An Introduction to Smooth Manifolds
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The Exterior Derivative 3
Lecture 62

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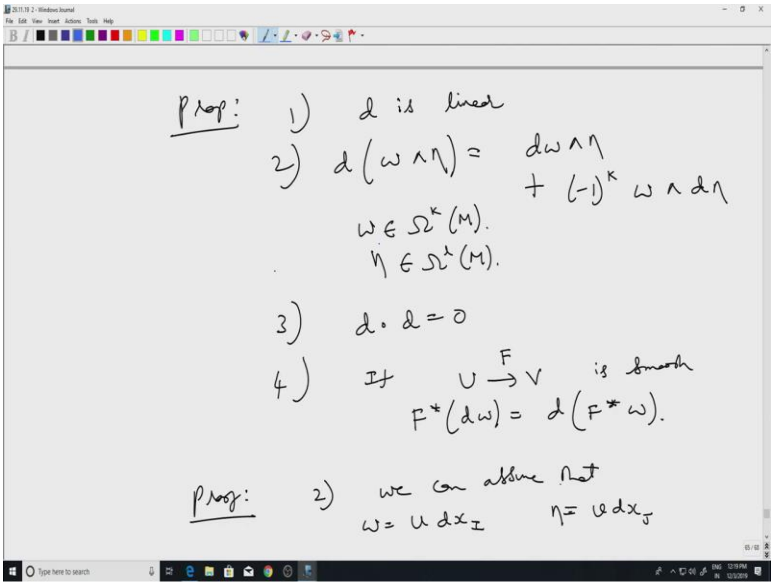

Hello, and welcome to today's lecture. Let us continue with our discussion of the exterior differentiation operator, which acts on a differential p form and the output is a p plus 1 form. Well, the very existence of the such a operator, operator or linear map, linear differential map is a theorem and towards this we first do some preparatory work on open sets in R n. So, what we needed was this proposition, we have defined d and then we have defined the exterior differentiation map on differential forms on open sets in R n. And we want to check that these properties are satisfied. So, at the end of last class, I had finished proving one, the second property. Now let us move on to this crucial and somewhat mysterious third property, d compose with d is 0.

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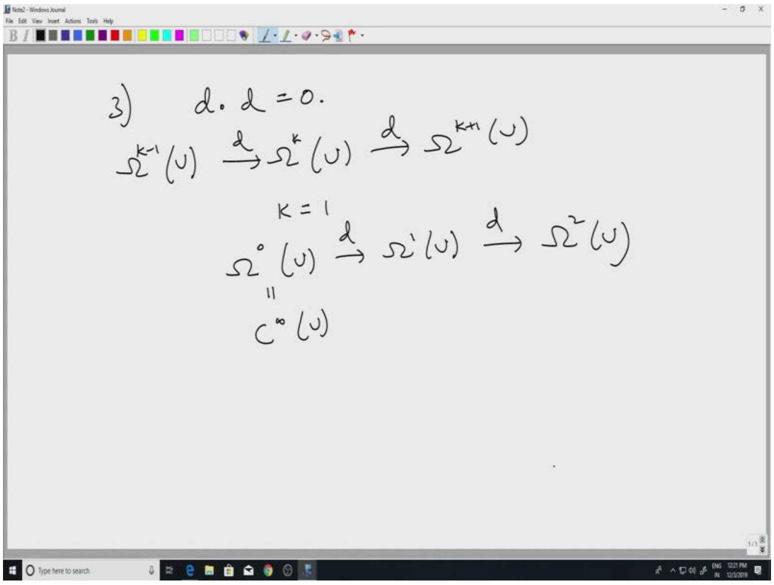

So, let me start with that. First, let us do it for 0 forms. So, remember that this d was supposed to be from omega k U to omega k plus 1 U and here I have omega k minus 1 U. The claim was the d composition is 0. So, let us start with k equals 1. So, in which case I have and recall that we had defined this to be just C infinity U, this to C infinity functions on U and so, if I want to check the d compose.

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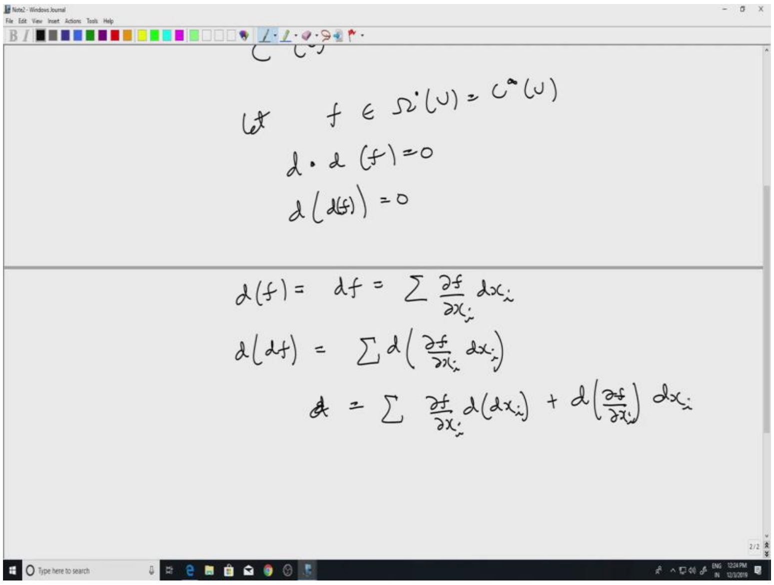

So, first what I have to do is let f belong to omega 0, U equal to C infinity U, then I would like to check whether this is 0 or not. Well, the first, so, this is d of, d of f. Now df the way we have defined d, this was just our usual derivative operator on functions and so, therefore, this is df, this is the same as del f by del xi, d xi. So, when I take d of this, we have already proved the product rule in part two. So, using that, I can simplify this expression. So, first I take it inside the summation sign. And then note that d, so, this is equal to, I will get two terms, one is the del f by del xi, d of dxi plus d of del f by del xi dxi.

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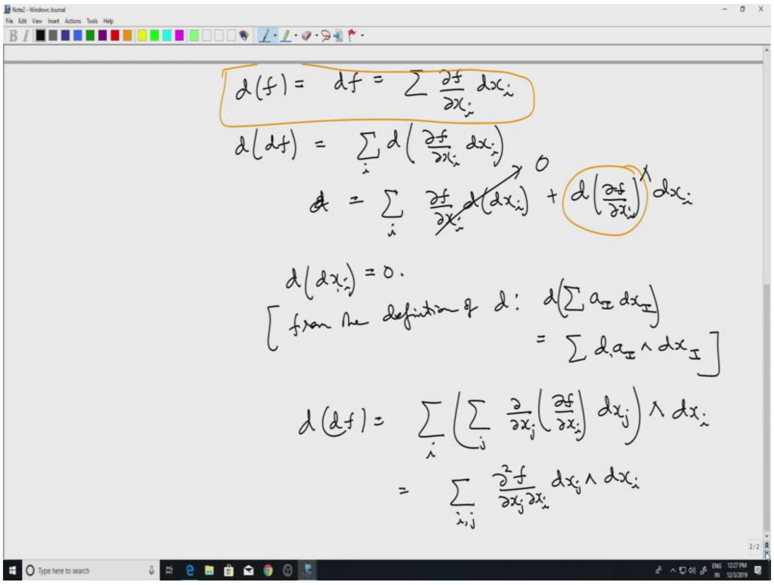

Now, well, d of dxi, d of dxi is 0. This is simply because the way we had defined d, what the way we had defined d was any form can be written as ai, dxi. And we said that this is the same thing as dai wedge dxi. which So, in other words, if these ai's are constants, then automatically d of that thing is 0. So, in particular d of dxi is 0. So, therefore, d df equal to so, this term goes away. And I am left with and as for this, this term here, again, I use this expression, except that instead of f now I have del f by del xi. So, I will have to use a different index when I take partial derivatives.

So, first, let us keep the, this sum is over i, i and then this will be over j, del by del xj of del f by del xi then dxj, wedge, Oh, here I should put a wedge. Wedge dxi and now the term inside the brackets, I will combine these two sums, the sum of both indices vary from 1 to n. And this is just the mixed partial derivative del squared f by del xj del xi, dxj wedge dxi. So, let us see why this is 0.

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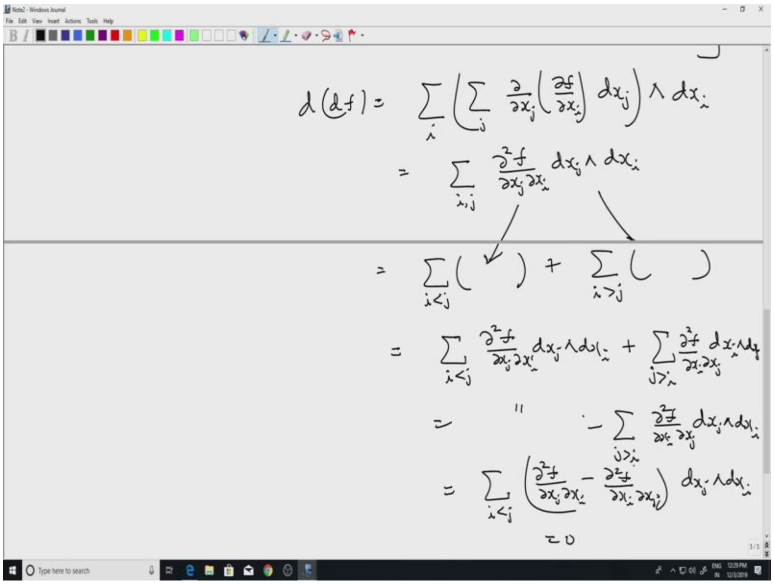

Well, when i equals j, I get dxi wedge dx, dxi wedge dxi that will be 0. So, we just have to consider two cases, i less than j of this expression plus i greater than j of this expression. So, it is the same thing here in both brackets. Now, what we do is let us just keep the first one as it is i less than j del squared f del xj del xi, dxj wedge dxi. As for the second one, I will change, interchange i and j. So, it will become j greater than i del squared. I do that so, that basically, I want to yeah, let us, we will see. So, del squared f by del xi del xj, and here I get dxi wedge dxj. Now the point is that dxi wedge dxj is the same as so, the first term I keep as it is.

The second term, del xi del xj but I interchange I swap it, swap i and j again, I changed the order of exterior product rather. So, I get dxj wedge dxi, but I obtained in the process I get a minus sign. So, now, the summation, both the sums involve summing over i less than j here towards the second one is also now i less than j. So, I can write it as i less than j del squared f by del xj del xi, the second one is del squared f by del xi del xj and then dxj wedge dxi, which is 0, since the f is smooth, its mixed partial derivatives are the same. So, ultimately, the fact that d composed with d is 0 boiled on to just the equality of mixed partial derivatives for a smooth function, at least for (one form), 0 forms.

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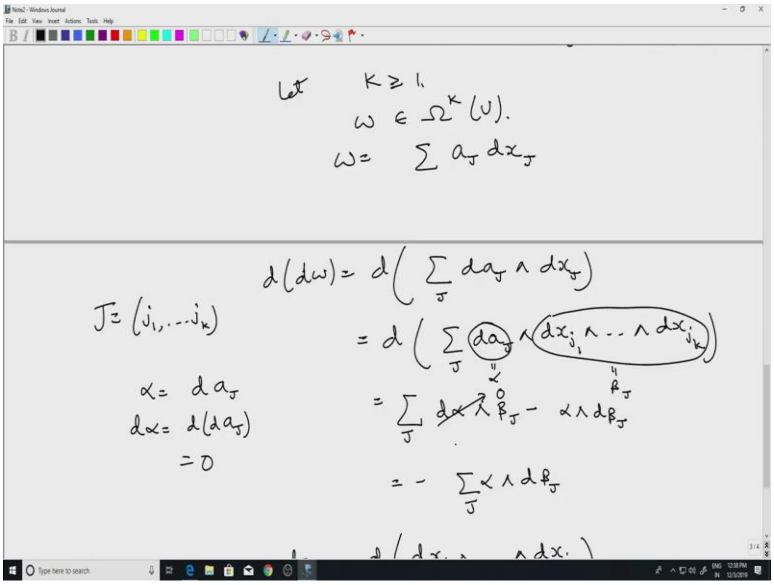

Now, let us look at any form of any, so, let k be greater than or equal to 1 now. Let us start with omega k form. And I want to so, we can write omega equals sigma aj dxj and this d d omega will be d of, so, d omega is by definition daj wedge dxj. And so, now, again, I plan to use, so, this sum is over multi indices j. So, let us, we, I plan to use this product rule, but for that let us recall that this is, I mean this is just dxj is just shorthand notation for dxj1 dxjk. So, here as usual, J is the multi index, j1, jk. So, well, now I will just use this step two where which says that I can take this d of a product is because it obeys a sort of Leibnitz rule except that the second term acquires a negative sign depending on the degree.

Well, in this case, if you, one has to apply Leibnitz rule, so, there are k plus 1 terms here. So, the first term will be, so, if I do this, let us this, let us regard this as a one form. Let us regard these two as omega alpha and beta, then I will get d alpha wedge this plus alpha wedge d of this remaining stuff. So, if I call this alpha and this is beta, beta j rather. So, then I get d alpha wedge beta j plus, well, here I will be, now this omega is, right.

So, here I will be getting a negative sign actually, because the first term here is a one form. Negative of then alpha wedge d of beta j. Now alpha is of the form d aj. So, d alpha will be where aj is, aj is just a function here. That is what we started with. And we already seen that for, so, in other words say 0 form. So, we know that by the what we just did here k equals 0 case, that will tell us that d alpha is d of, d of aj is 0. So, the previous step implies that this term goes away and you are just left with negative j alpha wedge d beta j.

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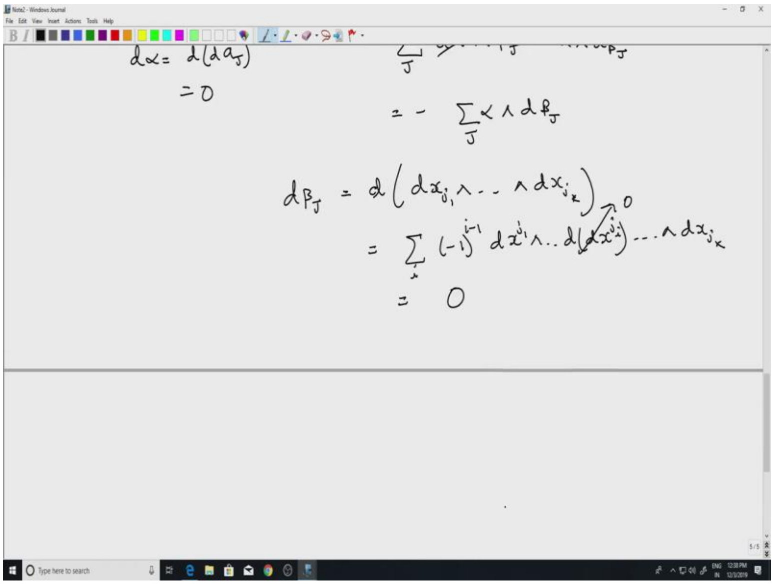

And one just has to check that each for every multi index j d beta j 0, and that's quite easy because d beta j after all, what is it, this is d of, again, it is one just keeps on applying Leibnitz rule. So, one can check that basically one will keep on getting, see the one will end up keeping all these one forms fixed and taking d of one of the in between ones. Of course one has to start with the first one and so on. And then it will just what the sign you will acquire will depend on how many forms will precede that. So, therefore, you can write this as negative 1 to the power i and then right. So, d of xj1 wedge you have to basically, you will end up taking d of xji. So, your summation is over i dot dot dot dxjk.

So, for instance, when I do the first, first one, I just get minus 1 to the power, minus 1, basically d of this actually here, yeah, it should be maybe I should change it, the index to i plus 1. I think that will be fine. So, or I might, yeah, let me write it as i minus 1 that is more clear, by minus 1. And then, yeah that makes sense. And then, right. So, one has this. And in the end, basically, one is again using this fact that if you have a, d of a one form, d of, d of a function rather then that will be 0. So, the previous step, so, all these things are 0. So, essentially using the previous step, you, one gets it for all k. But ultimately, the fact that d compose with d boils around to the fact that mixed partial derivatives are equal. So, that proves d compose with d is 0.

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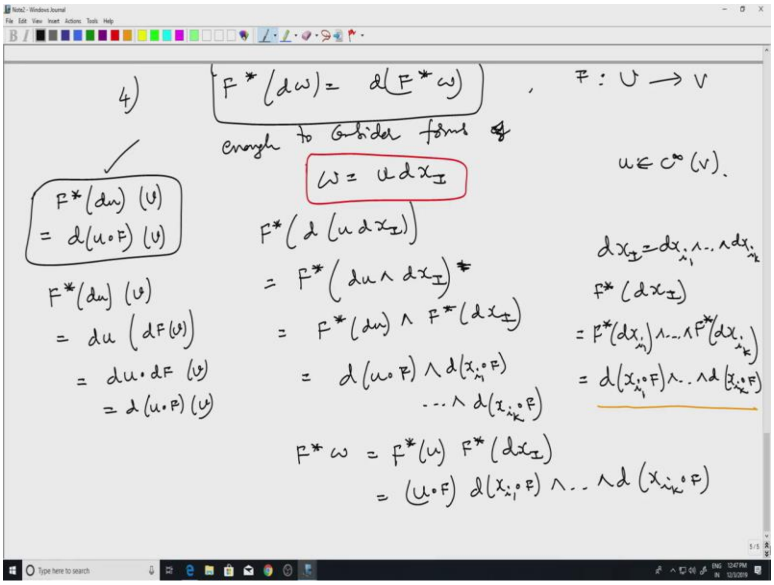

Now let us move on to the next one, which is the last one, which is the fourth one is F star d omega equal to d F star omega. Here F is a smooth map between two open sets in R n and R m, they may not be in the same Euclidean space. So, again enough to consider forms omega, which are just the function times dxi. So, here I am pulling back a form on V. So, this u will be a function on V. And the reason it is enough to consider forms of this type is the usual reason namely, any k form can be written as a sum of such forms and in this what we are trying to prove, in this expression, if we plug in, so, basically its both sides are linear in omega. Therefore, if it is true for certain omega 1 and omega 2, the equation will also be true for omega 1 plus omega 2. Therefore, we can just restrict ourselves to forms of this type.

So, let us look at the left hand side F star d omega, F star u dxi. And this will be F star, again Leibnitz rule, du wedge dxi plus, now here I will get minus 1 and then u times d of dxi which we already seen is 0. So, I will not even write it. So, the second term will be 0. So, I will just get this and well, what is this? So, this is F star du wedge F star dxi. Now F star d u is it is quite straightforward to check that if you, F star du is the same as d of u compose with F. And one can see this just by acting it on a tangent vector v.

Well, what is F star du after all? This by definition is du acting on dF of v and this is the same thing as du composed with dF of v, which is du composed with F of v by chain rule. So, therefore one has this. So, this is d of u compose with F. Well, I do not have to write the v here, wedge, now, as for this remember that dxi was dxi 1 wedge dxik. And we know that F star, the pullback behaves well with respect to exterior multiplication, this has nothing to do with manifolds or smooth maps, this was just came from the multi linear algebra that we talked about a few lectures ago.

So, when I do F star dxi, I get F star dxi 1 F star dxik. Let me write that a bit more clearly. So, this is F star dxi 1 F star dxik. So, I will have this and also, let us, from what we just this, this thing here. This is the same as d of xi 1 composed with F etcetera d of xik composed with F. So, what I end up with is, dxi 1 composed with F dot, dot, dot, dxik composed with F. So, this is the left hand side now let us show that the right hand side leads to this exactly the same expression.

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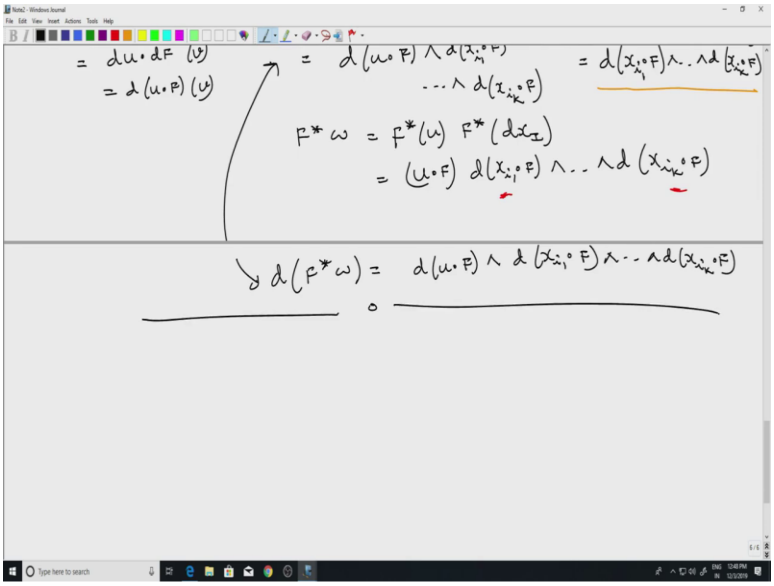

Well, F star, I have to start with F star omega and I go back to this expression here, so, this is F star. So, this is 0 form and this is a k form. So, this is the same thing as F star u, here the times F star dxi. This is the same as the pullback of a function, it is u composed with F, multiplied by again F star dxi, well I just use the same expression that I have here used this expression for F star dxi and then so, this will be dxi 1, oops, not quite dxi 1, dxi 1 composed with F, dxik composed with F. Well essentially end up with the same thing on both sides. Now, not quite, we are not yet done. So, I have to take d of this. So, d of F star omega. Now, again the same thing, so, when I have to apply Leibnitz rule k times, k plus 1 times actually.

So, but all this when I take d of this other terms, d of any of these terms this, this and anything in between, I will end up getting 0. So, Leibnitz rule just gives me two terms, one is d of u composed with F wedge d of xi 1 composed with F, well actually it gives me just one term which is non 0, which is this. So, just differentiating the function. So, and this and this if one compares they are the same. So, we have proved that pullbacks behave well with respect to exterior differentiation. So, once we have this in hand, now we can, this completes our discussion for exterior forms in Rn. Now we can move on, go to manifolds and prove the main result, which is basically the existence and uniqueness of this d operator.

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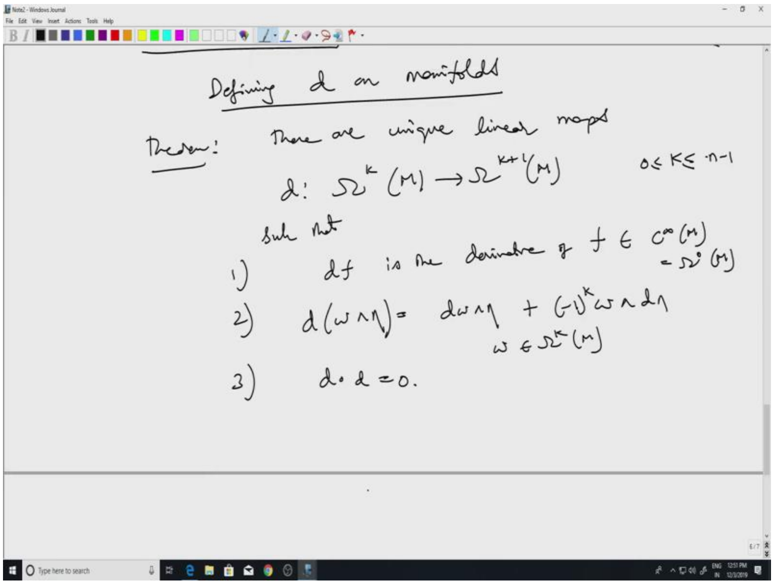

So, now, let us recall what we were trying to prove, the theorem was that there are unique linear maps d from omega k m to omega k plus 1 and so, here k varies between 0 and well, if the dimension of the manifold is m, we know that any n plus 1 form or basically any k form where k is strictly greater than n will be 0. So, I might as well say that k less than or equal to n minus 1 because here I have written k plus 1. They are unique linear maps such that df, so for a 0 form and we have the Leibnitz rule, d omega wedge eta plus minus 1 to the power k omega wedge d eta where of omega belongs to omega k m.

Third one is d compose with d is 0. So, we and we also have additional properties which satisfied by d. But these are the three main things the additional properties are consequences of these, well not quite consequences, the way they are constructed the additional properties will follow. The uniqueness just follows from these three actions. So, let us stop here. We will resume from next time.