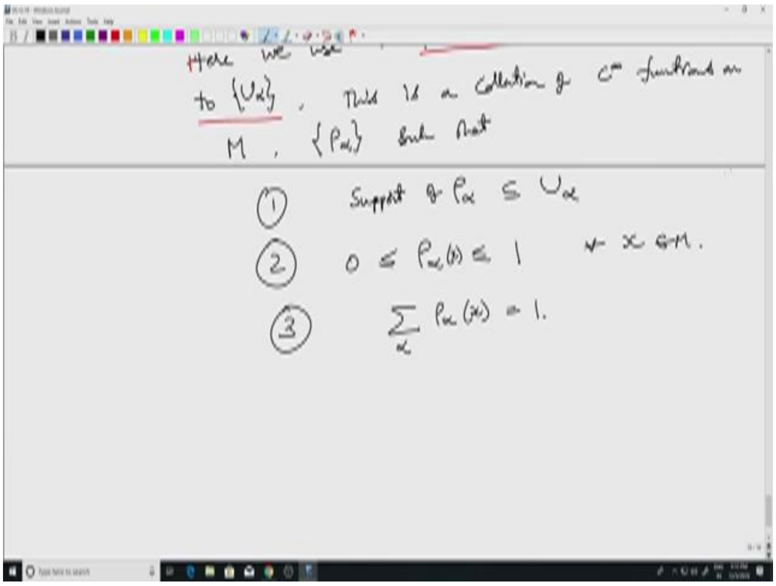

So now let me briefly say what this is, I will not go into the construction of this, so this is a collection of C infinity functions on M which we denote by rho alpha, the same index set as the this thing, as a cover, such that the support of rho alpha is contained in U alpha. Each rho alpha, values of rho alpha lie between 0 and 1 for all x in M and here is the last one is the reason why this named partition of unity, is the reason why these phrase is used, partition of unity. So, the summation of rho alpha x, for all alpha 1. Now, for this to even, the last condition for this to, for this condition to make sense, one needs this. First of all the we cannot ensure that the set of indices alpha is countable, let alone being finite. I mean if it is countable, at least one can interpret this as a series but alpha need not be a countable index. Alpha is basically keeping track of how many charts covering the manifold. Well, but what one can ensure is.

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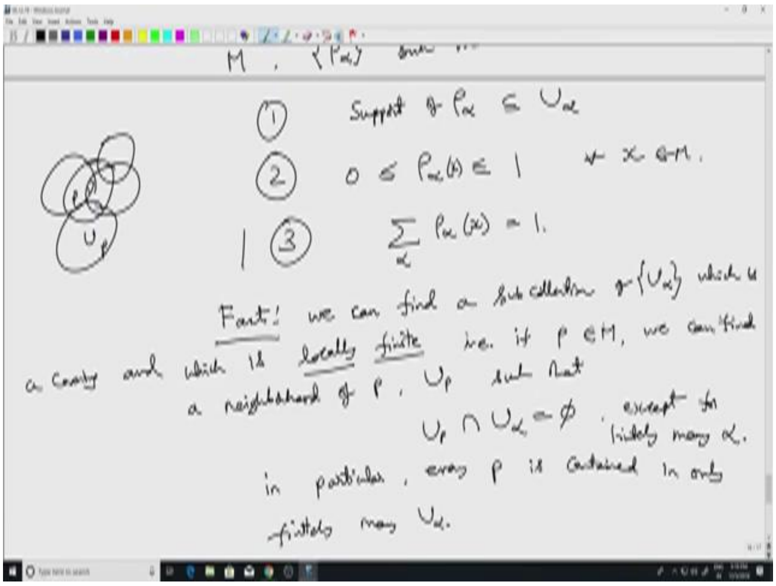

So note that we can find a collection, a cover covering of M by charts which is locally finite i.e. if P is in M, I can find a neighborhood of P, let us call it UP such that UO intersection this U alpha is empty, except for finitely many alpha. So, in other words, if this is P, this is UP which will intersect only finitely many. So in particular every P is contained in only finitely many U alpha. So, once one has this, the moment you take and so here to interpret the last condition 3, if you take a specific x we know we note that we can find a covering of charts which is locally finite. So here of course, actually we already stared with a system of oriented atlas.

So, what I need is that note that we can find a covering of M by, we can find a sub collection of U alpha, the same U alpha, which is a covering still a covering, sub collection meaning we can discard some of the original U alphas and which is locally finite. When I say note that, this still requires a proof, such that it is a topological argument and no time to go over it. So let us just assume it. So instead of saying note that, let me just say fact. Perhaps that is a better way of fact. We can find a sub collection of the U alpha which is a covering and which is locally finite, we can find a neighborhood such that finitely many U alpha.

Once we have this then every P is contained, so this becomes a finite sum the sum in part 3, because the P is contained in only finitely many U alphas and the support of rho alpha is in U alpha. So, the moment this x is not in U alpha, rho alpha will be 0. So, except for finitely many alphas, all the rho alpha x will be 0. So, therefore, this makes sense. And in fact, this says that, not just at a specific x in a neighborhood UP, also the same functions will survive and all the others will be 0.

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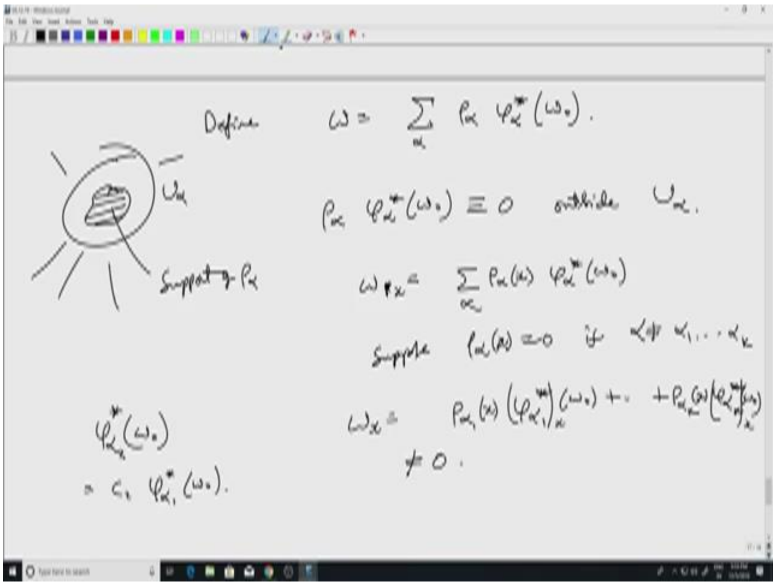

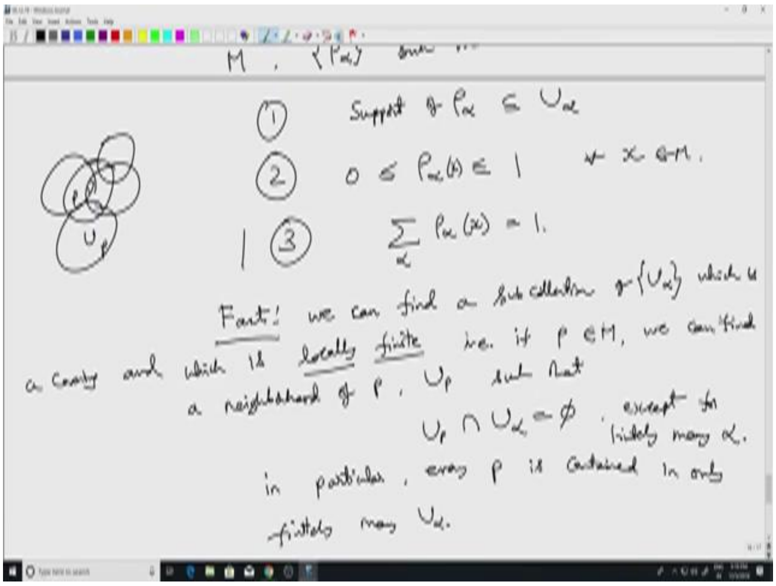

So, once we have this, then I can define our global form, define omega equal to rho alpha, then phi alpha star omega naught. And now the point is that here, so this is rho alpha phi alpha star omega naught of course, this phi alpha star omega naught is defined only on U alpha but the support of rho alpha is a close set inside this, this is support of rho alpha and this is equal to, so what we do is we define it to be identically 0 outside the U alpha. So elsewhere it is 0, on U alpha, this is given by rho alpha, this is what this notation means. It is understood that this thing is identically 0 outside. So, each term makes sense and then we, as a global form on M and then we add it all up.

The thing is that this is, now all one has to do is check that this is nowhere vanishing. So if I take a specific P, so rather x omega at x is rho alpha x and then this. Now, because of this partition of unity condition, only finitely many alphas will survive for which this rho alpha x is not 0, suppose rho alpha x is equal to 0 if alpha is not equal to one of these finitely many, this is then omega x will be rho alpha 1 x phi alpha 1 star omega naught etcetera, phi alpha k star omega naught. Well, that is we still have not used the condition that we stared with an oriented atlas. I could have done all this staring with any atlas.

Now the last step is to say that this is not 0 at any x. And here is where we use the oriented atlas condition. So, basically what I will claim, essentially I just want to relate phi alpha 1 star omega naught. Of course, this entire thing evaluated at x, everything at x, etcetera. So phi alpha 1 star omega naught, we know that any 2 n forms, multiples of each other at a specific point. So if I fix x, then this is some constant times, let us call it C1 times. Actually, let us use phi alpha 1 star omega naught as the base. So let me star with 2 onwards. So I want to compare everything else in terms of phi alpha 1 star and then alpha 2 is let us call it C2 phi alpha 1 star omega naught.

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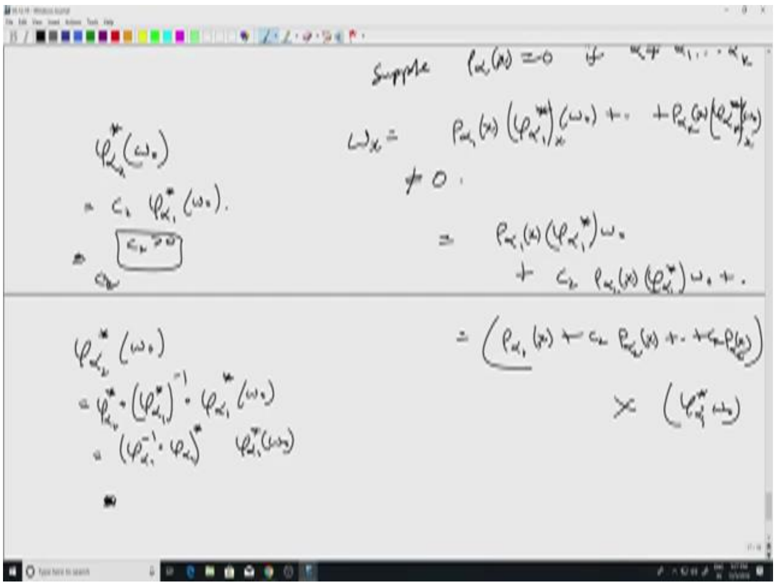

And here is the main point, the fact that it is oriented charts will imply that the C2 is positive, because after all, what is C2 is positive because C2, I can, to figure out what C2 is, just rewrite it like this, the usual thing of using a transition function. So, essentially I throw in the inverse of this and then this and then phi alpha 2 star then I take the inverse inside and so on and star outside and then finally I will get phi alpha 1 inverse composed with the alpha 2 whole thing star and then phi alpha 1 star omega naught. That is assuming that alpha 1, this U alpha 1 and U alpha 2 intersect, they I mean if x is, if rho alpha x is not 0, then they have to intersect.

So, I get it like this and therefore, and as we have seen many times, when I pull back by the, and here is the star means the derivative and so on. Here, it is not quite the transition function, but it is the sort of reverse of the transition function. But at any rate, what will play a role is the derivative, the determinant of the derivative of this will come into play. Then using the fact that transition functions are positive, then one can prove that this determinant is also positive. Ultimately, so one end up getting this. So, you have an expression like this. Yeah. So, the first term I keep it as it is phi alpha 1 star omega naught plus C2 times rho alpha 2 x phi alpha, again the same term and so on.

So, finally, I will end up with this plus C2 rho alpha 2 x dot dot dot plus Ck rho alpha k x, whole thing multiplied by, well rho alpha 1 star omega naught. Now, the point is that since this is a summation of the rho alphas is 1, at least 1 of these rho alpha 1, rho alpha 2 should be positive and the C's are already positive. And all the rho alphas are non-negative, so therefore the whole, the entire sum should be strictly positive. So therefore, this is not 0. It is just a possible positive multiple of this. So this completes the brief sketch and we will stop the course at this point. As I said one would then next talk about integration on manifolds and after introducing manifolds with boundary and then finally move on to Stokes theorem.

And when the whole theory of differential forms is crucial for even to talk about integration on manifolds. So thank you. So we will end the course at this point. A natural starting point for subsequent study will be a little bit more about orientation, examples and so on. And then one can read some material about manifolds with boundary, then integration on manifolds. Then finally, one will have Stokes theorem on manifolds, which is a very useful tool and generalizes all the classical Stokes theorem in R3, Green's theorem in R2 and so on. So, thank you.