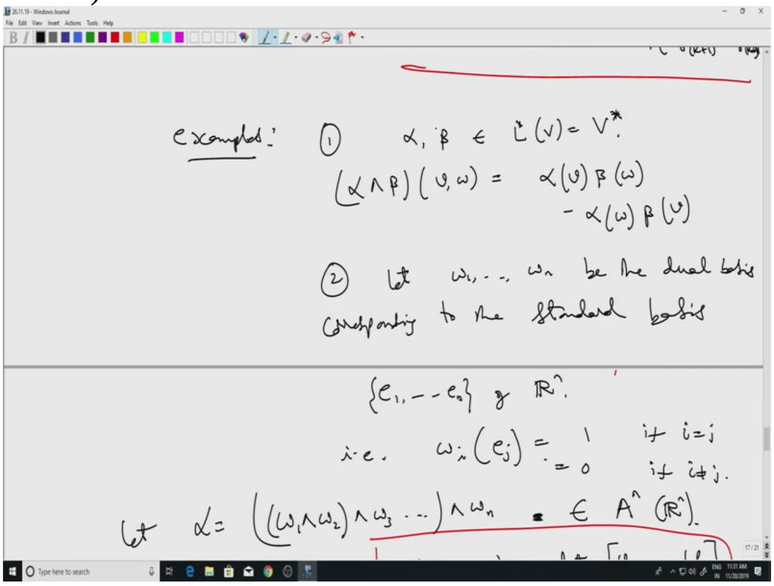
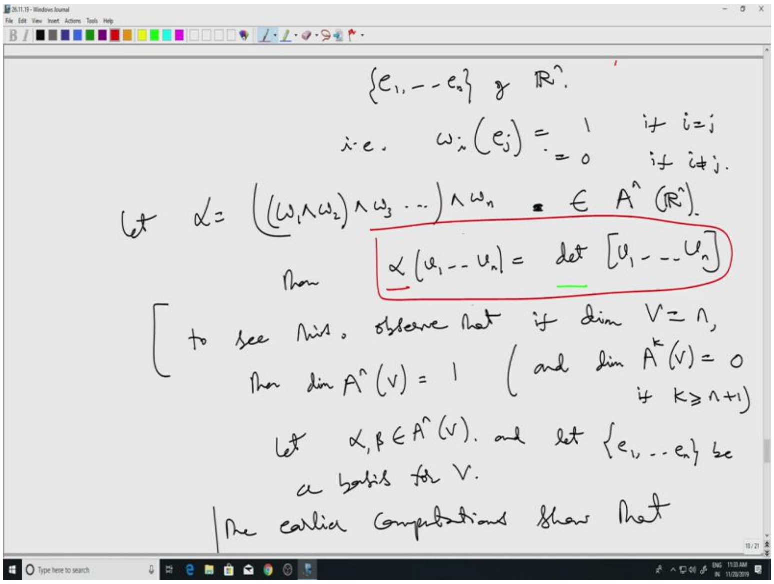
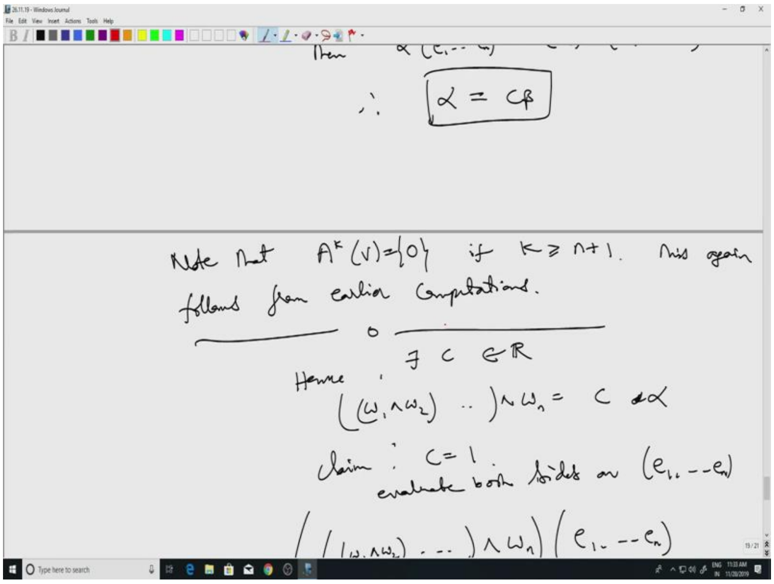
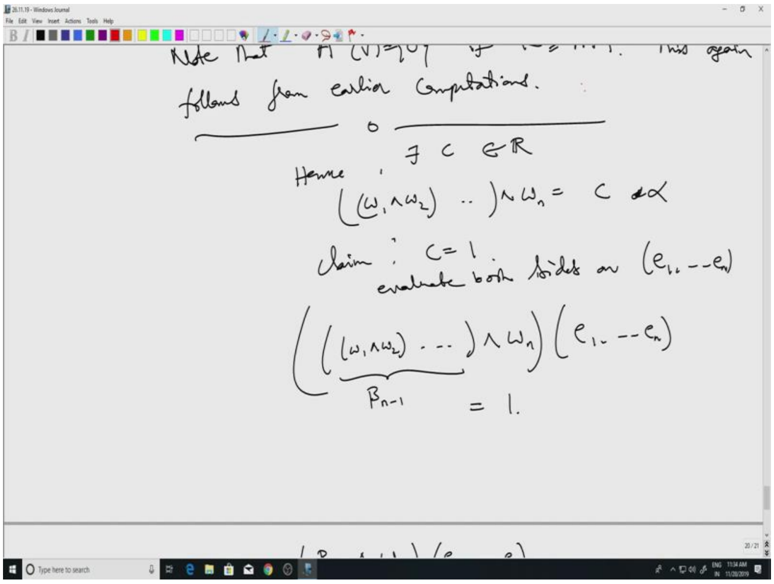
An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 53- Alternating Tensors – V

(Refer Slide Time: 0:42)









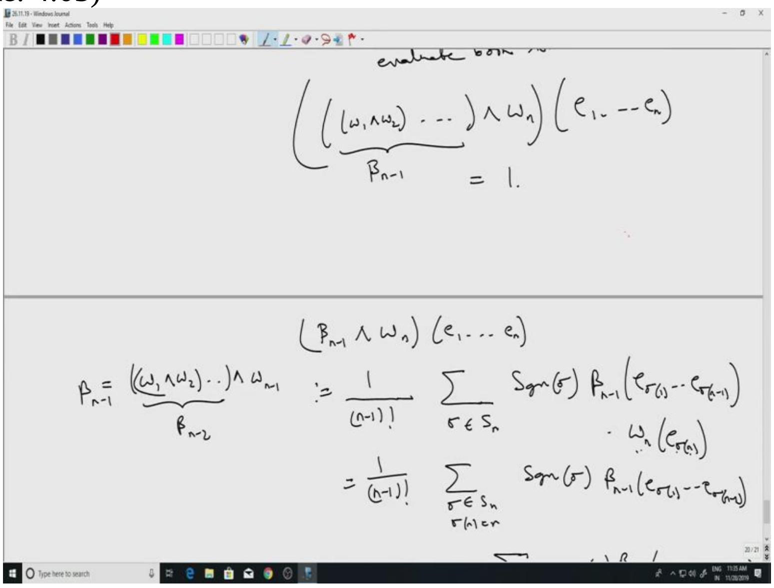
Welcome to the 53rd lecture in this series. So in this lecture, we will continue with our discussion of alternating forms or alternating tensors and, so let me complete the example that I had begun with last time. The second example was consisted of starting with a basis, omega 1 to omega n, which I took with a dual basis, the standard basis of R n. This is the basis of, this is a basis for 1-forms, for the space of one forms, namely the dual space of R n. And by definition, omega i e j is 1 and so on.

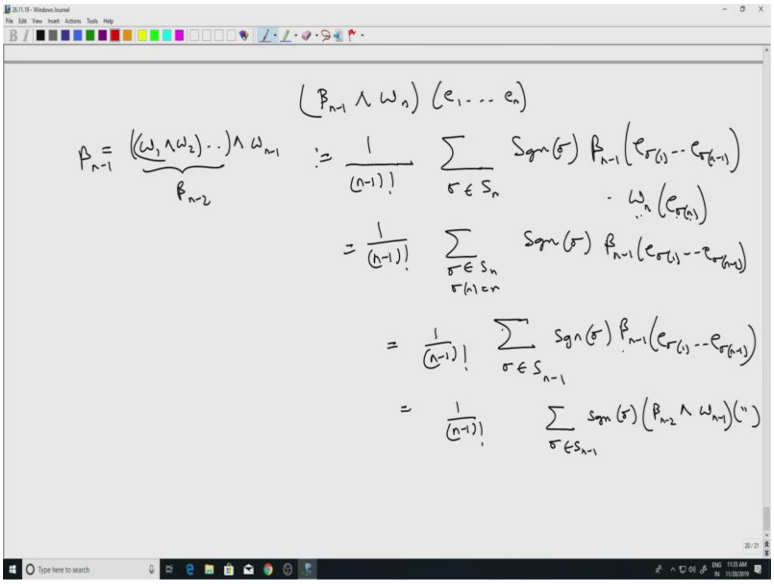
What I wanted to claim was that, if I take the iterated wedge product starting with omega 1 then omega wedge, omega2, et cetera, all the way up to omega n, I just get the determinant form on R n. And notice that, I have to be careful about putting brackets here when taking wedge product of multiple forms, because as of now we do not know that this operation of wedge product is associative.

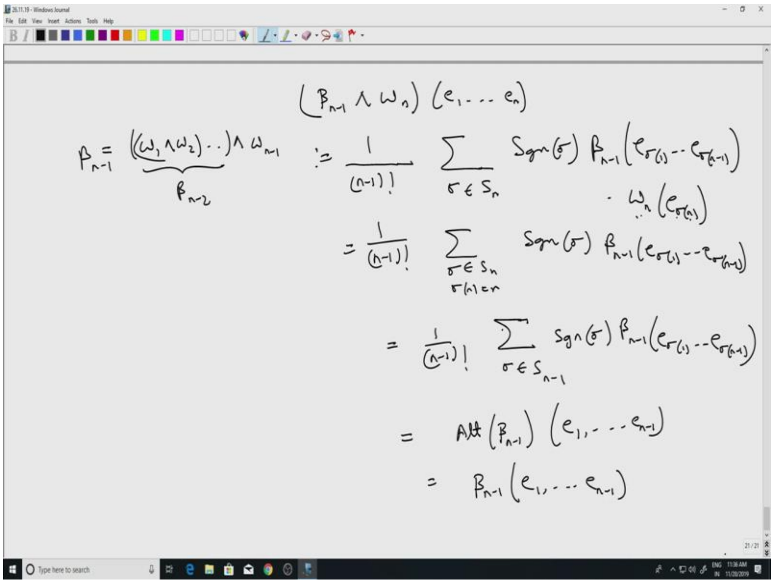
But soon we will prove that it is and I will not have to bother about putting brackets. Well, so the, in order to prove that this is the determinant, omega 1 wedge, omega 2 wedge, et cetera, first we made a general observation that if you have a n-dimensional vector space, then the space of n-forms as dimension 1. So, in other words, any two n-forms are just multiples of each other, scalar multiples of each other. And having proved this, we know that the determinant is already, this thing here, is already an n-form and the left hand side alpha is also an n-form since it is a wedge product of 1-forms. And, so both are n-forms and this general observation allows us to conclude that alpha is C times beta for any two n-forms, so therefore, this, the wedge product, the iterated wedge product is some constant times the determinant.

So applying this general observation to this specific case, now what we want to say is that, this C is 1, this constant is 1, then we would be done. So, we just (eval) to see that C is 1, just evaluate both sides on the standard basis e 1 up to e n. And, if we do that, the right hand side, the determinant of evaluated on e 1 up to e n is, we already know it is 1, where the e i's are written as column vectors. Now we are in the process of checking that, the left hand side is 1 as well, so I want to say that this iterated wedge product evaluated on this is 1.

(Refer Slide Time: 4:03)







And it is an inductive process, so let me call the first n minus 1 wedge products as beta n minus 1, the whole thing we can rewrite. And then we just evaluate, just wrote this as beta n minus 1 wedge omega n, and then use the definition of the wedge product to write it like this.

Now, the last, instead of going to the last step directly here, what I will do is, just notice that, this whatever I have here is by definition, Alt, beta n minus 1 is in n minus 1 form, in particular it is n minus 1 tensor, so this is Alt beta n minus 1 evaluated on e 1 up to e n minus 1. So and since beta n minus 1 is already an alternating form, again it is a wedge product of various 1-forms. So this is just the same as, we have seen that the Alt operation does not change the tensor if it is already alternating.

(Refer Slide Time: 5:41)

$$\beta_n(e_1 \dots e_n) = \beta_{n-1}(e_1 \dots e_{n-1})$$

$$= \beta_1(e_1)$$

$$= \omega_1(e_1) = 1$$

$$\therefore \quad c = 1 \quad \left(\text{since } \det[e_1 \dots e_n] = \det\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & & 0 \\ 0 & \vdots & & 1 \end{bmatrix} = 1 \right)$$

Claim: $c = 1$.

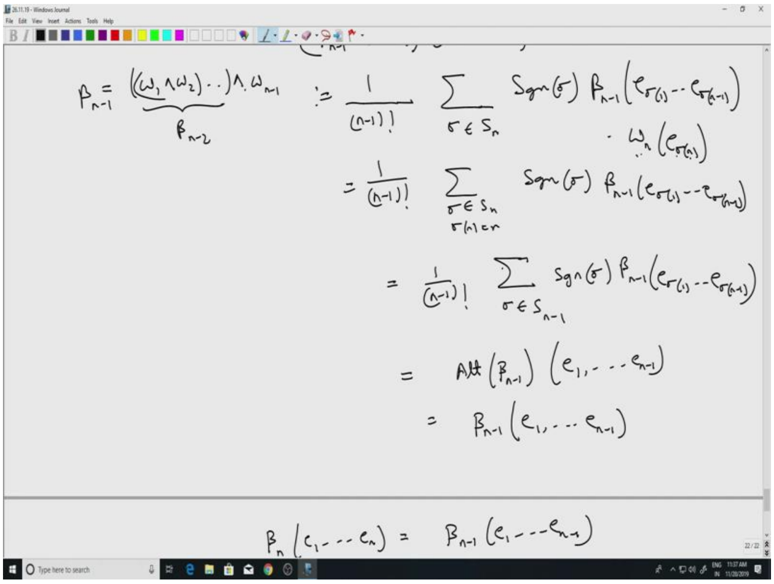
evaluate both sides on $(e_1, \dots e_n)$

$$\Big(\overbrace{(\underbrace{(\omega_1 \wedge \omega_2) \dots}_{\beta_{n-1}}) \wedge \omega_n}^{\beta_n}\Big)(e_1 \dots e_n) = 1.$$

$$(\beta_{n-1} \wedge \omega_n)(e_1 \dots e_n)$$

$$\beta_{n-1} = (\underbrace{(\omega_1 \wedge \omega_2) \dots}_{\beta_{n-2}}) \wedge \omega_{n-1} \qquad := \frac{1}{(n-1)!} \sum_{\sigma \in S_n} \operatorname{Sgn}(\sigma)\, \beta_{n-1}(e_{\sigma(1)} \dots e_{\sigma(n-1)}) \cdot \omega_n(e_{\sigma(n)})$$

$$= \frac{1}{(n-1)!} \sum \operatorname{Sgn}(\sigma)\, \beta_{n-1}(e_{\sigma(1)} \dots e_{\dots})$$

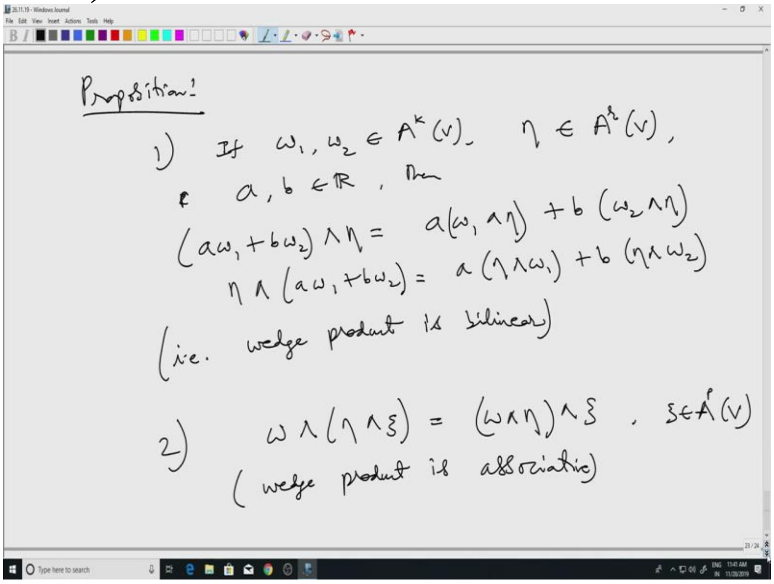


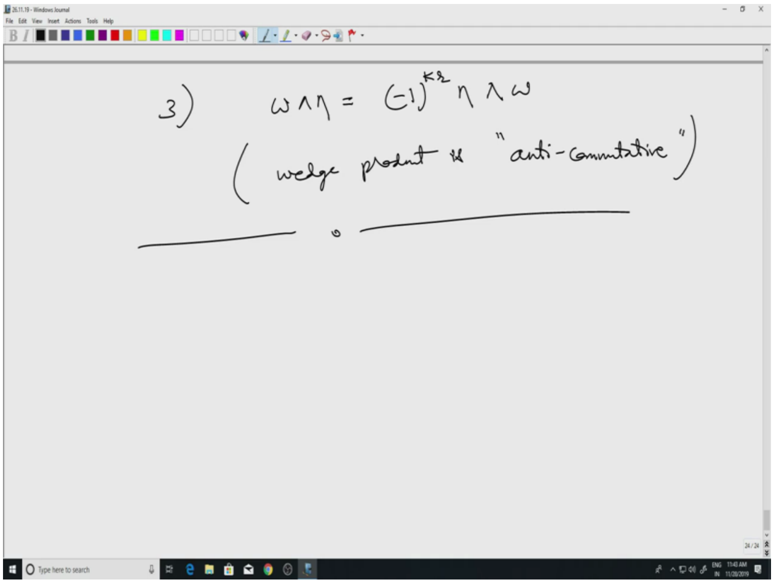
So, we started with this, beta n e 1 up to e n. Bet e n is the full wedge product with the same notation, this entire thing, well, this entire thing here is beta n, so we started with that and we see that this is equivalent, this is equal to this. And one can continue this process. At this stage you can again write beta n minus 1 as beta n minus 2 times omega wedge, omega n minus 1 and so on.

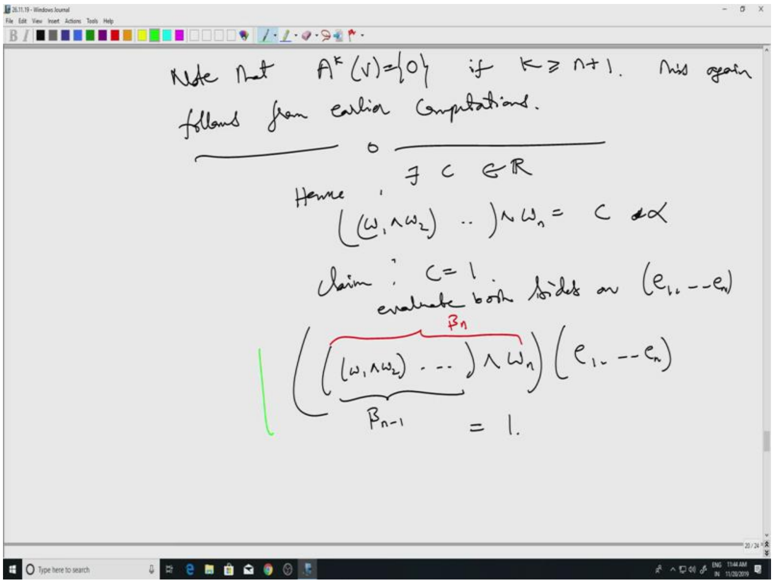
Finally, you end up with beta 1 of e 1 which is omega 1 of e 1 which is 1. So therefore, the left hand side is also 1 and therefore, the constant C equal to 1, since det e 1, e n as determinant of the identity matrix equal to 1. Right hand side is, yeah. All right. Now, I would like to, so this, this particular n-form which consisted of a wedge product of 1-forms, where the 1-forms have, are form of dual basis, this way of constructing forms, I would like to generalize to any k-form and prove various things.

So, what I want to prove, so, I will, this will take a couple of lectures, so I want to prove, first of all, some general properties of the wedge product operation.

(Refer Slide Time: 7:48)







So, proposition, first thing is that this wedge product operation is bilinear. So if, omega 1, omega 2 belong to A k V and eta belongs to A r V, and you have two real numbers a and b, then a omega 1 plus b omega 2 wedge eta equal to omega 1 wedge eta plus b times omega 2 wedge eta.

So this is linearity in the first slot. Similarly, one has the same thing in the second slot as well, eta wedge a omega 1 plus b omega 2 equal to eta wedge omega 1 plus b eta wedge omega 2. So it can be summarized by saying wedge, the wedge product operation is a bilinear operation. The second thing is, now this is the part which is require some work. The wedge product operation is associative. So here, eta is anything else, some other, I have already used k and r, let me call it p. The point is that the form, there is no restriction on the degrees of the forms, U1 has associativity and the third thing is also very important which is called anti-commutativity. So if omega wedge eta equal to minus 1 to the power k r eta wedge omega, wedge product is, I will put it in quotation marks, anti-commutative.

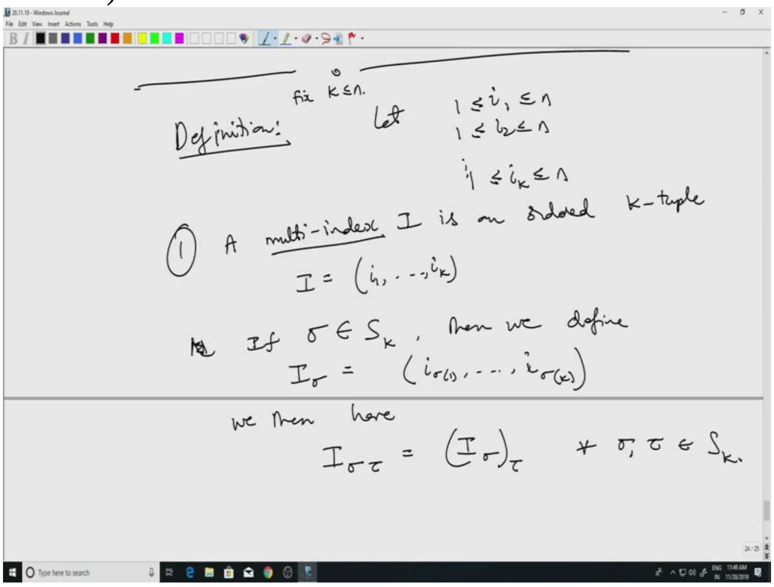
So the reason I have used quotation marks here is that, we do not always get a minus sign on the right hand side, it depends on the degrees, for instance, if k and r are even, so you start with the wedge, for example, if you start with the wedge product of 2-forms, both omega and eta are 2-forms, then omega wedge eta will be equal to eta wedge omega. One does not get a negative sign. So this, the fact that this power depends on the degrees is sometimes called anti-commutativeity. You know somewhat, this is not the usual anti-commutativity, it is in this sense. So these are the three properties that we are going to prove.

Now, of these three, the first is quite, is immediate from the definition, it is two and three which require a, which require some work, especially two. I would like to prove this and so the, to do this, I am going to introduce certain k-forms, specific k-forms and they will also help us to, help us obtain a basis of the space of k-forms and the way these forms are defined is exactly like the way we define the determinant in terms of these 1-forms. So I took, I iterated the wedge products.

Now, however, I am not going to define, so what one has in mind is this one wants to use this iterated wedge products but instead of going all the way to n, I would like to stop at some k and then, so in other words, take the iterated wedge product of k 1-forms is what I would like to do.

However, because of the, since we do not know that this operation is associative, somewhat inconvenient to work with this wedge products directly. So instead, we will work with determinants. Since we, at least in the, when case, in the case k equals n, we know that this iterated wedge product is actually the determinant. So, even when k is strictly less than n, one can get something involving the determinant which is what we will work with.

(Refer Slide Time: 14:10)



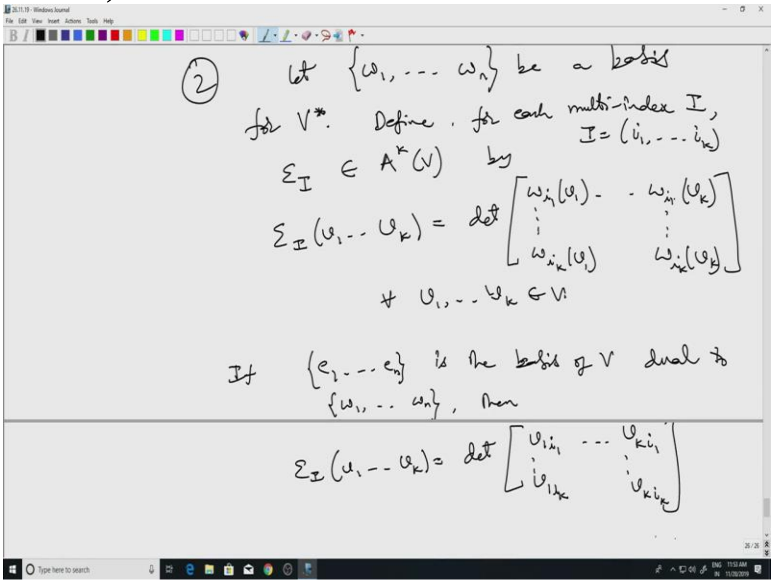
So let us start the proof. Along the way we, as I said, we will also obtain the basis for the space of k-forms. So here is the first definition, definition. Let i 1 be a number less than or equal to all these i 1, i 2, they are all, i k less than or equal to, so I fix k, fix k less than or equal to n. And look at some numbers in between 1 and n. The multi-index I is an ordered k-tuple. I equals i 1 all the way up to i k.

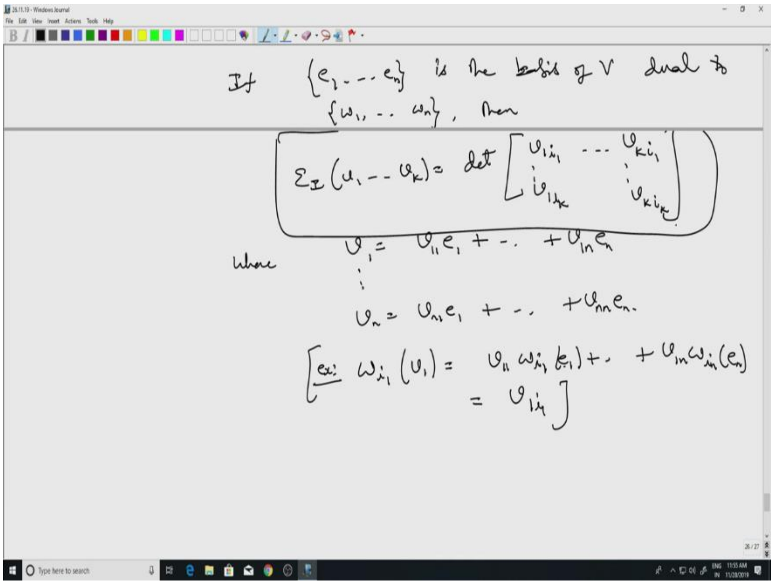
Now if sigma is a permutation on k letters then we define I sigma to be, I just permute this i 1, i sigma 1, et cetera, i sigma k. So I can do this, so essentially this I sigma, in terms of, as is if I forget the ordering, if I look at it as a set of numbers lying between 1 and n, it is same as i, it is just that the ordering may change depending on the permutation. And, we had done a similar thing on the, when we are dealing with k-forms, we could let a permutation act on a k-form and then obtain something, obtain a new k-form.

Here we are just dealing with index sets so and, as in the case of permutations acting on k-forms, here too, we have, we then have, if I have a product of two permutations, it is turns out to be the same as, you can first do I sigma and then whatever k-tuple you get, on that you do tau, and this is quite trivial just from the definition for all sigma, tau in S k.

This condition is what is technically called a, is one part of what makes a group action, so this group S k acts on the set of all I sigma's and what one means by act is one property is this, the other one is just that, the identity permutation just does not leave the I sigma equal to I if sigma is the identity permutation. These two properties make up a group action. So this is this, now the other thing is, so this the first definition.

(Refer Slide Time: 17:57)





Now, let us start with the vectors. Let omega 1, omega n be a basis for V star, the dual space. Define, this is the analog of that iterated wedge product that I was talking about earlier. Define, for each multi-index I, I am going to define epsilon I, A k-form by epsilon I acting on v 1 up to v k to be the determinant of the following matrix, omega i 1 v 1, omega i k v 1, omega i again i 1. No, omega i 1, yeah, it is v k, omega i k v k.

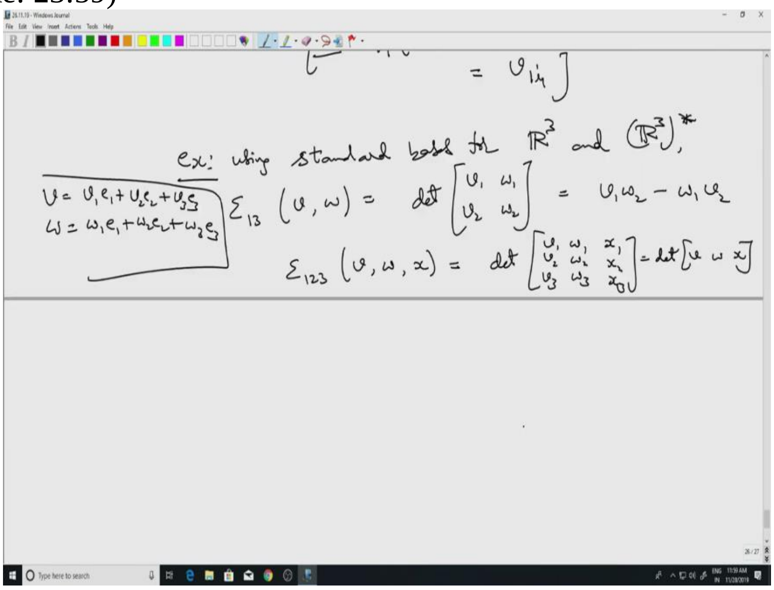
Where, so this multi-index as usual, the multi-index is i 1, i k. So I define it like this, the point is that the right hand side is a square matrix, so I can take the determinant and all these are just numbers, omega i are 1-forms so they act on vectors v 1 up to v k so I just get numbers. So for all v 1 up to v k in V. And after a while we will prove that this is actually just the iterated wedge product of omega i 1 wedge, omega i 1 wedge, omega i 2, et cetera, all the way up to omega i k, will turn out to be the same thing as the right hand side.

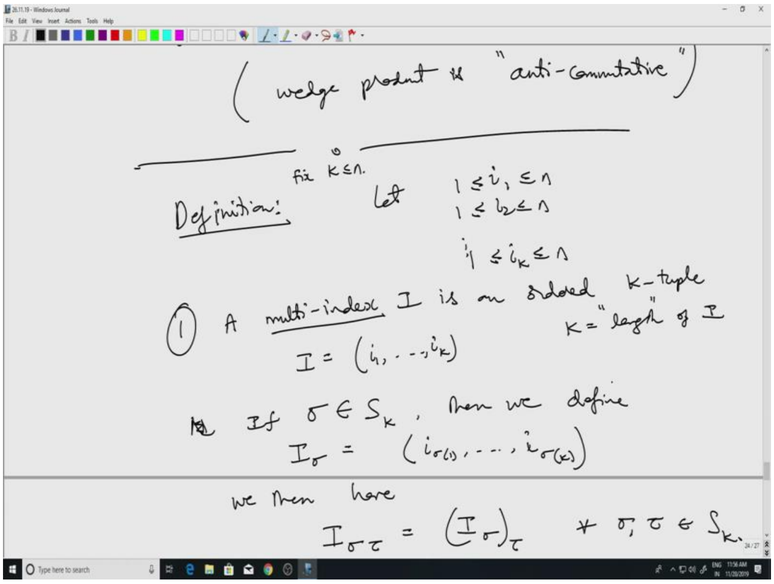
And also, let us notice one thing, if e 1, e 2, e n is the dual basis, is the basis of V dual to omega 1, omega n. In other words, as usual, omega i e j should be equal to 1 if i equal to j otherwise it should be 0.

So, if it is the basis of V then, this epsilon I acting on v 1 up to v k is nothing but determinant of v 1 i 1, v 1 i k, v k i 1, v k i k where, v 1, I have expanded v 1. If I expand v 1 in terms of this basis that I had, I would write it as v 11 e 1 plus v 1 n e n, et cetera. V n equal to v n 1 e 1 plus v n n e n. So, these, this is, if this, so the statement that this equal to this is immediate because omega i 1 v 1, for instance, if I do omega i 1, omega i 1 v 1 is the same as v 1 1 omega i 1 of e 1 plus v 1 n omega i n of e n, so I will write it as an example, so example.

So this is, right. So this is well, the only term which will survive is when this e i 1 is the term which will survive, so then as it omega v i 1 and then 1. All the other things will be 0 because we are working with bases which are dual to each other. So the first term becomes v i 1 and so on. Okay.

(Refer Slide Time: 23:59)





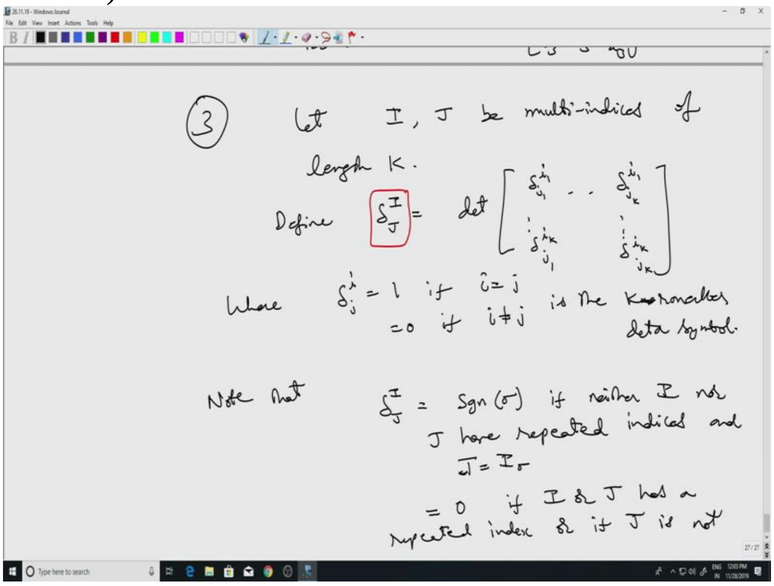
So, for example, another example, this construction is, this is not quite an example, I just did one specific index. Let us using standard basis for R 3 and R 3 dual, we have E 13, so this would be 2-form, the index. This k here is called the length of the index, is equal to length of I multi-index. How many entries there are? So here the length is 2 and so it would be a 2-form acting on v, w.

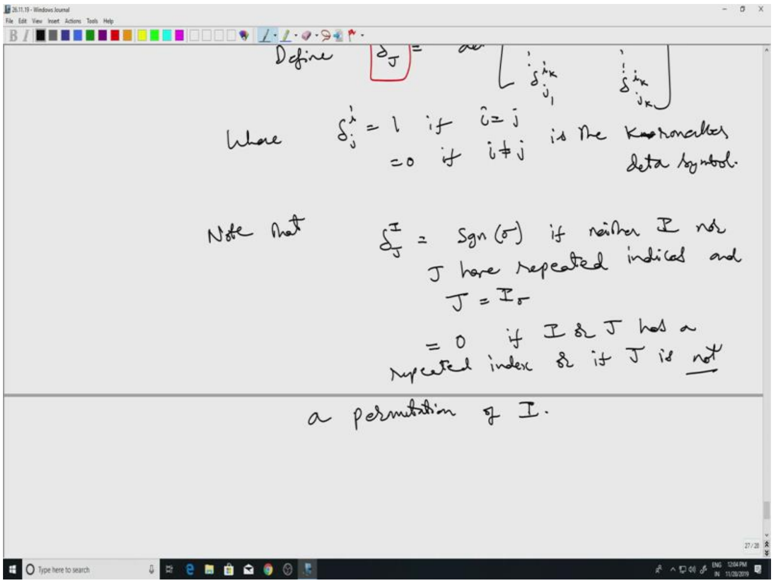
Well, according to what we have here, it is a determinant of, and I am working with standard bases, so if I expand V in terms of, so V is v 1 e 1 plus v 2 e 2 plus v 3 e 3. W is w 1 e 1 w 2 e 2 plus w 3 e 3, then I would be, the first column will be v 1, v 2 and here it will be w 1, w 2 which is v 1 w 2 minus w 1 v 2. So this would be my 2-form, one 2-form that I obtained this way.

There are others as well, of course I just took this index, multi-index 1, 3 and if I look at E 123 which is n-form, which is a 3-form on R 3, we know that we have already, know that anything in the top dimension, any two n-forms are multiples of each other. So in particular, E 123 would be evaluated on, we would expect it to be just a multiple of the determinant of the matrix formed by v, w and x. It is actually the way we have defined it, it is in fact the determinant itself. Also, determinant of v 1, v 2, v 3; w 1, w 2, w 3; x 1, x 2, x 3 which we, I mean, we write it as v, w, x. This is, I mean, there is nothing to prove here, this is just the definition we had stated.

So far we have not only brought wedge products into play. So in fact, all this, whatever I am doing now could be defined even before we introduce wedge (pro), we introduce wedge products. So that is one definition. And the other definition is, so this is definition two.

(Refer Slide Time: 27:50)





Let us introduce one more notation 3. Let I, J be multi-indices of length K. Define delta I, J equals determinant of delta i 1 j 1, delta i k j 1, delta i 1 j k, delta i k j k, where this is called the Dirac Delta function. So this is, no sorry, this is called the Kronecker delta, which is where delta i j equal to 1 if i equal to j, is equal to 0 if i not equal to j. This the Kronecker delta symbol. Just a notational device which keeps track of whether two indices are the same or not. Here, the difference is of course, this small delta i j where i and j are numbers is something which we might have seen before but the left hand side here, something which has to be defined because it is, these are multi-indices now, capital I and capital J.

And that I defined to be the determinant of this matrix, notice that every entry in this matrix is either 0 or 1 and in fact, the claim is that then, note that, one can say exactly what its value is, this multi-index delta, this is just the sign of sigma if neither I nor J have repeated indices and I and J equals I sigma. And is equal to 0 if I or J has a repeated index or if J is not a permutation, not, so here I should underline, not a permutation of I.

In other words, J is not equal to I sigma for any sigma, so let me write this clearly, so here J. And how does one see this? Well, that is couple of ways, one is just use the fact that, just use the definition of a determinant, that way one can see it directly, or the other way is, write permutation as a product of transpositions, and then one can see the stuff, this thing here as well. Okay, so we will continue with this next time, so let us stop here.