An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 56: Alternating Tensors 8

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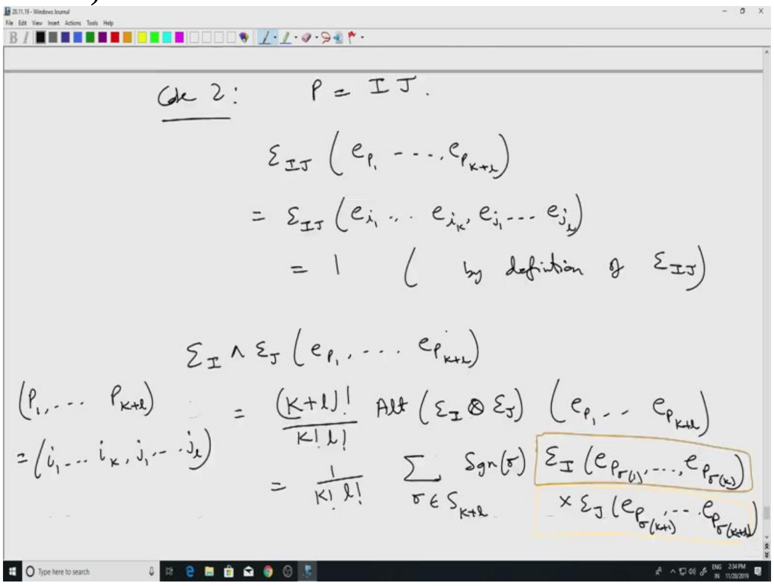

Welcome to lecture number 56. So, let me continue with the computation that I had started last time, this was with a view towards proving the associative property of, associativity of wedge product as well as the anti-commutativity.

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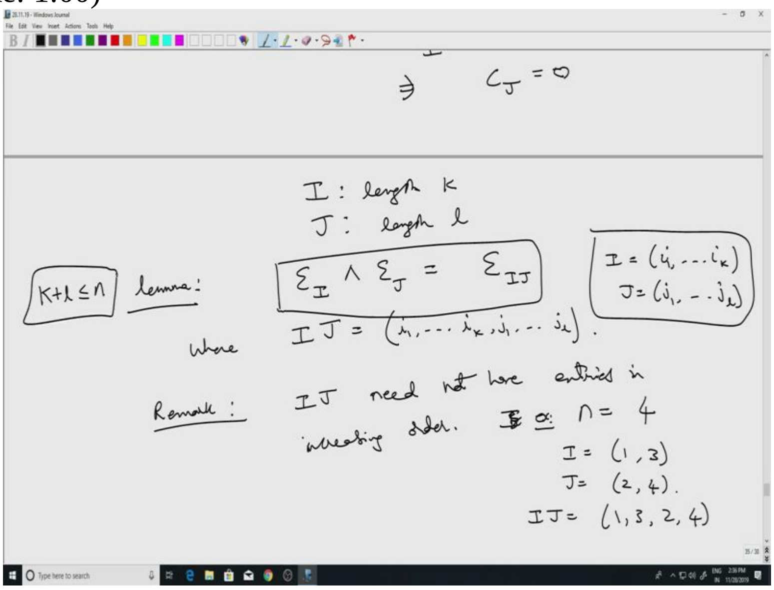

Proof: enough to check values as when both sides are evaluated on $(e_{p_1}, \dots, e_{p_{k+l}})$

$p_1 < \dots < p_{k+l}$

$I = (i_1, \dots, i_k)$

$J = (j_1, \dots, j_l)$

Case 1: P contains an entry not in I & J. Then

$$\varepsilon_{IJ}(e_{p_1}, \dots, e_{p_{k+l}}) = 0$$

(by definition of ε_{IJ}).

$$(\varepsilon_I \wedge \varepsilon_J)(e_{p_1}, \dots, e_{p_{k+l}})$$

$$\sum \mathrm{sgn}(\sigma)(\varepsilon_I \otimes \varepsilon_J)(e_{p_{\sigma(1)}}, \dots, e_{p_{\sigma(k+l)}})$$

$$\varepsilon_I(e_{p_{\sigma(1)}} \dots e_{p_{\sigma(k)}}) \times \varepsilon_J(e_{p_{\sigma(k+1)}}, \dots, e_{p_{\sigma(k+l)}})$$

Similarly, the L.H.S. $= 0$, since each term in the expansion (defining the wedge product) will be zero.

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Case 2: $P = IJ$.

$$\varepsilon_{IJ}(e_{p_1}, \dots, e_{p_{k+l}})$$

$$= \varepsilon_{IJ}(e_{i_1}, \dots, e_{i_k}, e_{j_1}, \dots, e_{j_l})$$

$$= 1 \quad (\text{by definition of } \varepsilon_{IJ})$$

$$\varepsilon_I \wedge \varepsilon_J(e_{p_1}, \dots, e_{p_{k+l}})$$

So now what we so, the main lemma as it turns out for proving that is this following a simple looking thing, namely this forms epsilon I that I defined last time. I have this property epsilon I by epsilon J epsilon I J, where I J is the concatenation of the two multi-indices I and J. So, we wanted to check this and in the process, we just had to as usual these are, these being alternating forms it is enough to check on basis vectors, their actions on basis vectors and we had two cases.

So we started with this a, this is, since this is both sides are k plus l forms, we started with basis vectors e p 1 all the way up to e p plus p k plus l. Of course, here it is we already know that the degree of the form is more than the dimension then an alternating form has to vanish. So, we might implicitly we are assuming that k plus l less than or equal to n; otherwise there is nothing to prove.

So, we started with k plus l basis vectors; e p1 e p k plus l and we assume they are ordered in the usual way and this concatenation, well, so we have two cases. The first case, P contains an entry not in I or J. In this case, it turns out both sides are 0, the second case is looks rather special where I am assumed that P contains the entries in this index set P, together they make up is exactly equal to I J so, in this specific order p1 up to p k plus l is this.

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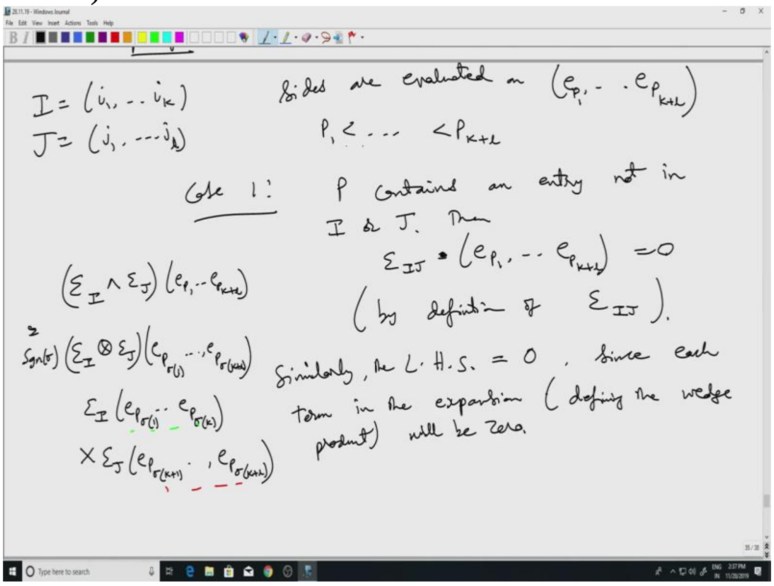

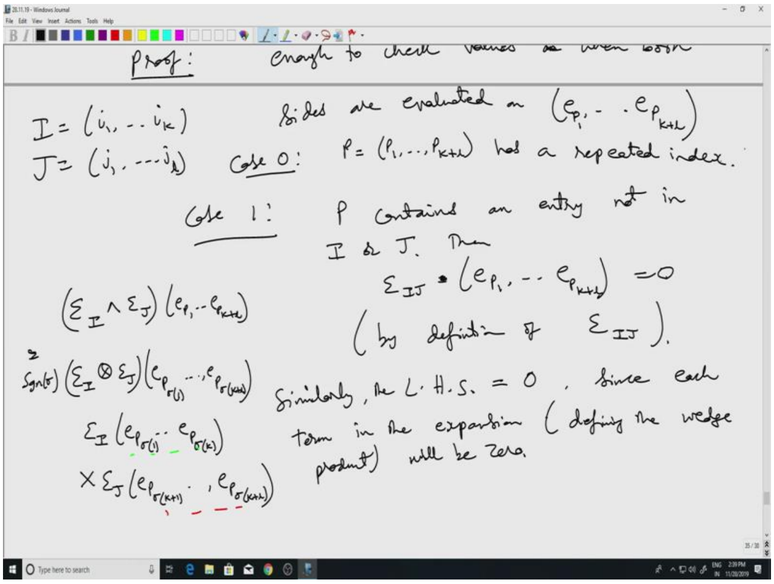

Now, here actually if I want to be consistent I have to be, I cannot insist on this strictly increasing ordering because that does not fit into this concatenation as we observed earlier. So, let me drop this condition that this, in fact I did not use that anywhere. So case one, but if I drop that condition then I have to be, I have to add one more case which is that case, let me call it case 0, this being the case that P equals P 1…. P k plus l has a repeated index.

So, when there is a repeated index, the right hand side is trivially 0, well because of the definition and similarly, the left hand side actually, it is you do not even need that definition, specific nature of epsilon just the fact that they are alternating forms will ensure that if two indices are repeated, you will have two vectors being the same in the input. So, therefore both sides will be 0. So, this case is kind of clear, case 1 and then case 2.

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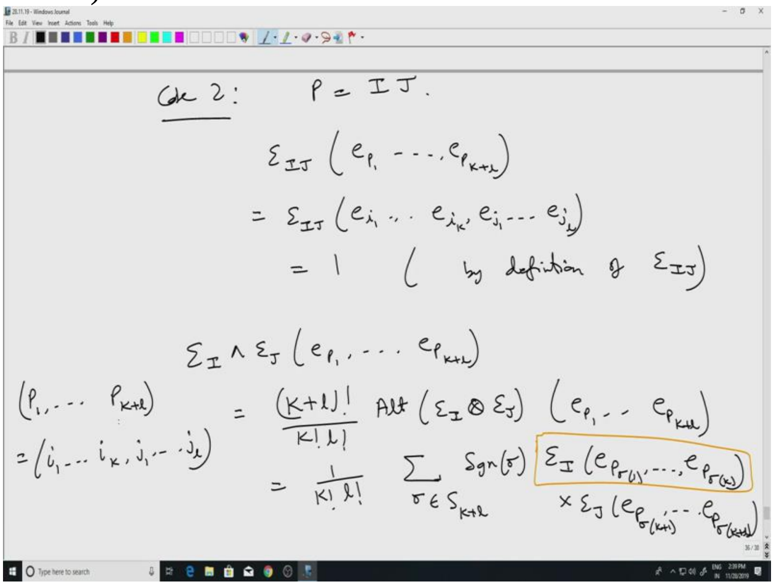

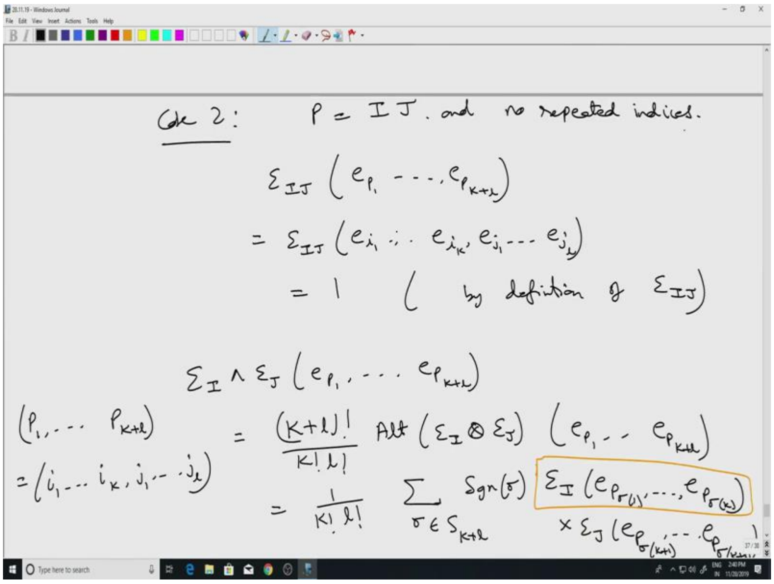

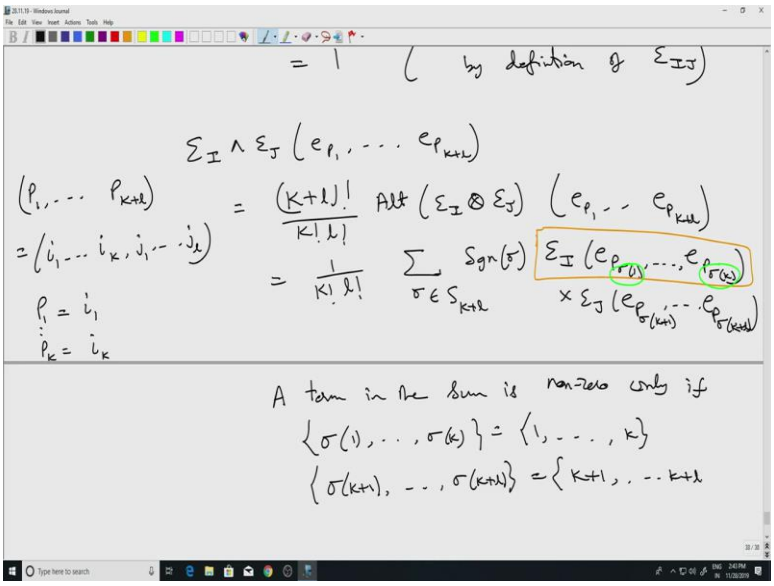

So, case 2, I have this. Now there is no conflict when I write P equals I J, it might very well happen that this is, these indices are not in increasing order but the point is that here I should say P equals I J and no index is repeated. So, in other words, what I mean is all the entries of P are distinct, another way of saying that is the entries of I and the entries of J there is no common entry in between these two.

So and then once just evaluates both sides so, epsilon i j k so, in this case when I evaluate it on P, I get 1 by definition of epsilon I J. Now, the left hand side is we have to go to the definition of the wedge product and then you expand it out like this. This was where we had stopped last time. Well, let us look at these terms in the sum, the only way a term can be nonzero is that the permutation sigma will have to, this be a permutation which takes the indices 1 up to k back to the same set 1 to k; otherwise if sigma takes one of these indices here, if sigma takes any of these indices to something other than 1 to k, something in other words more than k, then epsilon of i acting on that will be 0 for the simple reason that if you see this P 1 is i 1, P k is i k.

So, we know that epsilon i acting on certain basis vectors will be nonzero if and only if the basis vectors involved in the inputs are permutation, are just e i 1, e i 2, e i k in some written possibly different orders but essentially the index set has to be from this set I 1 up to i k. In terms of P, the index set has to be from P 1 up to P k.

So, essentially the sigma 1, sigma k has to be 1 permutation of 1 to k and likewise, sigma should take the index set k plus 1 etcetera back to k plus l. So, a term in the sum is nonzero only if the sigma of 1 etcetera sigma k, this set should be the same as 1 to k and sigma k plus 1, sigma k plus l should be the same as k plus 1, k plus l as set.

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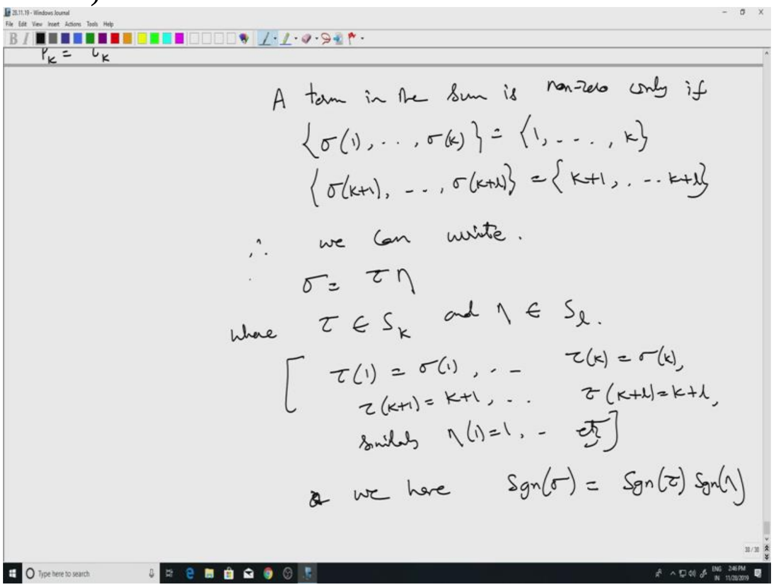

Therefore, what we can do is therefore, we can write sigma equals tau times eta, where tau is, belongs to S k permutation on only k letters and eta belongs to permutation only on l letters. So, it is quite clear what tau and sigma have to be, I mean it is essentially so, for instance tau is the permutation defined by 1, tau 1 equal to sigma 1, tau k equal to sigma k and tau k plus 1 onwards nothing happens, k plus 1, tau k plus l equal to k plus l.

Similarly, eta 1 equal to 1 etcetera so, eta does not do anything to the first k and then permutes all this and this even though both tau and sigma actually the way I have written it they act on k plus l, both of them act on k plus k symbols since, they are not doing anything, the tau is not doing anything to the last l letters, tau can be regarded as an element of S k and eta can be regarded as an element of S l.

Now, and in fact once we have this, of course we know that signature is multiplicative, also we have (signat), the sign, sorry, not signature, the sign of the permutation as behaves well under multiplication, composition of permutations. So, I have this.

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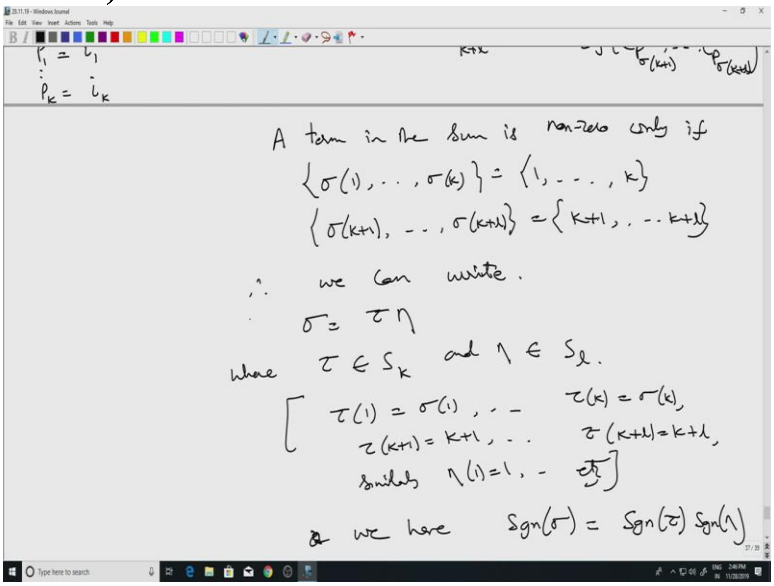

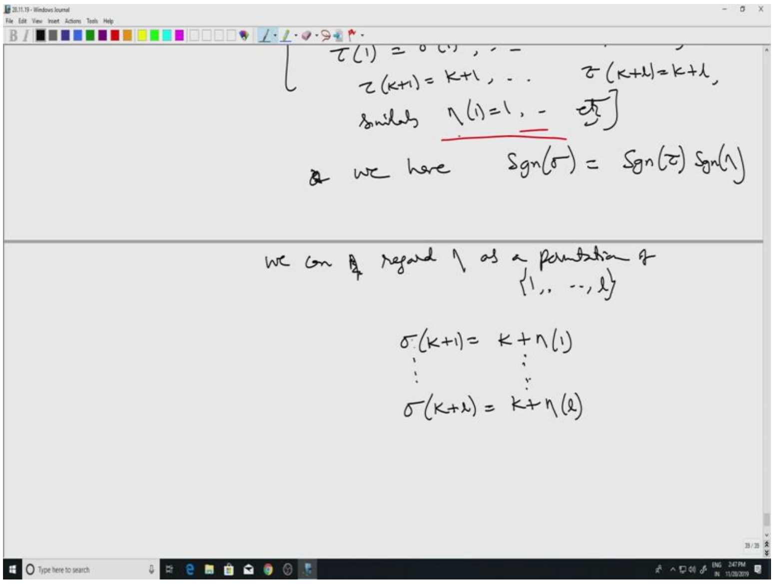

And another small change I make is the way I have written it, eta instead of notice that this for k plus 1 onwards I can think of instead of eta acting on k plus 1, k plus l, I can just think of eta as a permutation of, regard eta as a permutation of 1 to l in which case this for instance what I have here, what I was trying to write here, this stuff will become like essentially sigma of k plus 1 will be k plus eta of 1 sigma of k plus l will be sigma of l. The point being that anything in between k and k plus l will be of the form k plus something, what is changing is that something so, I might as well regard eta as a permutation of the various things that you add to k which is what I have written here.

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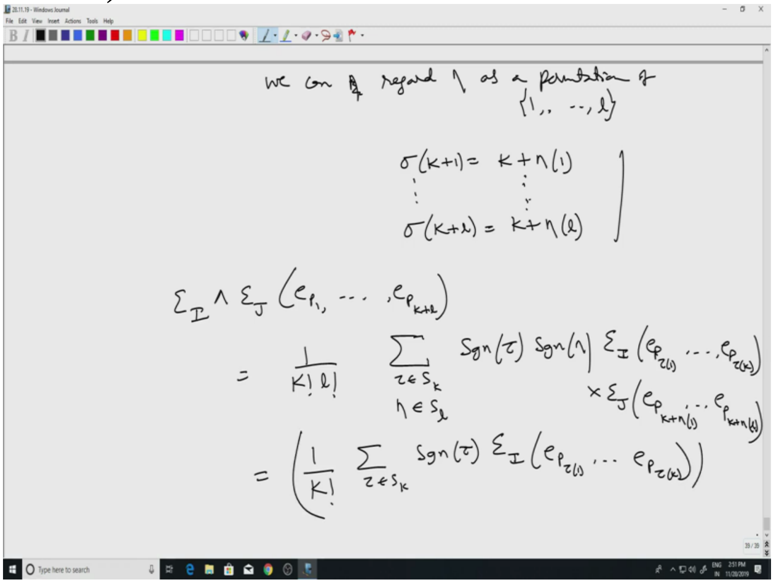

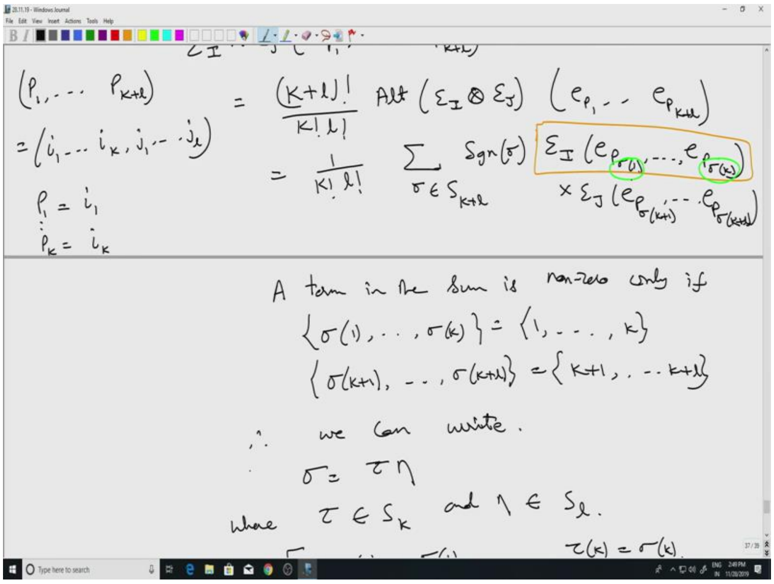

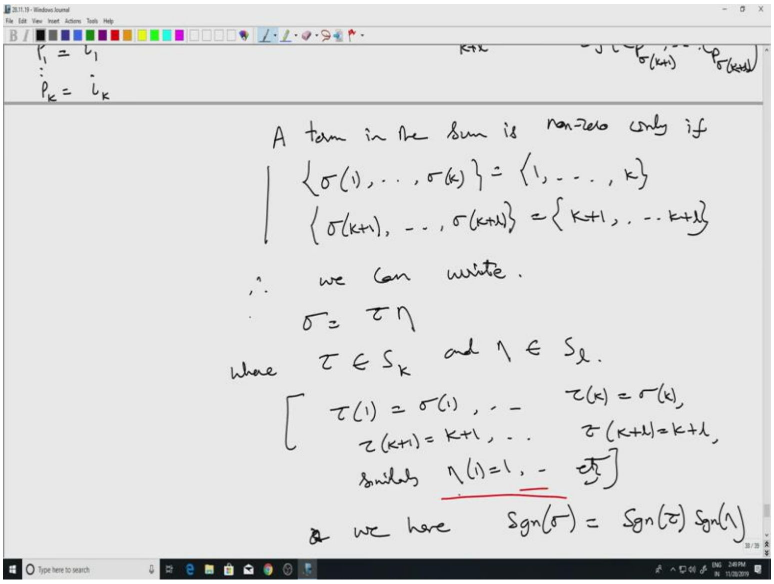

The advantage of doing that is now what we can write this is the quantity we are interested in: e p 1, e p k plus l, is I still have that 1 over k factorial l factorial. Now, I have tau in S k, eta in S l, the original sum was over S k plus l. But we saw that the original sum contains lots of 0 elements, any permutation which does not, which is not have this property will give rise to a 0 term. So, we just have to worry about those permutations, those elements of S k plus l for which this holds and once we restrict to those permutations in the sum, I can write it like what I am doing now.

So, tau in S k eta in S l and then I use the multiplicative nature of sine that e tau 1, e tau k and then times epsilon J then I use this as well e, oops, I forgot about the P, P tau 1, P tau k and here I will have e p k plus eta 1, e p k plus eta l. And so I can split it into, the point is that I can write it as a product, k on our k factorials summation over tau and S k sine tau, then I just keep the epsilon I e p tau 1 e p tau k, this is one term.

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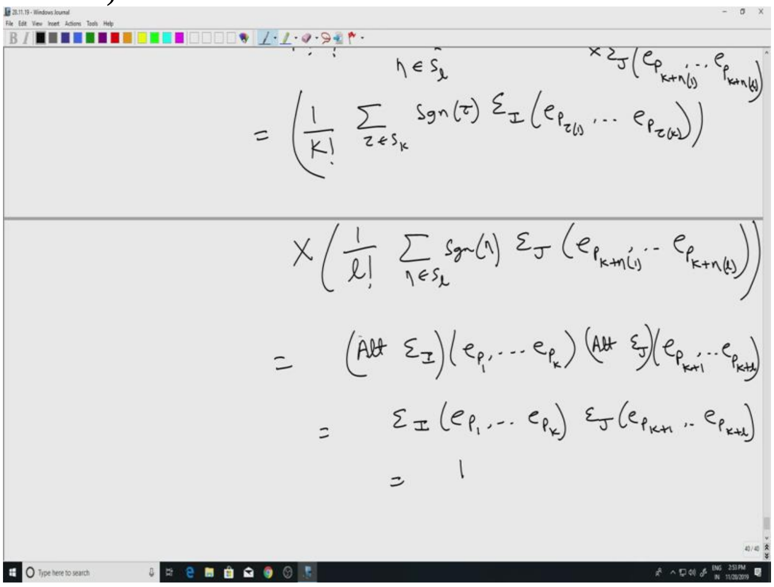

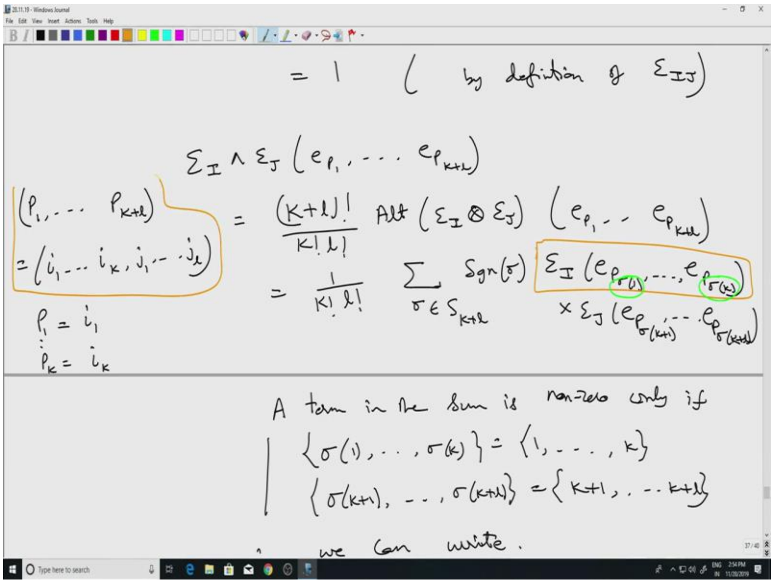

And other one is 1 over l factorial eta in S l sine eta epsilon J, e p k plus tau 1 etcetera, not tau, it is now eta, e p k plus eta of l and one whole and a big bracket around this and this thing that I have the first term, is by definition alt epsilon I acting on well, e 1 rather e p 1, e p 1…e p k. And then the second term similarly is alt epsilon J acting on e p plus, p k plus 1, e p k plus l, which is just I mean since, epsilon I is already alternating, alt does not do anything to it, this is one and likewise, the second term is also 1 the way because of the fact that P assumption that P is a concatenation of these two index sets. Both these terms will give me 1 and therefore one is done.

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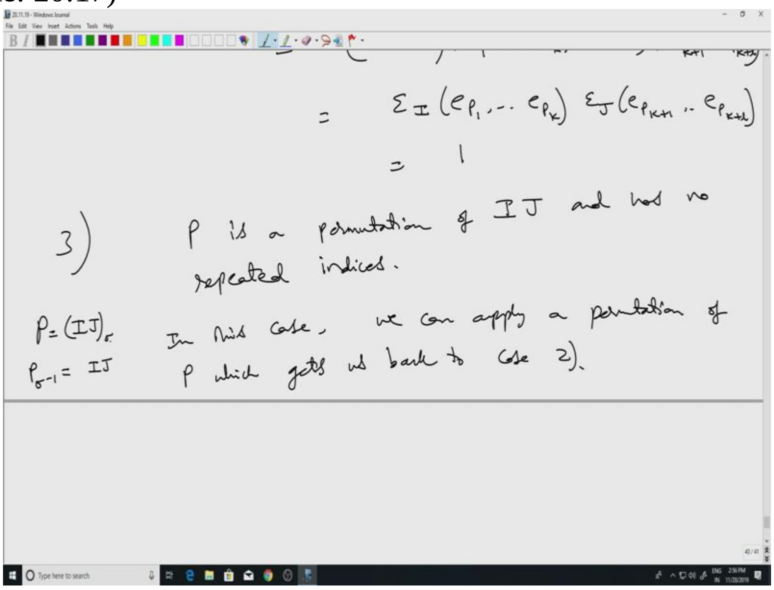

$\times\, \varepsilon_J(e_{p_{\sigma(k+1)}}, \ldots, e_{p_{\sigma(k+l)}})$

Case 2: $P = IJ$. and no repeated indices.

$$\varepsilon_{IJ}(e_{p_1}, \ldots, e_{p_{k+l}})$$
$$= \varepsilon_{IJ}(e_{i_1}, \ldots, e_{i_k}, e_{j_1}, \ldots, e_{j_l})$$
$$= 1 \quad (\text{by definition of } \varepsilon_{IJ})$$

$$\varepsilon_I \wedge \varepsilon_J (e_{p_1}, \ldots, e_{p_{k+l}})$$

$(p_1, \ldots, p_{k+l}) = (i_1, \ldots, i_k, j_1, \ldots, j_l)$

$$= \frac{(k+l)!}{k!\,l!} \operatorname{Alt}(\varepsilon_I \otimes \varepsilon_J)(e_{p_1}, \ldots, e_{p_{k+l}})$$
$$\sum_\sigma \operatorname{Sgn}(\sigma)\, \varepsilon_I(e_{p_{\sigma(1)}}, \ldots, e_{p_{\sigma(k)}})$$

$\ni \quad \varepsilon_J = $

I : length k

J : length l

$\boxed{k+l \le n}$ lemma: $\boxed{\varepsilon_I \wedge \varepsilon_J = \varepsilon_{IJ}}$ $\quad I = (i_1, \ldots, i_k)$, $J = (j_1, \ldots, j_l)$

where $IJ = (i_1, \ldots, i_k, j_1, \ldots, j_l)$.

Remark: IJ need not have entries in increasing order. Ex: $n = 4$

$I = (1, 3)$

$J = (2, 4)$.

$IJ = (1, 3, 2, 4)$

$P = (IJ)_\sigma$

$P_{\sigma^{-1}} = IJ$

In this case, we can apply a permutation of P which gets us back to case 2). The effect of the permutation will be to multiply both sides by the same sign.

Now, the last case, so the third and last case is P is a permutation of I J and has no repeated indices or entries. So, I mean this third possibility exhausts all the things that can happen so, we just and how do we deal with this case? Well, in this case we can apply a permutation of P which gets us back to case 3, case 2.

So, if P is a permutation of I J, in other words so, P in our notation I am assuming that P is I J sigma so, in that case what we can just apply sigma inverse for example, P sigma inverse is I J and then sigma multiplied by sigma inverse which would be 1. So, P sigma inverse would be I J so, then what one would do is one would start with this as in the proof of case 2, except that you will start with P sigma inverse and then you verify the formula.

But then you notice that P sigma inverse if you have it, if you have both sides equal, the final point of, final object of interest is this equation, if both sides evaluated on P sigma inverse are the same then both sides evaluated on P itself will be the same because you can go back to this just by moving things around. In other words, applying the reverse permutation, one can we can apply permutation of P which gets us back to case 2.

The effect of the permutation will be to multiply both sides by the same sign. So, this is essentially reduces to case 3, case 2 so, that completes the proof of this. Now, I should remark, perhaps I should have remarked on this earlier but this proof, this exposition I am more or less following exactly as it is presented in John Lee's text, Introduction to Smooth Manifolds.

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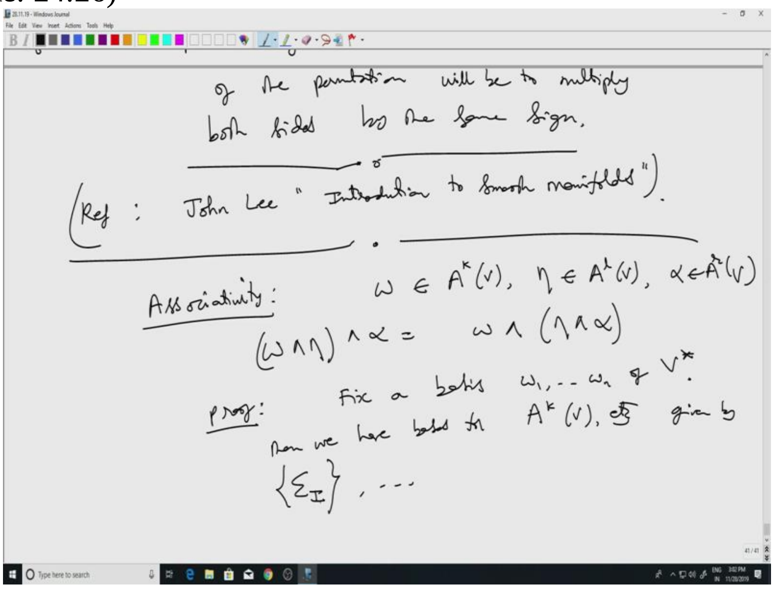

So, I should say a reference, John Lee, Introduction to Smooth Manifolds. So now, so once we have this lemma then we can move on to the main object of the theorem of main interest for us so, we wanted to prove associativity. So, I have omega A k V, eta n Al V and let us say alpha in A r V. So, one wanted to check that this is equal to this.

Now, what we can do is with all that we have discussed so far, we can, we know that we have a nice basis for this A k V, Al V and A r V as fix a basis omega 1, omega n of V star the dual space, then we have basis for A k V etcetera given by epsilon I dot dot. So, the length of the index dictated by what this the degree of the form one is considering, but we are using the same notation epsilon, the I index set will keep track of the degrees.

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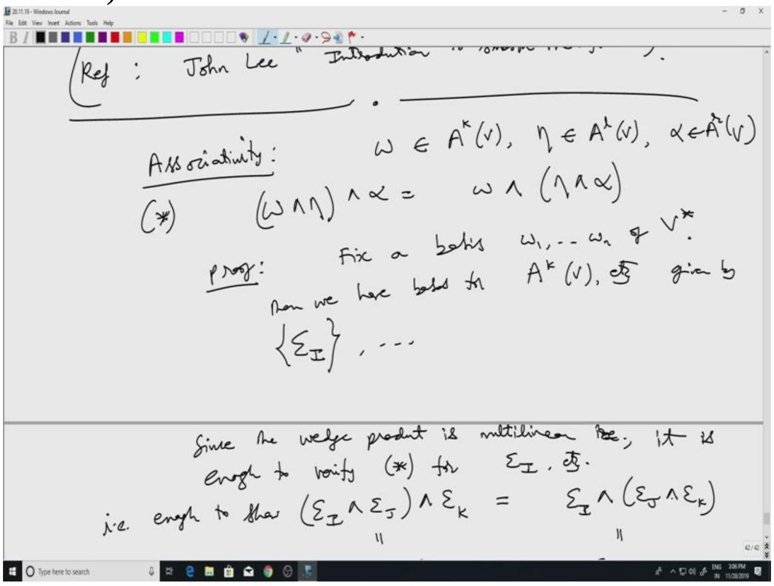

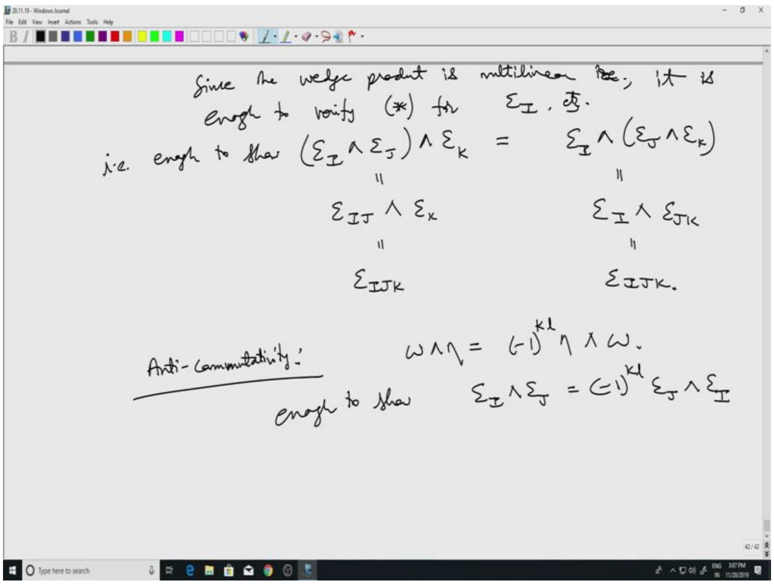

Well, once we have basis for all the spaces since, the wedge product is itself multi-linear i. e., so, we have, this was the first property that I wrote down, the bi-linearity of the wedge product but here actually we have three forms but it does not, it is the same thing carries over. And one can conclude that since the wedge product is multi-linear so, let us call this equation as star, it is enough to verify star for the basis forms.

So, what one does is you expand omega in terms of epsilon I, eta in terms of the epsilon J and alpha in terms of the epsilon K, where I, J, K are multi indices, expand them, in other words write them as linear combinations then use the multi-linearity of this wedge product and you will end up with a whole bunch of terms and individual terms in the sum will be of the form, of this form on the left side, on the right side they will involve terms of the form.

So, these terms will occur in the left side and the right side with the same coefficients but we want to know if this form is actually equal to this, i. e enough to show this. But this thing here on the left side is we have seen that this is same as I J wedge E k which is the same as epsilon I J K and this is also the similarly right side as epsilon I wedge E J K which is equal to epsilon I J K.

So, both sides are the same, therefore these star holds. So, it is a pretty straightforward consequence of this. Now, the other thing is anti-commutativity. Here, I should remark that it is, first let me write down the formula. So, one is interested in showing that omega wedge eta as minus 1 raise to K l eta wedge omega, here I should remark that when we are dealing with tensor products, associativity is extremely straightforward, there is nothing, I am it is almost like it is a triviality from the definition. Here it is not so obvious because the alt map is involved, which involves a sum and so on.

On the other hand, when one is dealing with tensor products, omega tensor eta has absolutely no connection with eta tensor omega, the two are not related at all so, no such equation holds. Here interestingly, there is an equation relating omega with eta and eta wedge omega, they justify by sign in fact. So, again, we can use our basis to write omega in terms of the epsilon I, eta in terms of the epsilon J and enough to show by the same logic we will end up with same coefficients, epsilon I wedge, epsilon J equals minus 1 over K l.

So, I will just write this and then stop. So, I have to show this. So, I will stop at this point, next time I will complete the small calculation, bulk of the work has been done. So I just, I finished this and then that will complete our discussion of multi-linear algebra tensors and forms, alternating forms on vector spaces. Then we will carry all these constructions to manifolds. Thank you.