An Introduction to Smooth Manifolds
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Lecture 58 - Differential forms on Manifolds 1

Welcome to the 58th lecture in this series.

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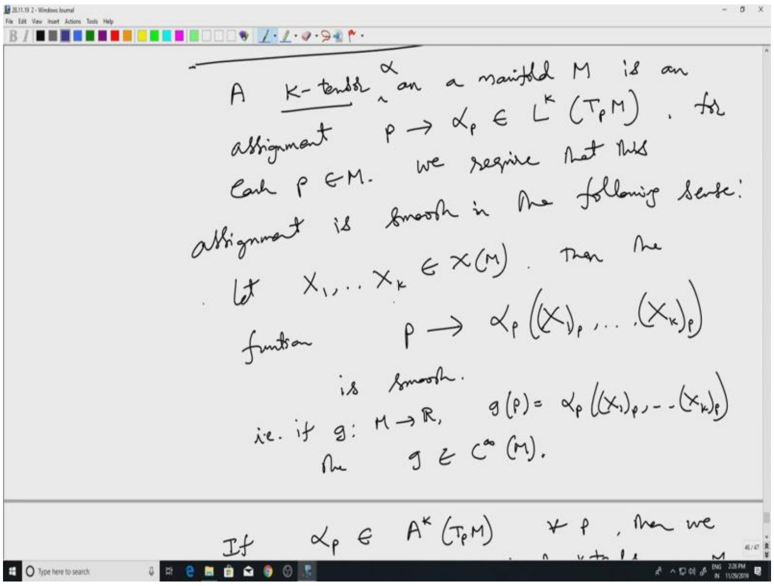

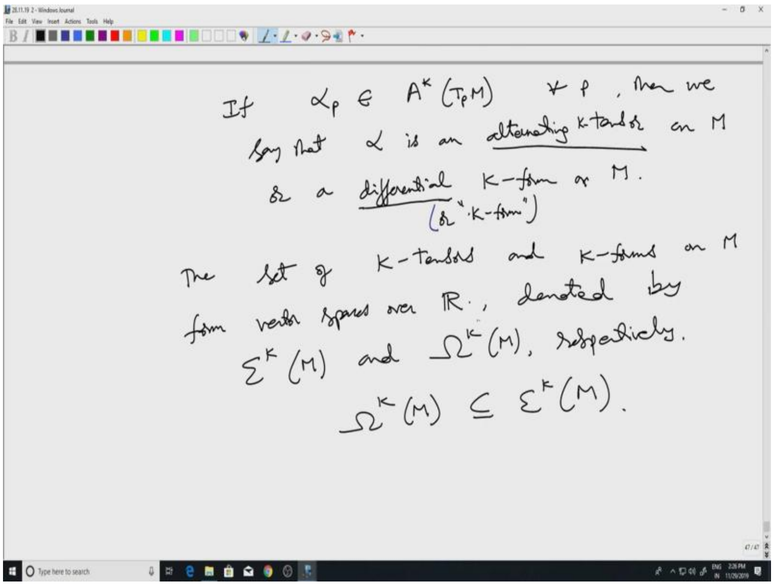

Today, we will be, continue our discussion of tensors and differential forms of manifolds. So last time, I started this topic a K tensor was at each point, one was assigning a multi-linear k multi-linear form on the tangent space. And as usual, we would like this assignment to be smooth, so the way we impose this condition is we insist that if we start with any k vectors fields or the smooth vector fields on the manifold then the function p going to alpha p x 1 at p etcetera, this should be a smooth function on the manifold.

And if we start with an, if alpha p is actually an alternating tensor at each point, we say that alpha is a differential k form on M. And normally, one just says k-form, so a k-form refers to a differential k-form which is the same thing as an alternating k tensor. So the set of k tensors and k forms on M form vectors spaces over R denoted by epsilon k M and omega k M respectively and we have this one as a subset of the other.

And so let us, the first thing we want to do is that a firstly we sort of recast this the definition of smoothness in terms of local data. And the analogy with, so whatever we did for vector fields pretty much the same thing, the same procedure is being followed here. And, so let us do this now, so local. Perhaps before I do that, maybe I should give an example of a differential k-form.

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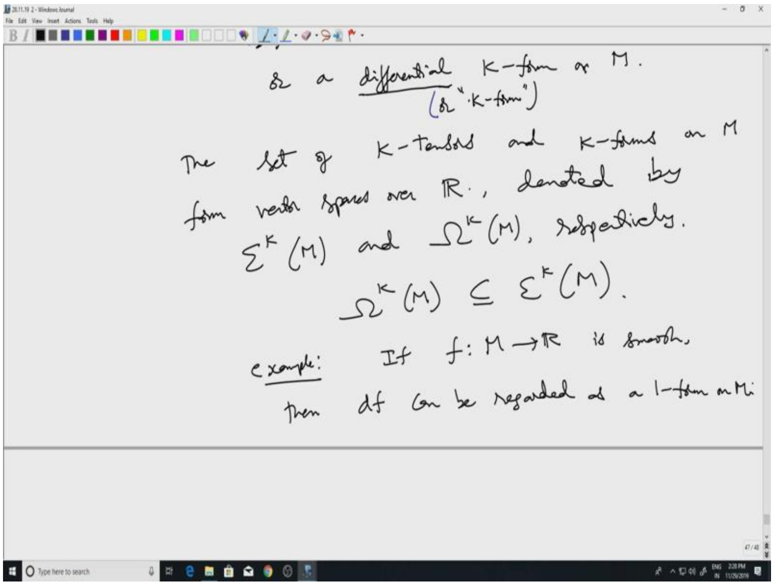

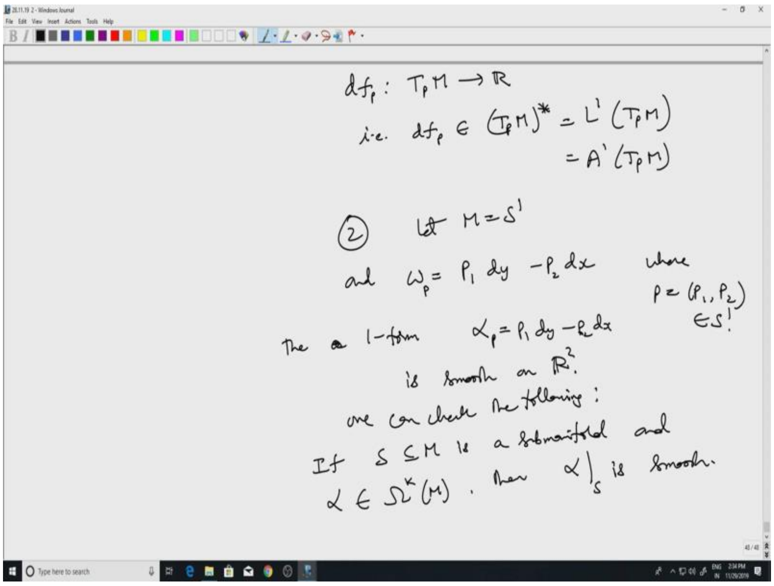

If f from M to R is smooth then the derivative of f can be regarded as a 1-form on M. Now here I say can be regarded and I use the term regarded simply because well, the derivative is actually a linear map from the tangents at any point t P M to t P R, t f of P R. But, we know that the tangent space at any point to Euclidean space can be identified with that Euclidean space itself.

So if we use that identification then the derivative at any point becomes a linear map from t P M to R. So which is the same thing as saying i. e. d f P is an element of the dual space at each point which is L 1 or the same thing as A 1. The alternating condition is vacuous in case of 1-forms, so every 1-form is automatically, every one tensor is automatically alternating. So this is all automatic, so the moment you have a smooth function the derivative is a 1-form, gives a 1-form on the manifold.

And we know that by taking wedge products we can get higher degree forms as well. Now let us do one more thing, so let M equals S 1 the circle and here I will define the form and omega equals, so omega at the point P is x dy minus y dx, where P equals x, y where is the point on the circle. Now the thing is, maybe I should just to be clear, let me use slightly different notation here. If I use P1, P2 then this would be P1, P2 and these two things, dy and dx.

And so the, if one want to check that this is smooth then as in the case of vector fields if you have object on the sub manifold, if you have a vector field on the sub manifold and one wants to check it smooth, if the vector field happens to be the restriction of a smooth vector field on a bigger manifold then we know that it is smooth. Well, same thing here, this form the 1-form alpha P is equal P 1 dy minus P 2 dx is smooth on R 2. And this can be seen directly just by plugging in.

We can go back to the definition of smooth vector field, plug in vector field inside this and one can directly see it is smooth. In fact, it is easy to see that k-form on R M is smooth if and only if certain coefficients that one gets then one writes in terms of the standard basis for k-forms when these coefficients are smooth. Well, I will come to that but for the moment let us just say that this is smooth on R 2, one can check the following if S is a sub manifold and alpha is a smooth k-form on M, then the restriction of alpha to S is also smooth. So this can be checked and hence one would conclude that this is smooth as well.

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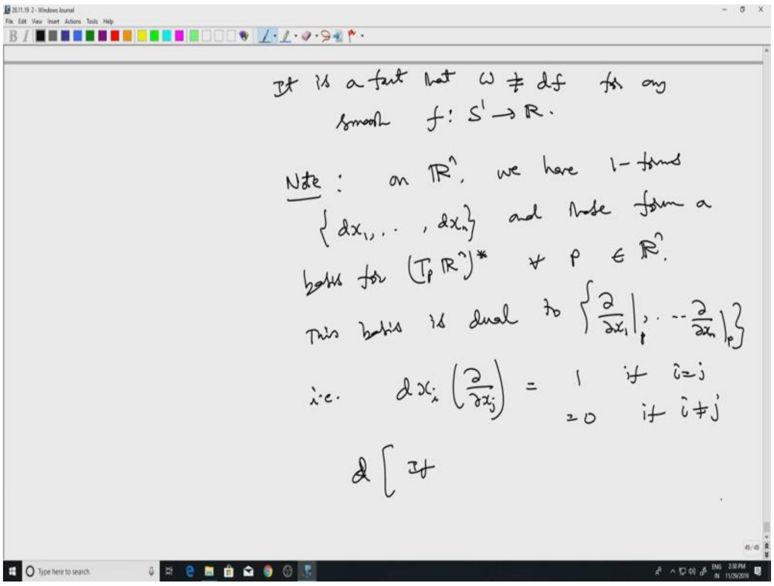

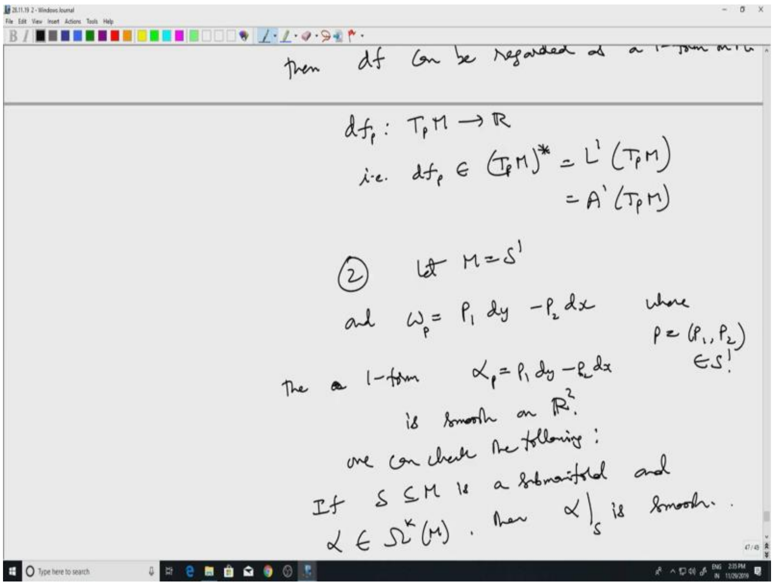

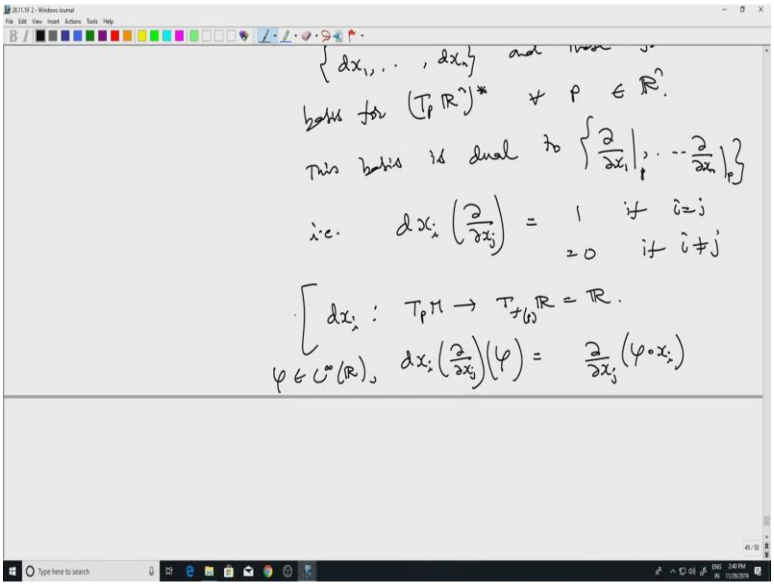

Now the interesting thing about this 1-form is that it is fact that alpha is not equal to d f for any smooth f from S 1 to R. Actually, fact that what I call here, I call omega is not equal to d f or any smooth from S 1 to R and this so, in other words in the first example, we saw that any smooth function gives rise to a 1-form. But here we have a 1-form on S 1 which is not d f for any smooth function f.

By the way here when I wrote omega P equal to P 1 dy minus P 2 dx, this dy and dx are 1-forms on R 2 which exactly come from example 1. x and y the coordinate functions give rise to dx and dy, we can, so we get a global 1-form on R 2 and we can restrict it to this thing S 1 and we get this.

It is fact that, so we should note on R n we have 1-forms dx 1 dx n and the point is and this form a basis for t P R n star for all P and R n. So just like we had vector fields which form a basis for the tangent space at each time, we have 1-forms which form a basis for the dual of the tangent space at each point. And in fact, these are just the dual, this basis is dual to del by del x 1, the standard basis del by del x n at P. Perhaps here I should write and this form a basis and this basis is dual to this i. e. del by del x i evaluated on del by del x, dx i evaluated on del by del x j is 1 if i equal to j equal 0 if i not equal to j. And this is almost immediate definitions because if I see after all the dx i, so if I want to see the validity of this, after all how is this dx i defined?

dx i is supposed to be a map from T P M to T as I said earlier T f of P R equal to R and so dx i of del by del x j would be, actually so this would be just an element of t f of P R. And if I acted on an infinity function, think of it as a derivation for a moment, act it on c infinity function, so f belongs to c infinity, phi belongs to c infinity R. Then this is by definition, it would be del by del x j and then phi composed with the function x i.

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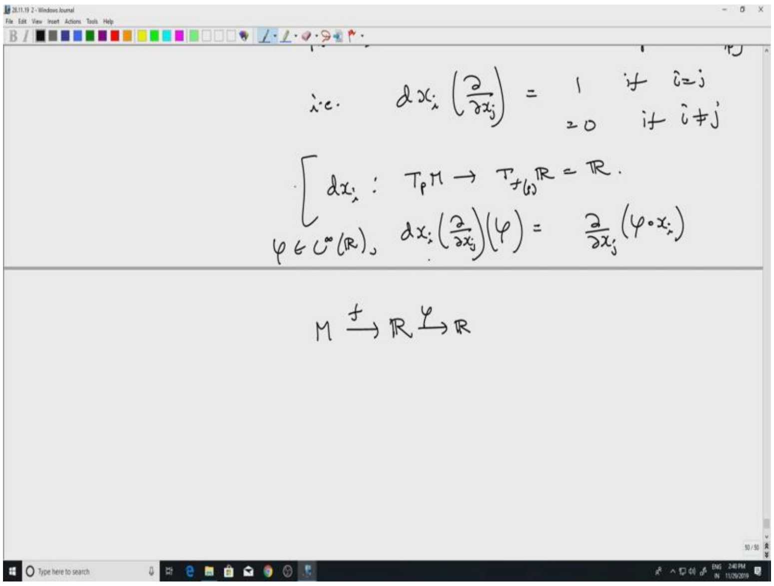

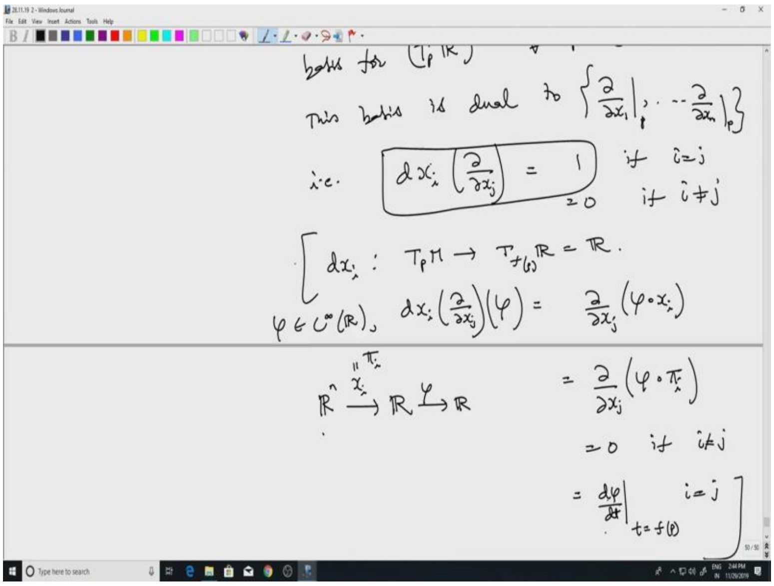

So here phi from M to R we have f and then from R to R we have phi. So sorry, this is into R, no, this is not f, this is and here also I should change it slightly. So from R n to R I have this function xi which I also referred to as actually the projection pi I, when I say x i it is that the projection function. And then I have phi. So when I want to do dx i of something here acting on a function, I get exactly at this del by del x j of phi composed with x i.

Now this is as I said, this x i refers to p i j, no p i . Well, this if i is not equal to j then it is clear that this is 0, I mean this is 0 if i is not equal to j and if i equal to j then this thing would be, it is if i equal j, it would be just the partial derivative of phi at del by del x i or del x j, does not matter, i equals j. So in other words, not quite the, sorry not quite the partial derivative, it would be just the, it would be just phi d phi by dt at the point t equals f of P.

And using the identification that we had, that is just saying that this after all 1 is the same as with the usual identification of tangent space with the Euclidean space. In the case of R, 1 corresponds to the derivation d by dt, so it is, it would be this. So this is a small thing and then also….

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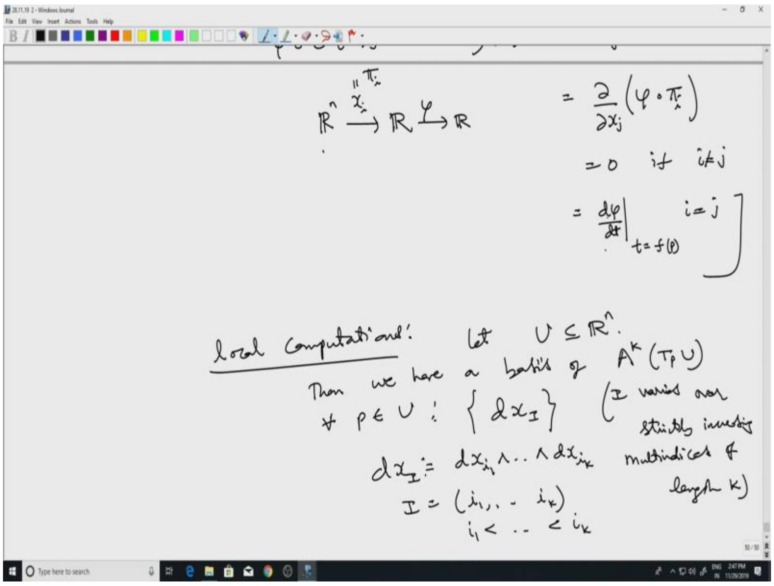

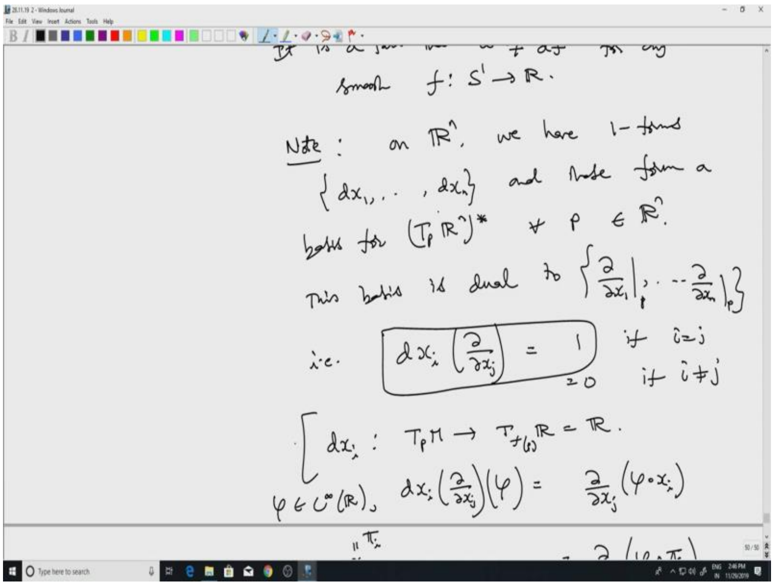

Now let us move on to local computations. So let U be an open set, so first we will work exclusively with open sets subsets of R n, then we will use charts to transfer the information from this open subset to back to the manifold. If U is a subset of, open subset of R n, then we have a basis of alternating K-forms on the tangent space of U for all P and U.

Well, this is, we, the previous example shows that we have a rather this remark, note in this note we have seen that this dx 1 etcetera gives the basis of the dual space and we know that in the last few lectures we have been seeing that once we have a basis for the dual space then we get a basis for K-forms.

Actually, whatever I am saying now could be carried over to tensors as well, but let me just stick to forms. So we have a basis for A k this thing, namely you just use the dual basis so and then use a multi-index, dx I, I is a strictly increasing so I varies over strictly increasing multi indices of length K. And this notation dx I, which I will be using later on is stands for dx i 1 wedge dx i k. So here I equal to i 1 i k and the strictly increasing condition is just this.

So this is from this stuff that we have done earlier. Now so this is what a basis for, so we get a basis for A k T P U. Well, what now what does a general differential K-form normally look like?

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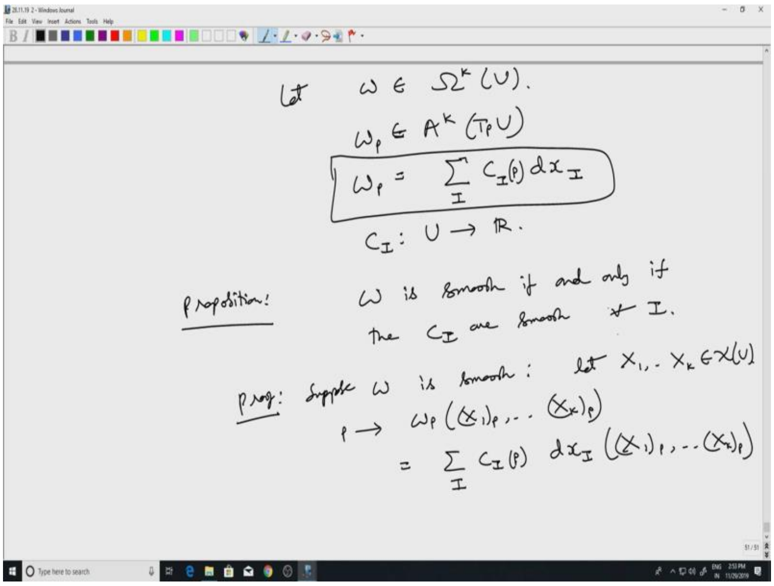

c_I ...

Proposition: ω is smooth if and only if the c_I are smooth $\forall I$.

Proof: Suppose ω is smooth: let $X_1, \dots X_k \in \mathfrak{X}(U)$

$$p \to \omega_p((X_1)_p, \dots (X_k)_p)$$

$$= \sum_I c_I(p) \boxed{dx_I((X_1)_p, \dots (X_k)_p)}$$

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assignment is smooth ...

let $X_1, \dots X_k \in \mathfrak{X}(M)$. Then the

$\boxed{\text{function } p \to \alpha_p((X_1)_p, \dots (X_k)_p) \text{ is smooth.}}$

i.e. if $g: M \to \mathbb{R}$, $g(p) = \alpha_p((X_1)_p, \dots (X_k)_p)$

then $g \in C^\infty(M)$.

If $\alpha_p \in A^k(T_pM)$ $\forall p$, then we say that α is an alternating k-tensor on M or a differential k-form on M.

(or "k-form")

The set of k-tensors and k-forms on M ...

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i.e. $df_p \in (T_pM)^* = L^1(T_pM)$

$= A^1(T_pM)$

(2) Let $M = S^1$

and $\omega_p = p_1\, dy - p_2\, dx$ where $p = (p_1, p_2) \in S^1$

The ~~a~~ 1-form $\alpha_p = p_1\, dy - p_2\, dx$ is smooth on $\mathbb{R}^2$.

One can check the following:

If $S \subseteq M$ is a submanifold and $\alpha \in \Omega^k(M)$, then $\alpha|_S$ is smooth.

It is a fact that $\omega \neq df$ for any $f: S^1 \to \mathbb{R}$.

So let omega belong to this, so it is a smooth K-form on the open set U, what that means is that, at each point it is an omega each P by definition belongs to A k T P U. Now since we have a basis for A k T P U, I can write omega equals some coefficient C I dx I, now the summation is over, all strictly increasing multi indices of length K.

Now the point here is that as P changes over U, these coefficients can change. So actually, I have to put C I P and then dx I. So in effect, the CI are functions from U to R. Now if you recall how we went about rewriting the definition of smoothness of a vector field in local coordinates, it amounted to expressing the vector field in terms of some standard vector fields and then saying that the coefficients in such an expression are smooth. Here too the same thing holds.

The smoothness of omega in the sense that I defined here this smoothness, so this thing here is amounts to saying that in the case the manifold is an open set U in R n, it amounts to saying that C I are smooth. So let us write that it is a small proposition. Omega is smooth if and only if the CI are smooth for all I.

And this is what I was referring to when I was talking, when I discuss this the case of the special one form on S1 or in fact, the form from which it arises, namely the form alpha P equal to P 1 dy minus P 2 dx, I said is smooth on R 2. And the point here is that the coefficient functions here are just the projection functions P 1 and the negative of the projection function P 2 which are smooth on R 2. So that the smoothness which is asserted there follows a corollary of this more general statement, where any K-form and you write like this it is smooth.

Well a, so let us see why this is the case. So we have to start with suppose omega is smooth, suppose omega is smooth, let X 1, X K be smooth vector fields on U. I am supposed to consider this function P going to omega P, X 1 P….X K P. Now, this is the same thing as, let us use this thing here, this is same thing as summation over I C I P dx I X 1 P X K P.

So in effect, we just have to check that this it is enough to conclude that these functions are smooth. Since, well not quite, I mean I already, I am assuming that omega is smooth and I would like to conclude that this C I is smooth. So in that case, so I want to get hold of this C I of P. So we already know that this assignment is smooth and from that I want to get smoothness of C I. So here the point is that this is supposed to be smooth for any choice of vector fields.

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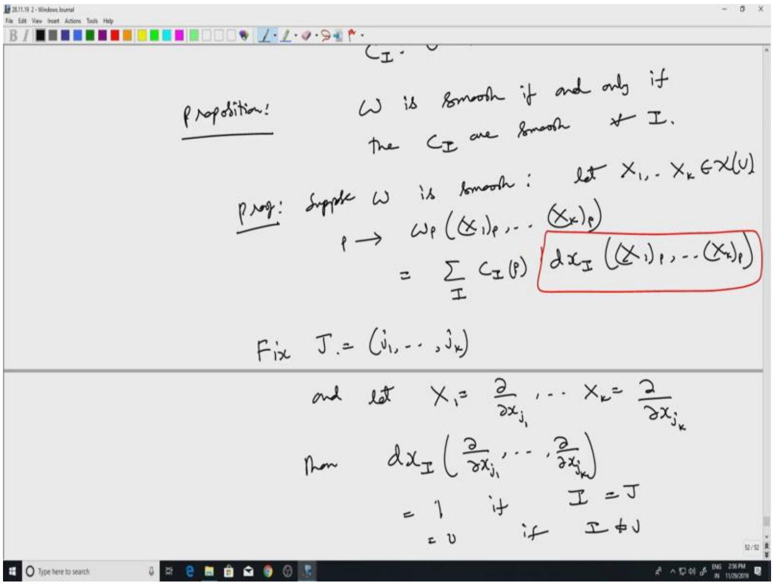

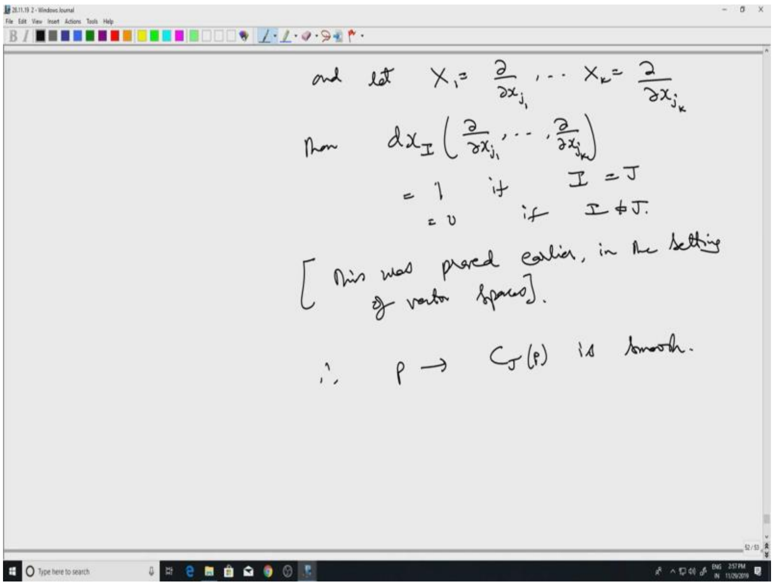

Let us take, let X 1 equals, so let us start with some, I want to prove each of these C I R smooth, fix a, so let us fix a multi-index. Fix a multi index J, j 1, j k and let X 1 be just del by del x j 1, x k equal to del by del x j k. Then we have dx I acting on del by del x 1 del x j is equal to 1 if I equal to J equal to 0 if I is not equal to J.

And this is something we have seen earlier whenever we have a basis for 1-forms. And if you look at the corresponding dual basis for the vector space then one has this formula here. So this was proved earlier in the setting of vector spaces. The moment you have this, you are pretty much done because what one, all the terms here are 0 except for the one where the multi index I equal to J, therefore and then I will just get 1 on the this expression.

And so I am left with….So therefore, we conclude that P going to C J P is smooth, and that is precisely what we wanted. So let us stop here. In the next lecture, I will just talk about the other direction and then move on to more general other properties of differential forms. Thank you.