An Introduction to Smooth Manifolds
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Lecture 60
The-Exterior-derivative-1

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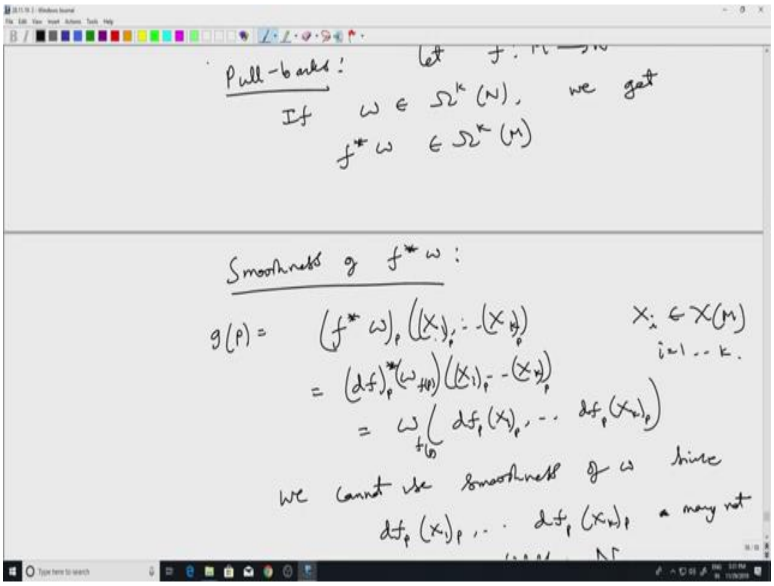

Hello and welcome to the 60th lecture in the series. So let us continue our discussion of differential forms and the various operations associated with them. So, one thing which we had for vector spaces was we could pull back differential whenever we have a linear transformation between two vector spaces, it would it gives rise to a pullback map pullback linear transformation from the corresponding spaces of alternating forms. And we can do the same thing when we have a smooth map between manifolds. The derivative is a linear map and we can use that to define this f star Omega for any k form Omega on the target.

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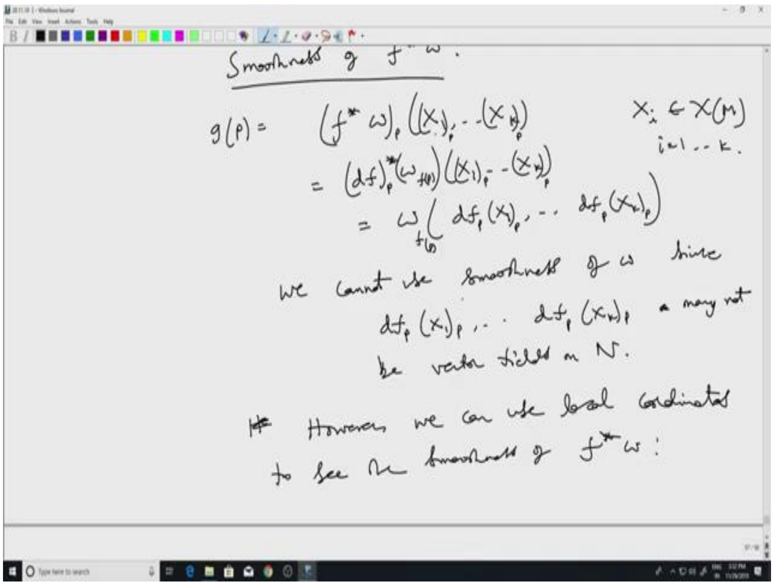

But as we saw last time smoothness becomes a bit of an issue if you try to work directly with the definition, but in terms of local coordinates, it becomes quite easy to see this so let us do that.

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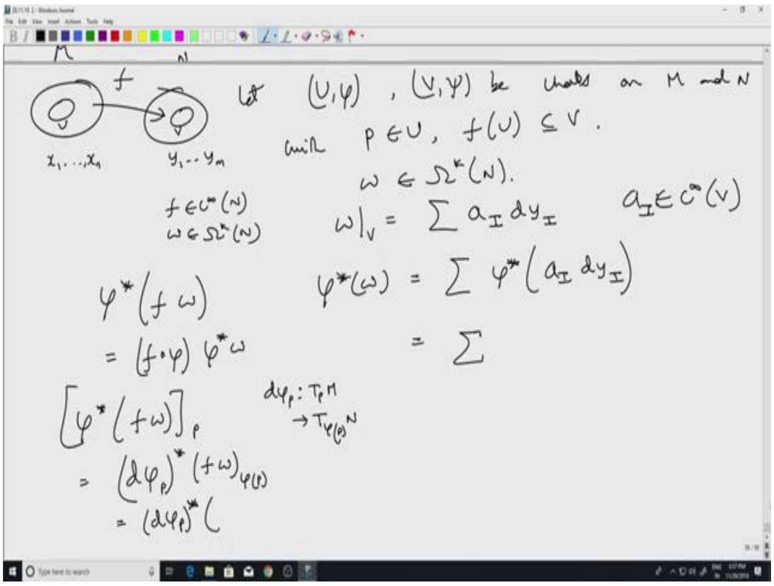

So, let U, Phi, V, Psi be charts on m and n with P in U f of p in V. Actually with P in U and more than f of p in V, I want to say that the whole thing f of u itself contained in V, charts on M and N with this property. So let us denote coordinates on this, so this is M and N and have a map f U, V. So, these coordinates here, I will denote by x 1 to x n and here y 1 up to y m. I mean they are the coordinates of the corresponding Euclidean spaces, with P in U, and f of p in V. So now what we can do, so Omega is now a form Omega k form of N, and I will restrict

Omega to this chart V, Omega restricted to V by know can be written as a I d y I where a I are smooth functions C infinity V.

Now let us apply the pullback map Phi star Omega. I will drop the restricted notation. So, let me just write Phi star Omega be Phi star of a I d y I, Phi star of a I d y I this would be, let me look at a more general thing where Phi star. Suppose I have something like this, f is a C infinity function on N, Omega is a k form on N. I claim that this is the same as f composed with Phi multiplied by Phi star Omega f composed with Phi star omega.

And the reason is, it is just that after all, what is this Phi star of F Omega at this form at a point p is by definition d Phi at P star f Omega. Now, d Phi at p this thing here, after all d Phi p is a linear map from T p M to T Phi of p N. So, the corresponding form here is being concerned as Phi of P. Now, even though I did not mention it earlier, just like for vector fields, if you ever have a differential k form, you can multiply it with a smooth function and get another differential k form.

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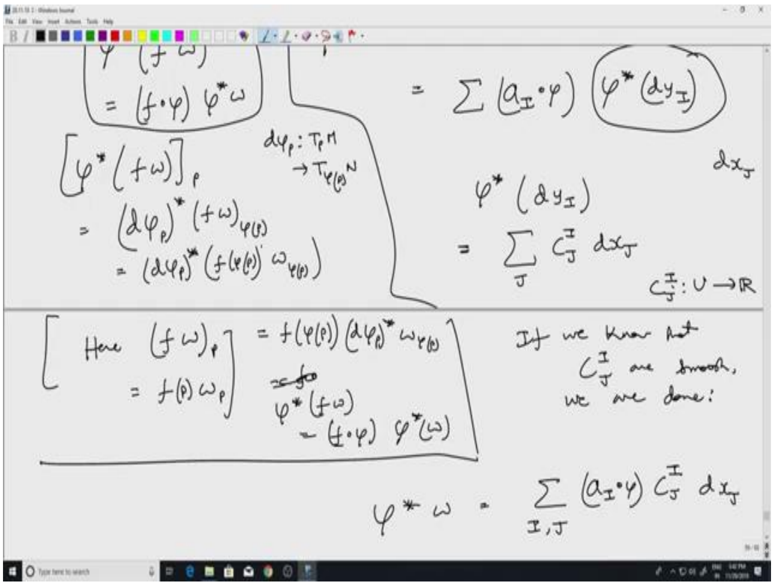

And here f Omega is its value at a point, is just f of p times Omega at p. Do the obvious thing, so here what I will be getting is f of Phi of p Omega at Phi of p. Now this is just a constant f of Phi of p so that comes out and then I am left with d start Omega at Phi of p. So, if I drop the dependence on p, convenient way of writing it is Phi star Omega f Omega is f composed with Phi times this is just Phi star of Omega.

So, that is how one gets this thing that I heard here. So using that here, I get a I composed with Phi and then Phi star of d y I. So, remember that we are trying to prove that Phi star

Omega is a smooth form one M. This does not quite show it yet, but we are almost there because to say that Phi star Omega is a smooth form, all I have to do is write Phi star Omega in terms of the standard basis for k forms, namely d x j.

So let us do that, so here I would like to express this in terms of this d x j. And then all one has to do is look at the coefficients of the final expression and claim that there are smooth functions on M and then one more to be done. Well, what is this d x Phi star of d y i? This will be some linear combination, also this will be some combination of C j, but it will also depend on I, C J I d x J. So as usual, the C I J are functions from U to R. For the moment, let us I mean if we already knew that if we know that C I J are smooth, we are done.

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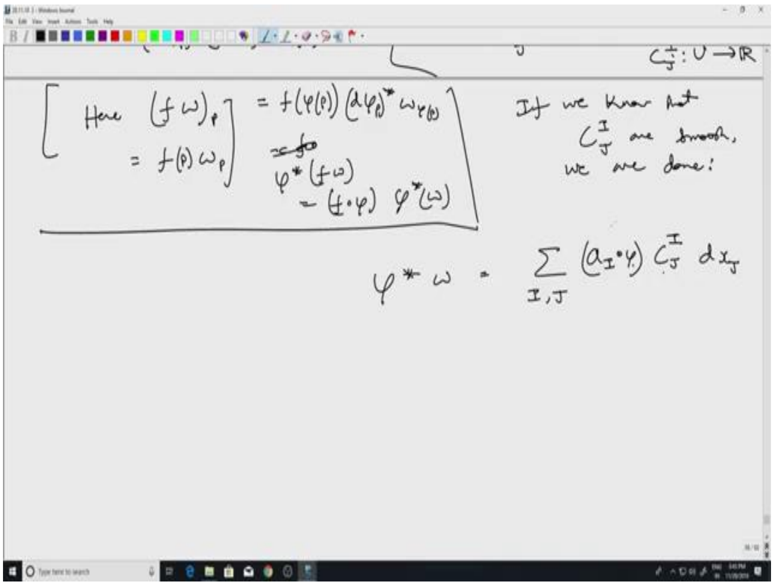

We are done in the sense that we can show that F star Omega is smooth. Why? Well, it is just a matter of plugging in so, here the summation is over J. For each of these Phi star d y I one has an expansion like this and just go back and plug it in there and what one gets is Phi star Omega equals summation over I as well as J. Now, this thing is still sticking around, a I composed with Phi, there is also this C I J dx J.

Now, as I said if you already if you know that C I J are smooth, then in each this is smooth, well a I composed with Phi is smooth because the map Phi between manifolds is by assumption smooth. And this is not F, this is Phi. So Phi is smooth, and a I are smooth therefore this composition will not be smooth function on U. And see if we knew this C H is smooth, then the whole thing would be, this product could be smooth so, one would be done.

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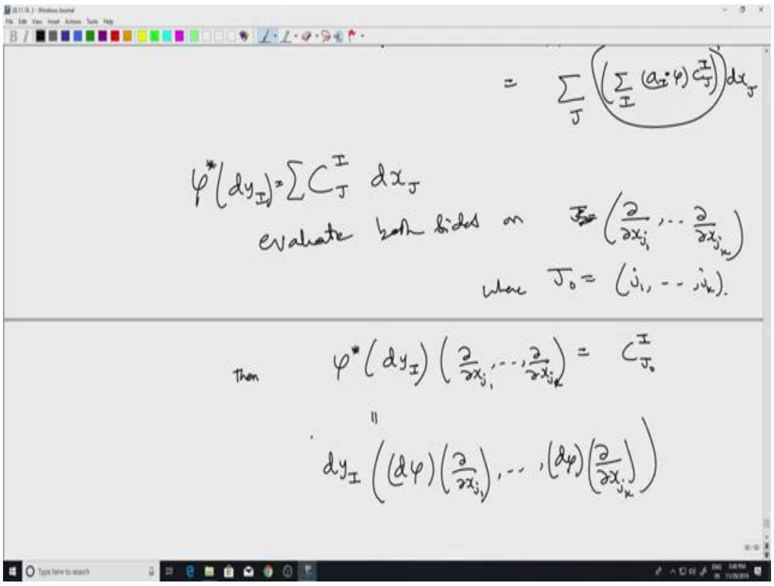

Well, actually, ideally one should write it as summation over J then summation over I a I composed with Phi C I J d x J, these are the coefficient functions one is interested in and at any rate one will have to end up showing that C I J have to be smooth. But what was C I J ? C I J we have f star, no the Phi star d y I equal to C I J d x J, this was the definition of this function C I J. Now, as we have been doing all along in order to get hold of these functions, all we have to do is evaluate both sides on.

So I want to show all of the C I J are smooth so let us do one thing. Fix I 0 and J 0, evaluate both sides on J 0 equal to on del by del x g J 1 del by del x J k, where j 0 equals J 1, J k. When we do that, all the terms in the sum disappear, except one when J equals J 0. And that will be Phi star d y I so this one evaluated on del by del x J 1, del by del x J k, this is our C I J 0, so I do not have to worry about fixing I 0, that is anyway fixed that.

So, for C I J will be this, and I am interested in whether this thing is a smooth function or not, the left hand side. Well, what is this after all? This is this thing here, it is more of the same thing. So, d y I and then Phi star just means I will be doing d Phi of del by del x J 1, etc d Phi of del by del x J k. That is what the left hand side is.

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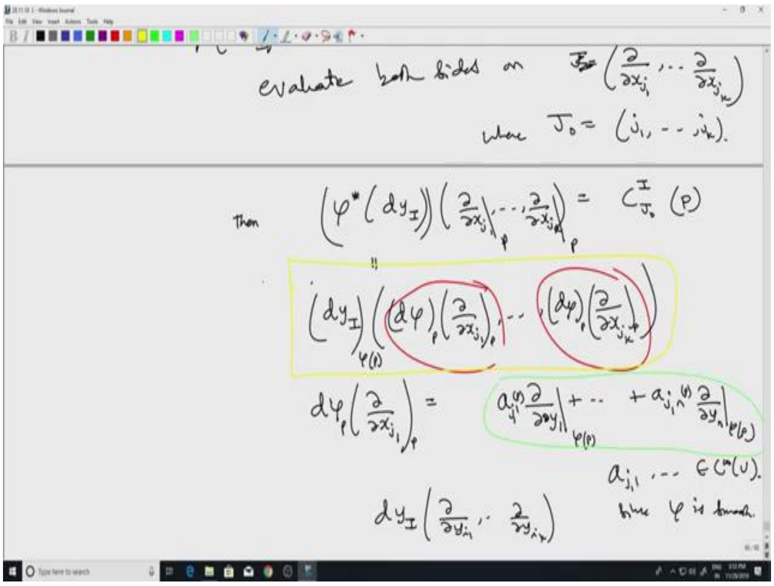

Now, d Phi of del by del x J 1 can be written as some linear combination of. So, here we are working in local targets and so d Phi of del by del x J 1, etc can be written as a linear combination of del by del y 1. So, I have to keep track of indices J 1 1, J 1 n del by del y n. Now, at this point so one has to be a bit careful about this, so let us take a point P. So, this is P, if I take a point P, this will be P, this would be P and this thing is at P.

So, this would be at Phi of P, so we are pulling it back from Phi of P, d Phi at p and this is at p, etc, this is fine. So this is P, this is P, as I am saying, so this would be at P this would be Phi of p, etc. So, the fact that Phi is a smooth map means that these coefficient functions a j 1 etc, they are all smooth functions, a j 1, 1, etc are all to C infinity U since Phi is smooth.

Now, it is quite straightforward because as before what one can do is, one can just in all these expressions here, one can plug in these corresponding expansions, use multi linearity and then again use the action of d y I, we know exactly what it does to things like del by del y 1, del by del k, etc. So, we know that this is going to be either 0 or 1. So essentially, this expression that I have here will be product of these a functions and then some sum of products of these a functions. And these are smooth, the overall thing is smooth.

So, it is actually it looks a bit complicated but it is nothing literally. One is using multi linearity at every step. So, that proves that pullbacks essentially pullbacks are smooth, that is all that we have done is prove that.

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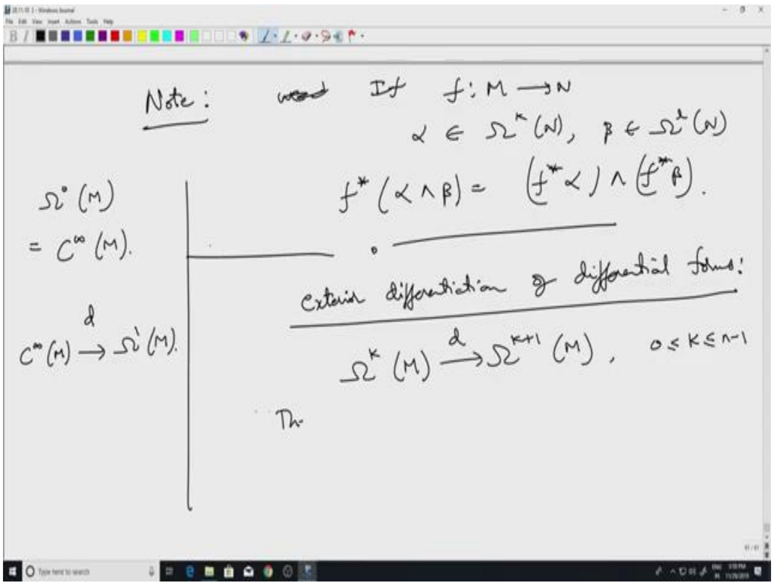

Now, we also have another thing we have is that regarding pullbacks, note if f from M to N Alpha belongs to Omega k N, Beta belongs to Omega L N, f star alpha wedge beta f star Alpha wedge f star Beta. There is no proof needed for this since it is a direct consequence of the corresponding property for vector spaces and linear transformations so this continues to hold right. Now what I want to do next is something which does not so far other than the issue of smoothness, which did not arise for vector space since there is only a single space involved.

There is other thing like exterior product, pullbacks and so on, all of these and the fact that we could get a basis for a k, the space of alternating forms, all of these things we directly carried over from vector spaces, the only new thing was smoothness and in connection with smoothness, there is another important perhaps the most important operation involving differential forms which does not arise for in the context of vector spaces and linear transformations. So this is the notion of exterior differentiation, differentiation of differential forms.

So, what we are going to define? A map a linear map from Omega k M to Omega k plus 1 M, which we denote by d. Strictly speaking, I should put a d subscript k because this map is defined. It is it obviously, when I just write d, the chance of confusion if I am using different Omega k, Omega k plus 1 and Omega k plus 2, I will be still using the same D but usually one does not put a subscript. So, let us just follow the convention and write it like this. So, this will be defined for each k between k greater than or equal to 0 less than or equal to n minus 1. Incidentally, I should mention that Omega 0 M is a space of 0 forms.

Now, by convention, one takes it to be just the space of functions on M. And the reason for this convention actually, it has to do with this d operator. So, according to what we are going to define, this d will take C infinity functions on M, to one forms on M. And we already know how to do that. Given a C infinity function, I get a one form just by taking its derivative.

So this d that we are going to define when k equals 0, it will be exactly this operation just taking derivatives of functions. When k is small than greater than or equal to 1, then we have to do something new. And so, the way we go about it is, let me first give the statement and then we will, so theorem maybe I will write it in the next page.

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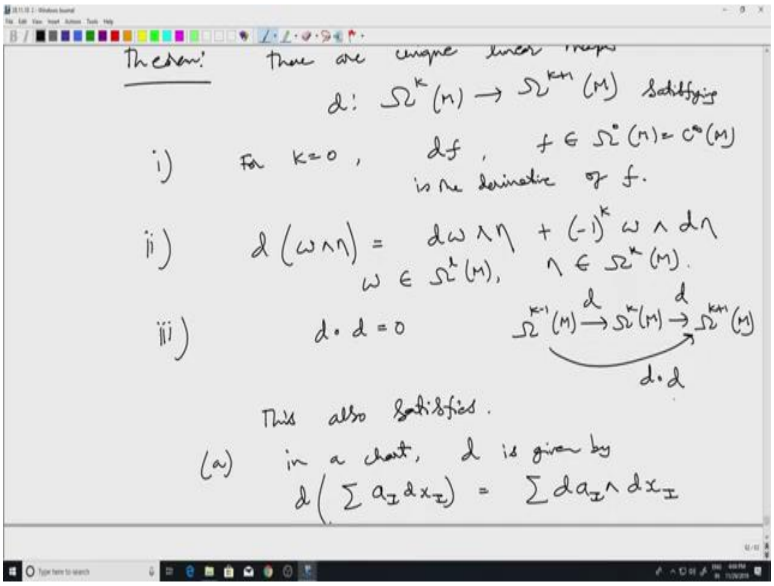

So, this theorem asserts that there is a map d like this, which has various properties. So, there are unique linear maps d from Omega k m to Omega k plus 1 m satisfying for k equal to 0, d of f, f in Omega 0 M equal to C infinity M is the derivative of f. In other words, whatever we had defined earlier, and d of a product. Now, this d is supposed to be some kind of derivative so, whenever we have derivatives we expect product true.

So here it takes this form d Omega wedge Eta plus it is not quite the Leibnitz rule a sign comes into play, Omega k Omega wedge d eta where Omega belongs to Omega L M and Eta belongs to Omega K M. So, only the sign of the, only the degree of the second term comes into play. And a crucial property of this d and the other crucial property is that d composed with d is 0. So, this is Omega k minus 1 to Omega k M Omega k plus 1 M. I have a d here and a d here.

As I said earlier, strictly speaking there are different apps in several different domains and so on. But if we use the same notation d for both, then what we want is that this map composed with this map, the resulting map is this, this map should be just identically the 0 map. Now this reason this is kind of hard to explain if you think of d as a derivative operator. So naively d composed d would be the second derivative and we would be saying that the second derivative of any k minus 1 form is 0, which does not quite make sense.

So, it is somewhat a subtle thing, but at any rate d cannot be regarded as just a derivative in any usual sense, it is something more than that. In fact, the reason for this, natural interpretation for this d compose with d arises from topology rather than the study that we are engaged in right now. This also satisfies, in a chart d is given by, in a chart we know that every form can be written in this way, then this is d a I wedge d x I. Remember, a I were functions so d a I is one form. So one form wedge, this d x I was already a K form so, this is a k plus 1 form.

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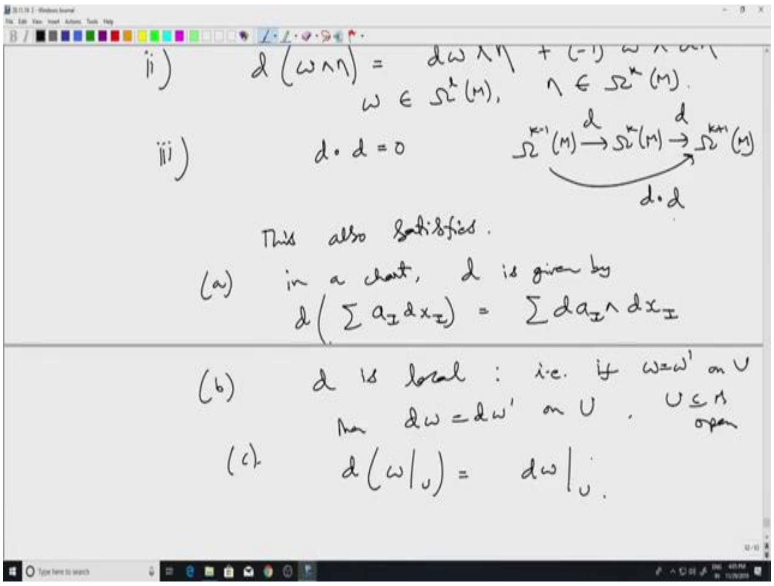

Then d is local, i e if Omega equal to Omega prime on U then d Omega equal to d Omega prime on U, so it does not matter what Omega and Omega prime are outside U, if they happen to coincide on an open set, the d will also be the same. Then d of Omega restricted to U so, here U is any open set, same thing here d Omega restricted to U as d Omega restricted to U. So I will prove this next time. We will first do the construction for R n and then get it. Alright so we will stop here.