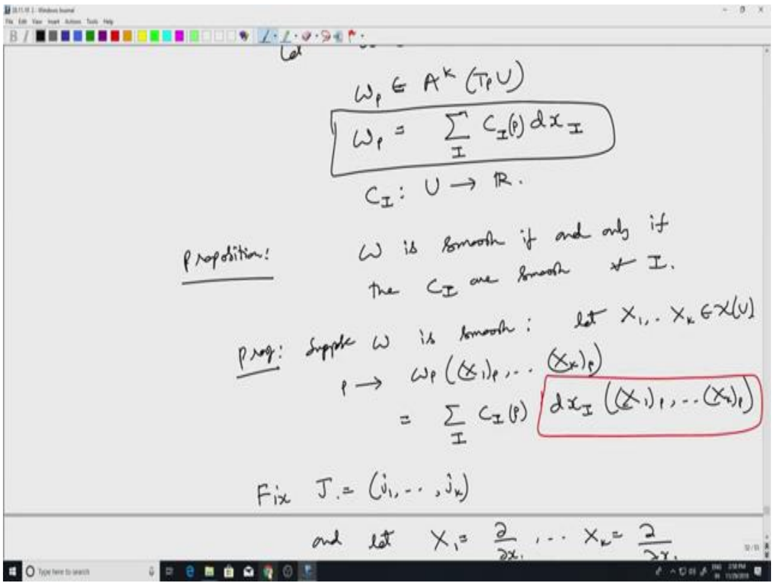
An introduction to smooth manifolds

Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru

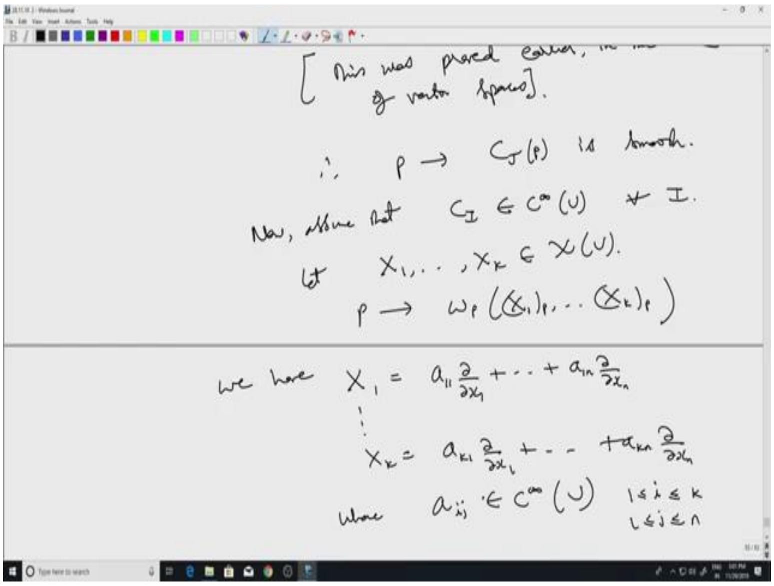
Lecture - 59

(Refer Slide Time 0:47)



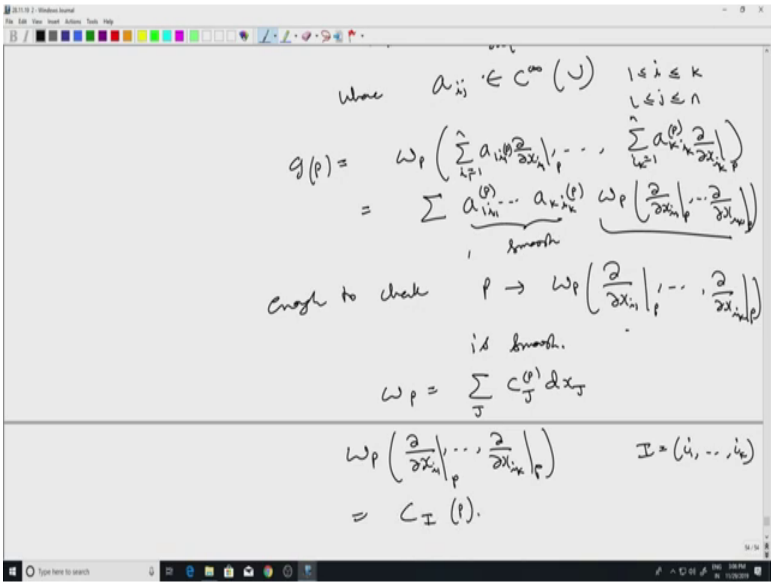
Hello and welcome to the 59th lecture in the series. Let us continue our discussion of differential forms and local coordinates. We are talking about actually we just talking about open sets and R n and we saw that here we have a simple way of checking smoothness of a differential k form defined on an open set in R n. The main observation is that we have a natural basis for the space of alternating k tensors. And in terms of the basis, smoothness amounts to saying that the coefficient functions are smooth and we are told one part of the statement now, let us prove the other part.

(Refer Slide Time 1:31)



Now, assume that C I belongs to C infinity U for all increasing strictly increasing multi indices I. So, I would like to say that and then this is smooth. So, now I have to start with arbitrary vector fields, you look at this function P going to Omega p X 1 p, X k p. Now the what we can do is, we can write each of the X i as in terms of the standard basis. We have x 1 equals a 1 1, del by del x 1 plus a 1 and del by del x 1, etc. X k equals a k 1 del by del x 1 a k 1 del by del x 1, where the and we know that the smoothness of this excise amounts to the smoothness of this functions a i j. So, these coefficient functions belong to so I varying between 1 and k, j varying between 1 and n.

(Refer Slide Time 3:38)



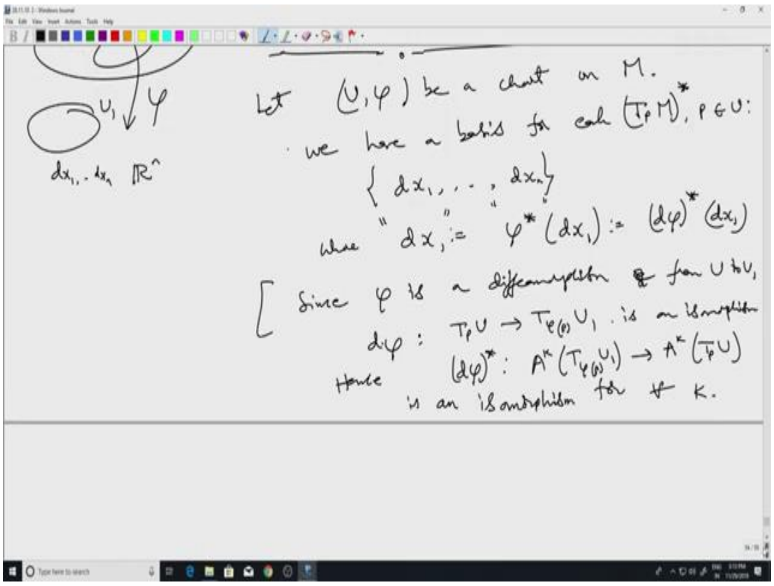
So, we have this smooth coefficient functions and then just use multi linearity. So, this function let us call it so this is called g of p. So, then g of p is equal to Omega p, instead of x 1, I just write a 1 i del by del x I, etc, a k i del by del x i. Now, strictly speaking, I should not use the same index i in different sums, but just for notational simplicity, let us just keep it like this. Now, the thing is that this using multi-linearity what one gets us this will be summation over so in fact now I will be forced to use different indices. So, here I will have to say i 1 i 1 i k i k, where all this i 1 runs from 1 to n, i k also runs from 1 to n.

Now, so using multi linearity, one gets a 1 i 1 a k i k, product of all this coefficient functions, then Omega p, so here since I have taken p, so I have to put a p here as well. So this is P, this is p, so a 1 of p and p and here as well p. So, I will end up with this product and then Omega p del by del x i 1 at p del by del x i q at p. Now, so these are by assumption these coefficient functions coming from the vector fields are smooth.

So therefore, the product here this entire thing is smooth, so one just has to check that these things are smooth. So, enough to check P going to Omega p del by del x i 1 p, del by del x i k at p is smooth. Again one is left with the similar thing so, here just like in the previous step previous calculation, you write Omega p as C I dx I again at p and then this thing is also at p, so I will not write at P, notation becomes a bit messy.

So, but if I plug in this coordinate vector fields, del by del x i k at p as in the previous step, I will end up evaluating dx I on all these things. Maybe let me just say just to not keep track of the indices. So, let me use j here as the summation multi index, so this will be, and if we call I equals i 1, i k, then dx j evaluated on these factors will be 0 unless I equals j. So, then essentially only one term will survive and that term will be C I, C I at p. So, these are smooth so this is just C I p and by assumption, we are given that C I are all smooth. Therefore, this thing here what we wanted to check is smooth, so this is just C I and one is done.

(Refer Slide Time 8:53)



So, that takes care of open sets in R n, and in the manifold setting, let U, Phi be a chart on M, again so we have just like we had a basis for the tangent space, we had given a chart we get n linearly independent vector fields defined on U so it is dimension of the tangent space is also n, these n vector fields form a basis. We call this the coordinate vector fields and we get similarly we can do the same thing for forms. We have a basis for each T p M P U namely htp and star, the dual space. So d x 1, dx n and where just like for vector fields when I write d x 1 what I mean is, so this is u and this is v, so this is mapping to something in R n.

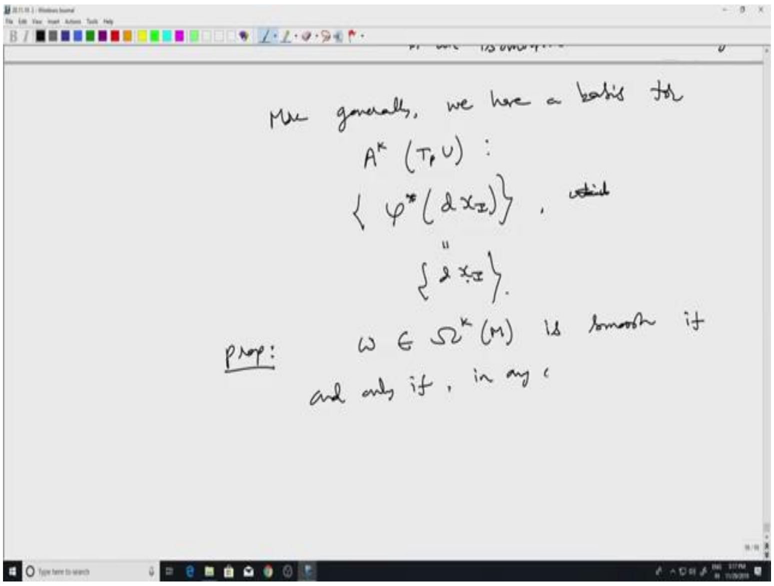
Here, I have d x 1 on R n and what I want what I am doing is, I am just pulling these back to, so Phi upper star d x 1 etc. So this notation, I put it in quotation marks, this means actually just the pullback of the corresponding d x 1 back to the. And since Phi since Phi diffeomorphism of from, so this is what we have been calling U 1 U to U 1, the derivative, the pullback map, by the way, Phi upper star here, I should clarify this notation. It is slightly misleading notation but this is standard when one Phi upper star. What one means is the Phi? So actually Phi is just a map of open sets. The derivative is a map of vector spaces, and we are using the derivative to pull back so it is actually d Phi star of d x 1.

So here also I should put quotation marks, Phi upper star is nothing but d Phi upper star, but one normally does not write d Phi upper star, one just writes Phi star. Since Phi is a diffeomorphism from U to U 1. We know that the derivative is an isomorphism from T p U to T Phi of p U 1 is an isomorphism hence, whenever we have an isomorphism of vector spaces, the pullback maps induce isomorphisms of corresponding between corresponding spaces of multi linear forms. So, d Phi upper star is in A k T Phi of p U 1 to A k T p U is an isomorphism for all k.

So therefore, the point is that tr in particular for one forms. If I start with the basis for the dual space of the tenant space in R n and I pull it back via d Phi, I get a basis corresponding basis for each T p and star. And in fact we can do this for since this map is here is an isomorphism, we can also conclude more generally, we have a basis for A k T p U namely Phi upper star where Phi upper star again, I mean d Phi upper star, Phi upper star of dx I, which we write as.

So these things again I just write as just d X I. So in short, we use the same notation, even though we are on the manifold, we use the same notation as if we were, as long as we are inside the chart we will use the same notation as if we are on open subsets of R n.

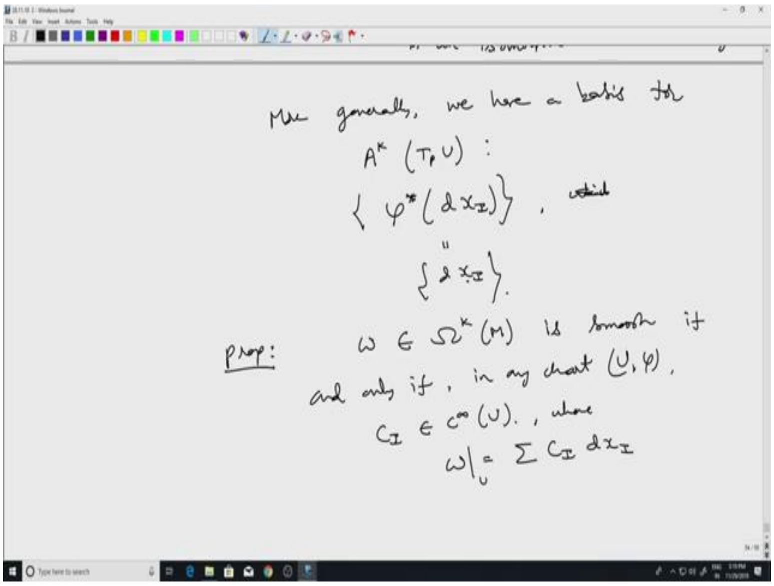
(Refer Slide Time: 16:58)



So everywhere however, if you want to know what it literally means, there is this chart map involved which is omitted in this standard notation and you just write d X I. So, if we carry over this discussion whatever we proved for open subsets of R n, then at least we can see that we have the following proposition Omega and Omega k m is smooth if and only if, of course this notation and this statement is kind of odd because when I write capital Omega k m, automatically I am assuming that Omega is smooth.

So, what I want to say is that any assignment from k to this thing, any assignment P 2 alpha p in L k T P M or what we are talking about is A k T P M is smooth if and only if something happens and that something is exactly what we had for open subsets of R n.

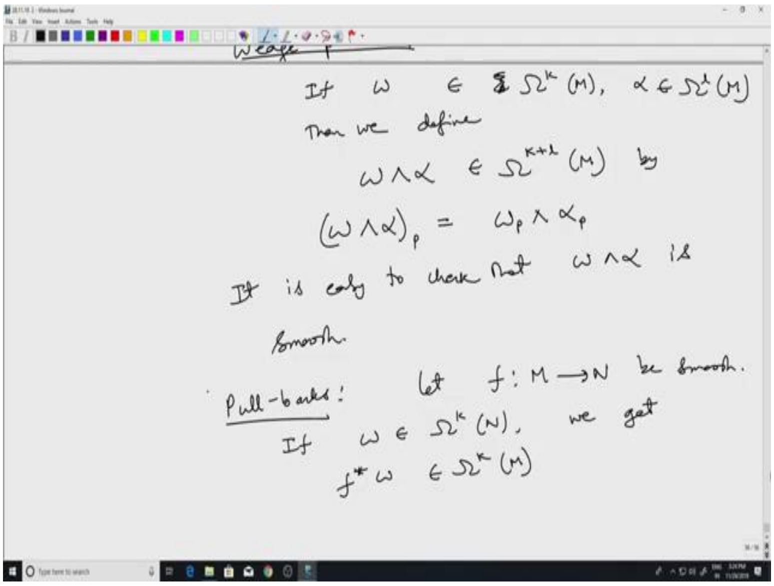
(Refer Slide Time 18:36)



If and only if any chart U Phi C I are smooth where Omega is C I dx I. Omega restricted to you is C I d X I. So, I will not prove this and I mean the prove I have already, there is nothing much to prove. Essentially, we have proved the statement for open subsets of R n. And what we have to observe is that when we have a diffeomorphism, the smooth forms will go to smooth forms.

So, you pull back essentially, if you start with the smooth form on the manifold restricted to the open set, then you pull it back to U 1 using Phi inverse which goes in the opposite direction, and then this expansion of Omega in this form, so you will be applying Phi inverse d Phi inverse star to everything here. And the way d X I was defined when you apply Phi inverse star, actually you will get the actual d X I on R n and then you will end up concluding C I composed with Phi inverse is smooth and so on and the other way also is similar. So, this is a useful criterion to have.

(Refer Slide Time 20:49)

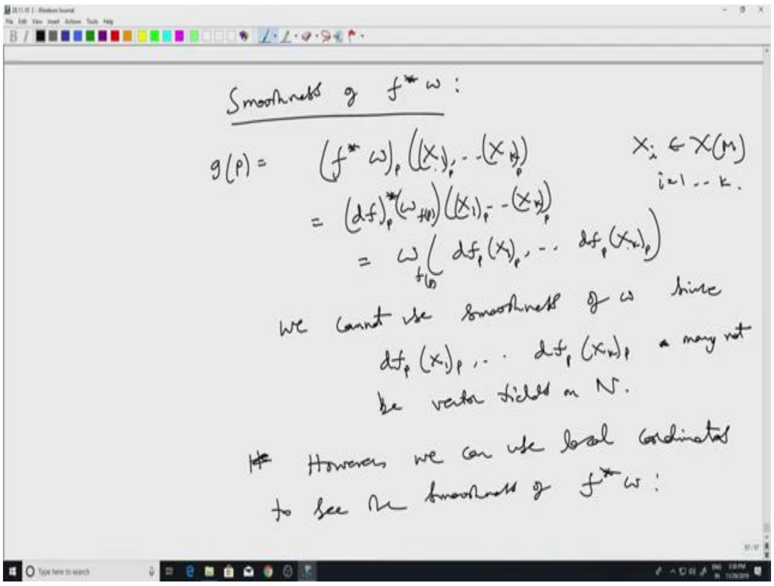


And then we also had these operations tensor and wedge products, Omega alpha belongs to let us say let us just stick to wedge products right now, I will have no occasion to talk about tensor products anymore, for us tensor products, wedge product. So, if I start with two k forms Omega k M, then we define Omega wedge alpha, this is a new, this is a, actually I do not need them to be the same Omega n this thing, Alpha and Omega L of m. So, this is k plus L of M by so, I have to say what its value is what I get at a specific point. So, I just defined this to be Omega at p wedge Alpha at p.

So, Omega at p will be a alternating k form on T p M and Alpha t will be an alternating L form, so I can take the wedge product. It is easy to check that Omega wedge alpha is smooth. Now, this can be done either using the local using a coordinate chart and expressing Omega and alpha in terms of the charts or one can directly go to the definition and use vector fields as well. Both ways will show that this is smooth.

So one has this, now the other thing is that we have the pullback operation, let f be a smooth map between manifolds. If Omega belongs to Omega k N, we get f star Omega in Omega k M. And again, what one is doing is, one is using the derivative to pull it back from here to here.

(Refer Slide Time 25:01)



Now, at this point I have to be, there is one small subtlety which is the smoothness of f star Omega. This subtlety arises because of the following reason; suppose I want to go to the go by the definition of smoothness, then I will have to plug in k vector fields on M. So each of these X i are vector fields on M. Well, this by definition, first of all let us write down what this really means, it is d f upper star Omega acting on X 1, X k and by definition, this is supposed to be or Omega d f acting on X 1, d f acting on X k. Now here comes the problem that we know that Omega is a smooth form on n.

So if you plug in k vector fields, I should get something smooth a smooth function on n. So actually, let us see let us take a point p here, p equal to f star p X 1 p, etc, X k at p. So, this would be by definition d, f of p and I would be pulling back the form at f of p X 1 of p X k at p and this is Omega at f of P, d f of p X 1 p, well, let me just write it like this at p. So, as p changes on M, I would like to say that now I know that Omega of when the input to Omega or smooth vector fields on n, I know I get something smooth on n.

However, the problem is as we have seen before, X 1 and X 2 all the way up to X k are vector fields on M, but when I do of those vector fields, I may not get actual vector fields on N. After all, the extreme case of f being a constant map will show what can go wrong. So in short, the input here on this right hand side for Omega are not actually vector fields on the target. So, we cannot directly use the definition of smoothness of Omega.

But we cannot use smoothness of Omega since d f p X 1 of p, etc, d f p X k at p may not be not be vector fields on n. For each p of course I get a tangent vector and to some tangent space on n, but I may not get a vector field. So, the problem is easily overcome just by working with local coordinates local coordinates to see the smoothness of f star Omega. So let us stop here. In my next lecture, I will talk a bit more about this. Thank you