An Introduction to Smooth Manifolds
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Lecture 52
Alternating Tensors 4

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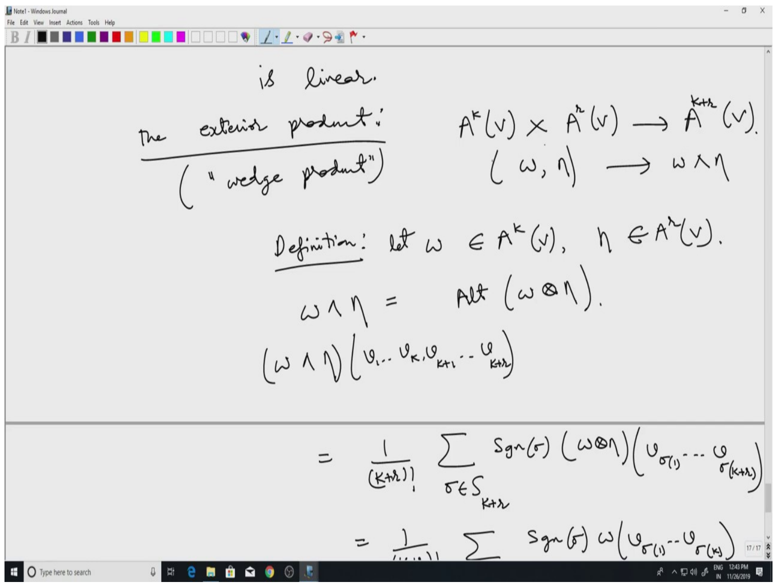

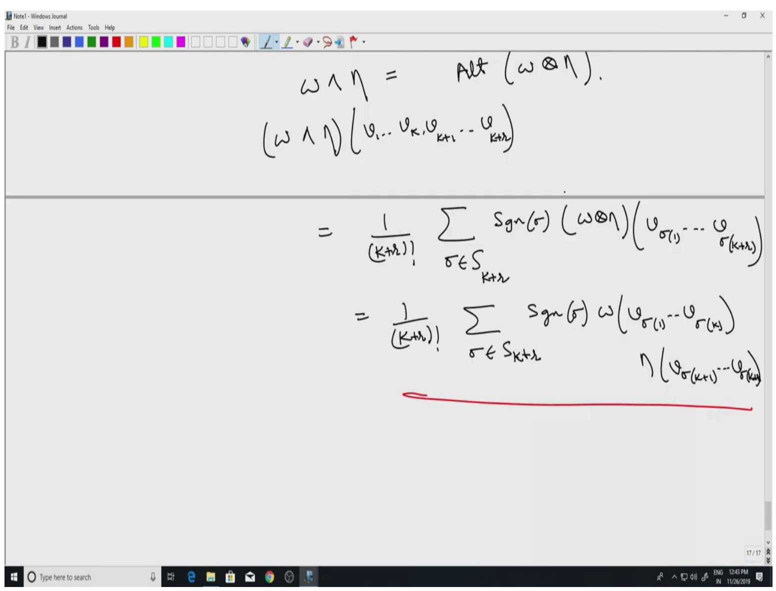

Welcome to the 52nd lecture in the series So, I would like to continue where I had stopped last time, so the, I had introduced this operation of wedge product. You start, one starts with two alternating forms and one gets an, by taking the product, one gets an alternating form whose degree is, by degree, I mean the number of input variables, this K plus r, if they happen to be K and r to begin with. And the, we essentially use 2 things to define this, the tensor product and the anti symmetrization map Alt.

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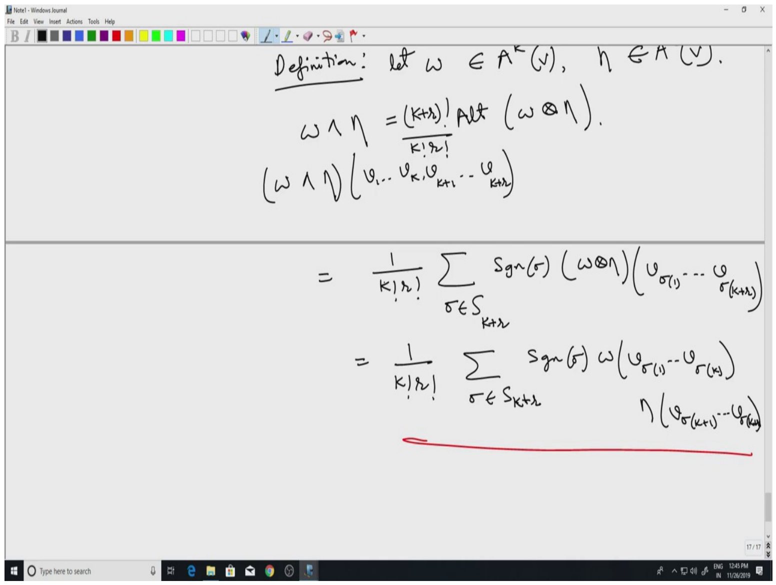

Now, last time I had not written the constant which occurs in the definition. So this time, I will write the constant. So the constant here is actually K plus r factorial K factorial r factorial. And the reason it is defined like this is, it becomes immediately clear because Alt itself as a 1 over K plus r factorial, that cancels with this, and one is left with, well that still does not explain exactly why it is defined like this. But, at least that part cancels and you end up with this 1 over K factorial r factorial.

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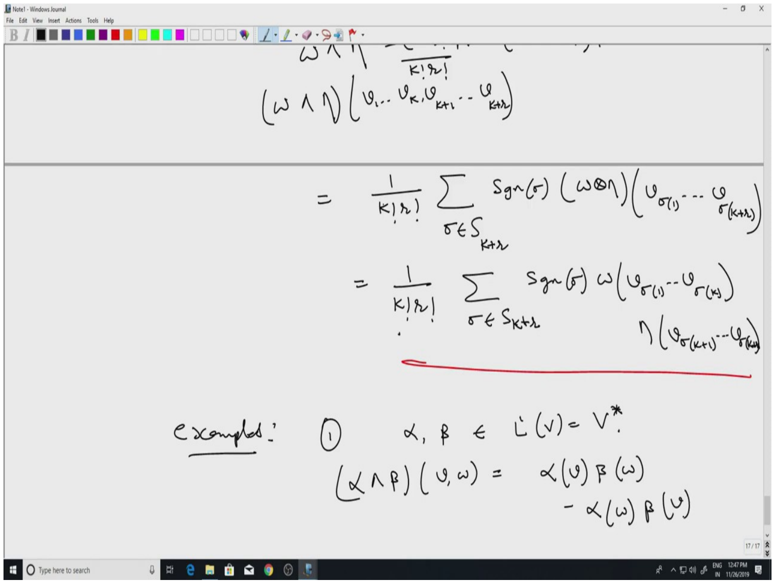

Now, right, so this, let us look at some examples before we move further. So let us take 1 forms, so let us take a alpha and beta to be in L1 V which is V star. I would like to see what alpha wedge beta is, so alpha wedge beta, this would act on 2 vectors. So, this should be a alternating 2 tensor and I want to see what this is. Now, if I just go by definitions, so I will, can use the last expression here.

So, this is, so both K and r are 1 here, so this constant in the (def) definition of omega wedge eta does not play a role. So, I will end up with, and again K plus r is 2, so essentially I will be looking at the symmetric group on 2 letters and just the 2 permutations. The first one is the identity. The identity gives me alpha v beta w and the second one gives me, it is a transposition interchanging 2 and 1, its sign is minus 1, therefore I get minus 1, and then I will let alpha of w and beta of v. Right, so, that is it. So, the (wed) wedge product of two 1 forms is rather easy to describe, it becomes, it quickly becomes more complicated as we take higher order forms.

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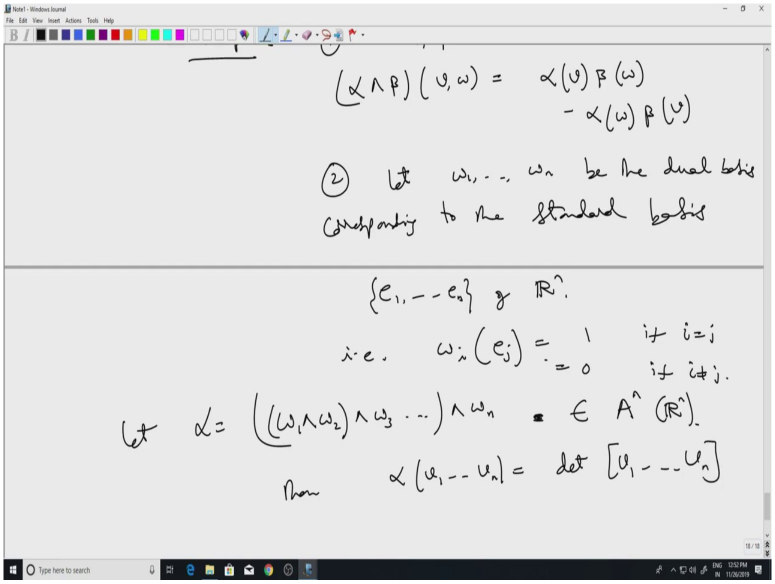

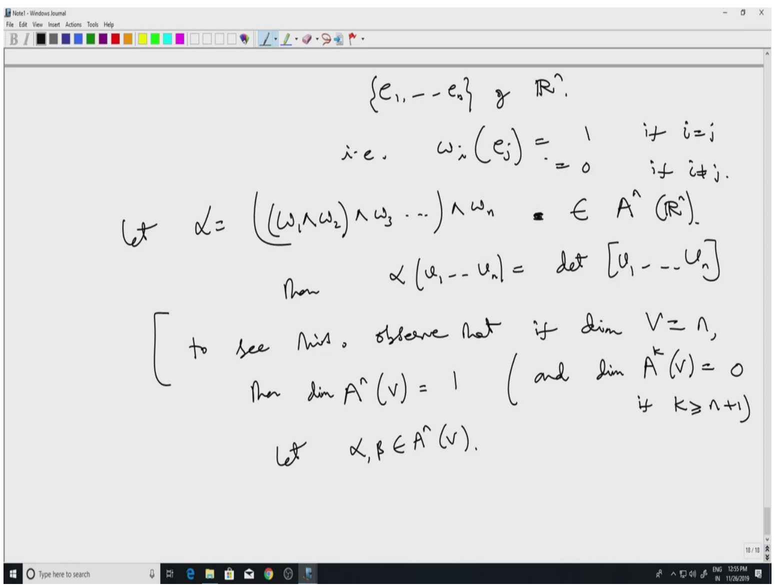

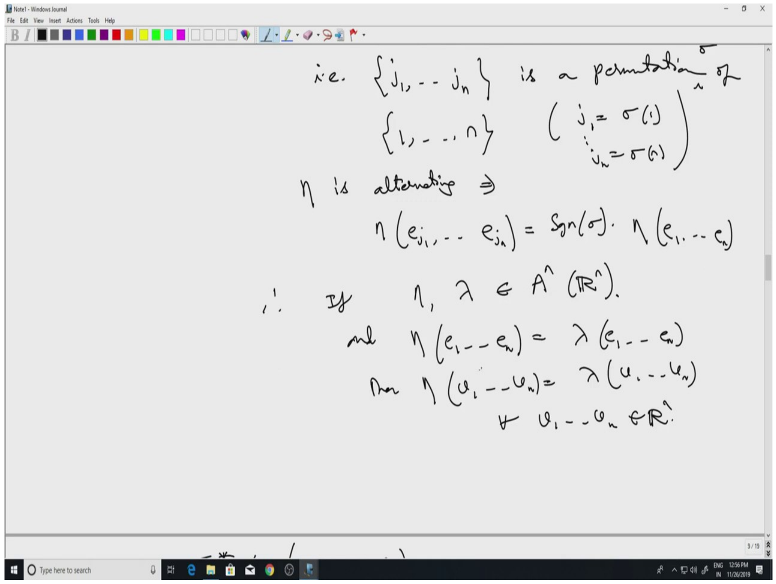

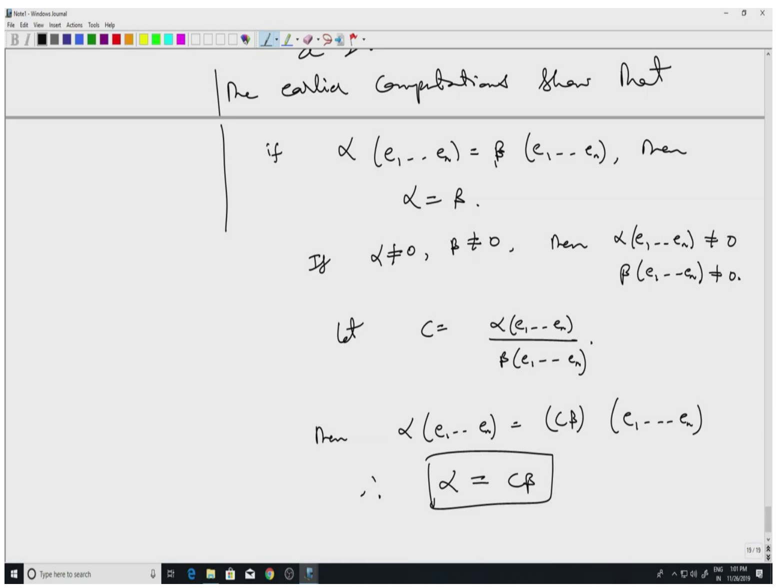

Now, the other thing is, the other thing we can do is, let omega 1, ah no, okay, omega 1 omega n be the dual basis of, dual basis corresponding to the standard basis e1 up to en of Rn. So, i.e. omega as usual, omega i eta j equal to 1, if i equal to j, equal to 0 if i not equal to j. So, let us look at omega 1 wedge omega 2 wedge omega 3 et cetera, all the way up to omega n.

Now, I am being careful in putting brackets around each of these product, because, at this point we have not proved that this wedge product is associative. So, in fact that turns out to be the slightly non trivial point here, unlike tensor products where it is quite straight forward to check that it is an associative operation. Here it requires a bit of work and, but it is in fact true that the wedge product is also associative. So, I do not have to be concerned too much about where I put the brackets. But at this stage, lets us keep the brackets and proceed with the calculation.

So, omega 1 wedge omega 2 and so on, this, I claim that this thing here is nothing but, so this first one, let us notice that omega 1 wedge omega 2 will be a 2 form, alternating 2 form and then when I take omega 3 it will be 3 form and so on. So, ultimately I will end up getting, so I do not have to put these brackets at the end. So, ultimately I end up getting something in An Rn.

And the claim is that this, thing here is exactly the determinant in fact. So let us call this as alpha. Let alpha equal to this. Then alpha v1 up to vn is nothing but the determinant of v1, the matrix obtained by putting all the columns of, all these columns, vis as columns. So, and how does one prove this? One can try to do this explicitly. But there is a nicer way of going about it, namely first prove the associative property first, prove the associative property of the wedge product and then observe that once one has the associative property, it will follow that.

The space of n forms, is essentially there is only one element up to multiple of, up to scalar multiples and that is the determinant and then it becomes a question of determining what the multiple is. That is one way, but the other way is that one does not really need that associativity. One can directly try to prove that. Let us check this, the following. To see this, observe that if dimension of V is n, then An V is, so dimension of An V is also, is not also, it is 1.

So, on n dimensional vector space, the form, the space of forms of degree, alternating forms of degree n is 1 and in fact, even though we do not need it right now, and dimension of Ak V equal to 0. In other words, all of these, the only 0 form is contained in these spaces if K is bigger than or equal to n plus 1. So, unlike tensor products, you can, even on an n-dimensional space, can keep on taking tensor products of 1 forms and you will get as large a degree as you want, order as you want.

But for an alternating form, you have to stop at K. If it is more than K, there is nothing other than the 0 element and let us see this. So, the point is that, let, this is (cal), this competition is very similar to what we did earlier. Let alpha belong to An V, alpha and beta. To say that a vector space is one-dimensional amounts to saying that given any two non zero elements, one is a multiple of the other, so let us check that.

So, alpha beta, so I start with 2 elements here and we have already seen in my, when I did this pullback calculation, the pullback of the determinant function. Then, yeah, right here, this, in this calculation, so I said that, what we in effect proved was that, if eta and lambda are 2 elements of An Rn and if they agree on this standard basis in the specific ordering, then they agree for all vectors.

So, in other words they are identical forms. Here the role of e1 e2 en is the standard basis. In other words, the fact that we started with the standard basis, as the proof shows is not that important. One might have very well started with any basis and still the proof works. So, in short, if you use the same proof there, we end up concluding that, so let alpha beta belong to this and let v1, I have already, I am going to use V for arbitrary vectors. Let e1 en be a basis for V, I mean, since V is any vector space, does not make sense to say is standard basis, just any old basis, I will just call it e1 up to en.

So, I want to claim, our earlier calculations show that if alpha e1 up to en equals beta e1 up to en, then alpha is equal to beta. And, so now if alpha is not equal to 0 and beta is not equal to 0, not only that, if these two are equal then this is equal to this. And the value of alpha of v1 up to vn or beta v1 up to vn is essentially, it is a multiple of alpha e1 up to en. So, if these are non zero forms, then on this, this specific numbers have to be non zero. Then alpha e1 en is not equal to 0, beta e1 up to en is not equal to 0.

Right, so, if this equal to, so what we can do is, if these two are non zero numbers, in order to ensure that alpha of this, alpha e1 up to en is beta en, we can just, let C equal to alpha e1 up to en divided by beta e1 up to, and let omega equal to C times beta. Then omega, then oh, I do not even have to give it a new name actually. So, let us just take C, then alpha e1 up to en equal to C beta acting on e1 up to en, because the way I set it up, these two automatically, this beta will cancel out and I am left with alpha. So then this, therefore now you go back to this remark. The earlier computation show that if this happens, then alpha. Therefore alpha equal to C beta.

So, what we have done is, shown that, as far as the top vector spaces, the top degree is concerned, it is a one dimensional vector space. The space of alternating n forms is one-dimensional. And since we are discussing this, let me also make this point clear that Ak V is 0 if K is bigger than or equal to V, that is also a very frequently used thing.

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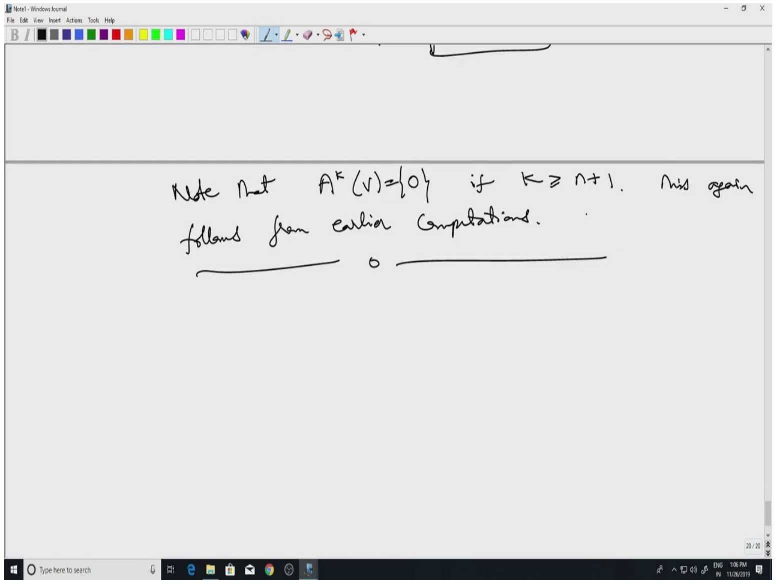

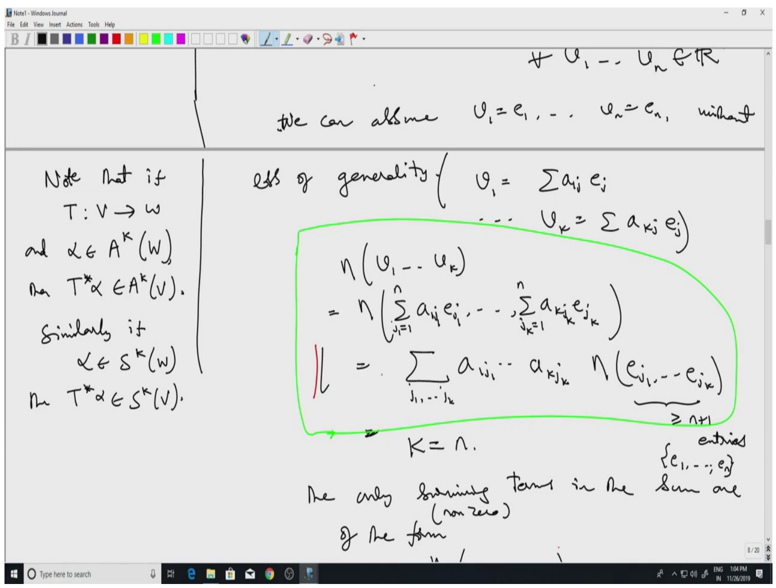

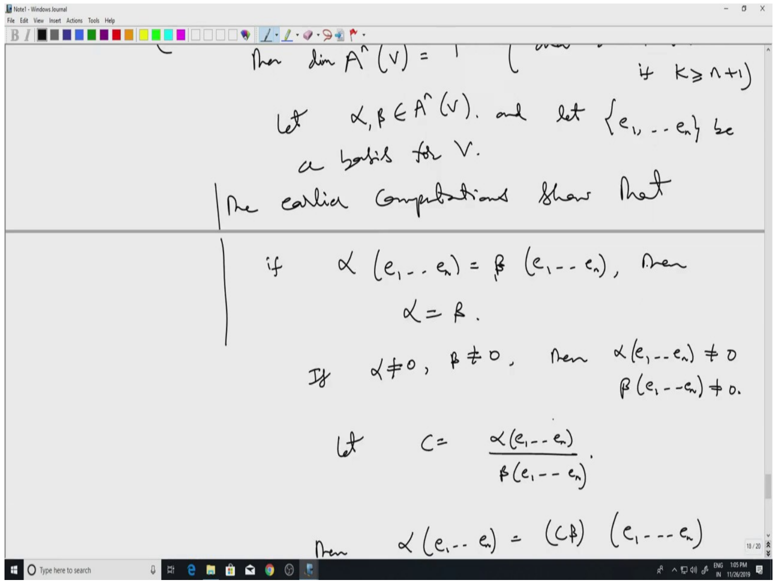

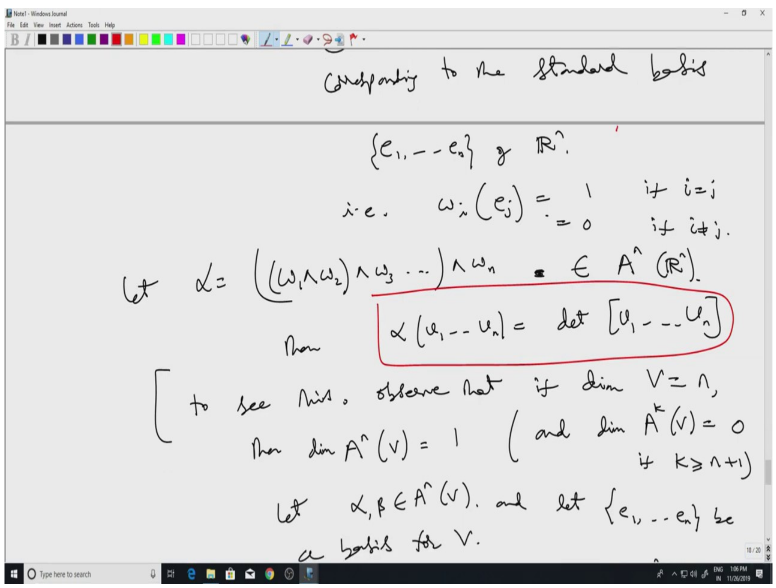

Notice that Ak V equal to the 0 vector space, if K greater than or equal to n plus 1. This again follows from earlier computations. The reason is, let us go back to that pullback example that I did and let us look at this thing here. This, what I had here. I started with a form of degree K and then I wrote eta v1 up to vk in terms of some constants and the specific values eta of ej1 ejk. Now if in the number of variables K, is bigger than or equal to n plus 1, then what will happen is. here there would be at least greater than or equal to n plus 1 entries inside the brackets.

On the other hand, since we have assumed that the dimension of the vector space is n, there are only n of these e1 up to, n of these basis vectors. But there are n plus 1 slots here, which have to be filled out. So there will necessarily have to be repetition in each of these terms. Some ei will have to equal ej in this. But since eta is alternating, the repetition will force the term to be 0. So that is the gist of the reason why. for K bigger than n plus 1, Ak V is 0.

So, it just follows by looking at reducing to basis vectors. And there are only n basis vectors, but they are more than n plus 1 slots, so, and so that earlier competition showed us two things. The dimension of A and V is 1, and this we showed by saying that any two non zero vectors are multiples of each other, any two non zero elements are multiples of each other. That is the same thing as saying the dimension is 1. So, and it also showed us that if K bigger than or not equal to n plus 1, Ak V is 0.

Now, but we are, at this point I want, I wanted to claim that, use the wedge product to claim that determinant is the wedge product of 1 forms which are dual to the, which is dual to the standard basis. So, I was interested in this actually. So, let us see. Right, so, what all this we have discussed so far just shows us that, this, any two 1 (for), n forms on Rn are necessarily scalar multiples of each other. So, there is a constant so that alpha equal to this. But I want to say that it is actually, that constant is 1, so I started with this alpha which is the wedge part.

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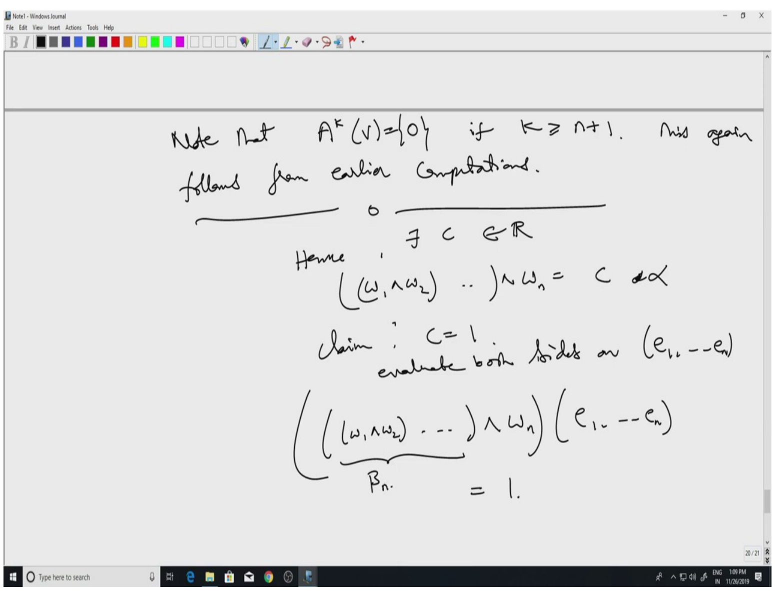

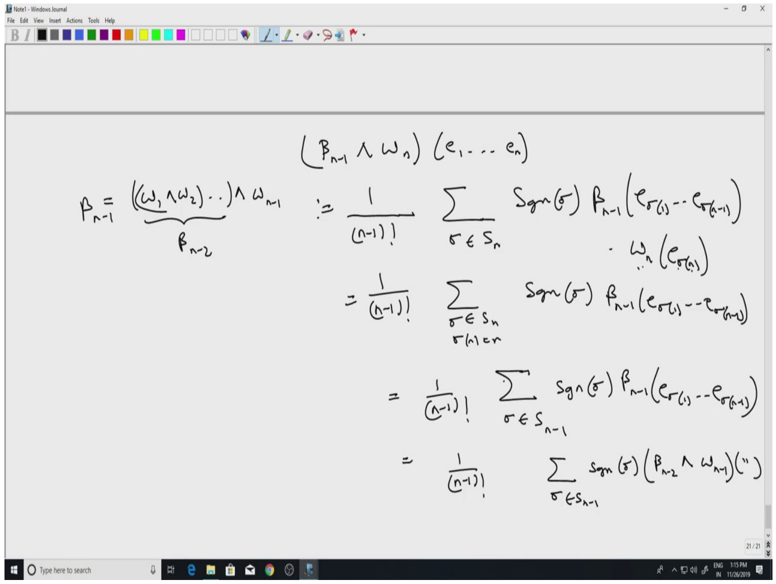

So, hence what this previous (discuss), omega 1 wedge omega 2, the successive wedge product is some constant times. Hence there exists C in r such that this is equal to times alpha, where alpha is the determinant n form. So, I want to say that, claim is that this constant is actually C equals 1. And this again, to see this, you just evaluate both sides on the standard basis vectors. So, when I do alpha of e1, evaluate both n forms, both sides on e1 up to en.

In the right hand side, of course determinant of the matrix even by e1 up to en is 1, so I will just get C. On the left hand side I will get omega 1 wedge omega 2 omega n, this entire thing evaluated on e1 up to en. And the claim is that, this is actually equal to, this is equal to 1 as well. So this is 1, and the right hand side will give me C, so therefore we done. Now to see that this is 1, you just, one proceeds step by step. So call this as beta n minus 1. So beta n minus 1 wedge omega n, so this is n minus 1 form. That is why this index is that, and e1 up to en. This is, by definition, n factorial by n minus 1 factorial.

And then of course, this is a 1 form, so this is sigma in Sn sign sigma beta n minus 1, e sigma 1 et cetera e sigma n minus 1 multiplied by omega n of e sigma n. Right, so this is just the (define), so here I can just write n times some stuff here sigma n. Right, oh actually there is no n factorial. That got cancelled with the alternating thing. So it is, actually it is 1 by n minus 1 factorial and then, now notice one thing here. Omega n is the dual basis. So the, and this term here, will be non zero if and only if sigma n happens to be n, for, if sigma n is not n, then this is 0.

So, it will only survive for those sigma such that sigma n equals n, and then sign sigma and then it will be 1. Beta n minus 1 e sigma 1 e sigma n minus 1. So what one ends up with is essentially 1 by n minus 1 factorial things. So this (corres), all those permutations of Sn which fix n, this essentially can be identified with sigma in Sn minus 1, 1 lower down and then sign sigma remains the same, then one has beta n minus 1. Everything is the same, except that I am now looking at this, so in other words, so, right I end up with this.

Now, you go back to the, again, so it is a inductive process. So you, now you go back to beta, what does the definition of beta n minus 1? It is omega 1 wedge omega 2 all the way up to omega n minus 1. So, what one wanted to claim was that this thing here, omega 1 major this entire thing is equal to 1, and what we have done is we evaluated it on e1 up to en and then reduced it to sum involving beta n minus 1.

And we have restricted the permutations to Sn minus 1, the point being that I can again write it as this thing here as a wedge product, so this would be 1 by n minus 1 factorial sigma in Sn minus 1 sign sigma, and then this beta is, so, I can call it again, this as beta n minus 2 beta n minus 2 wedge omega n minus 1, acting on this the same stuff.

Again, so when one looks at this one will get exactly one term which will (survi), no not one term, if this, only certain terms will survive here and one can proceed like this. However, so we will stop here and in the next class, I will resume at this point and then I will list down the basic properties of the wedge product that we will be needing. I may not be able to prove all of them as I said. Alright, so thank you.