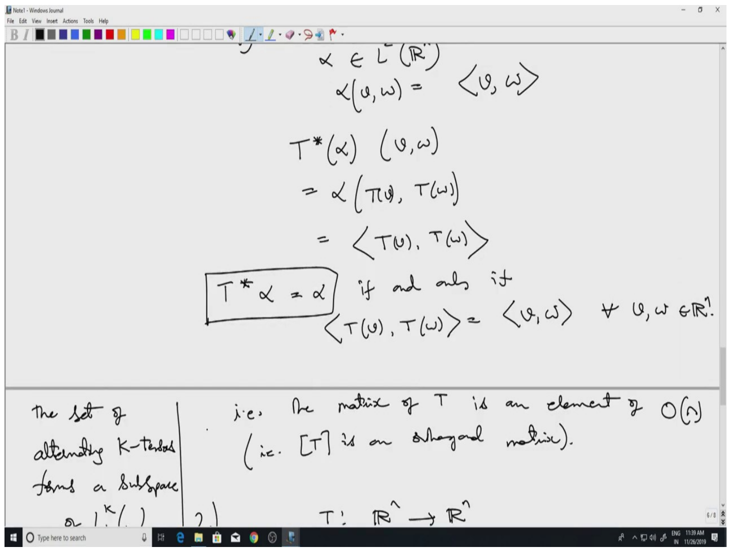
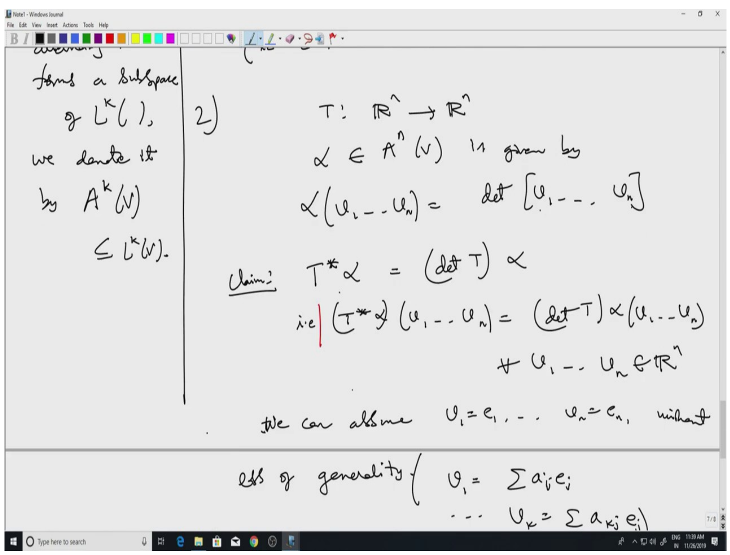
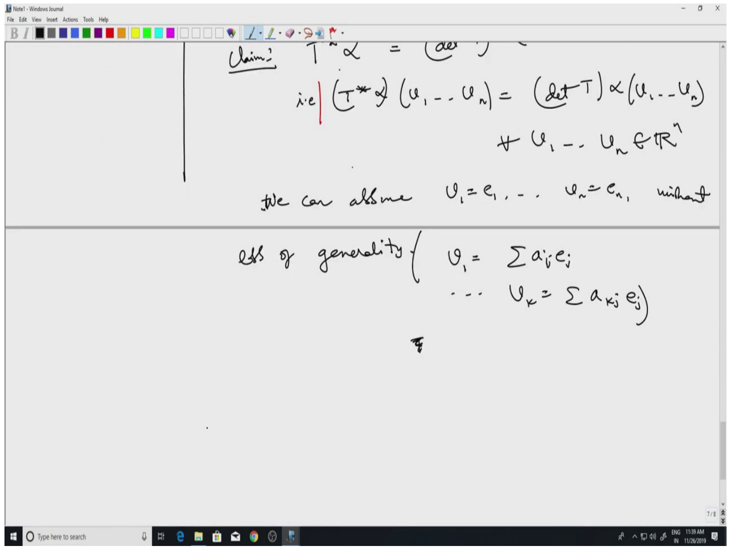
An Introduction To Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 50
Alternating Tensors 2

(Refer Slide Time: 00:32)



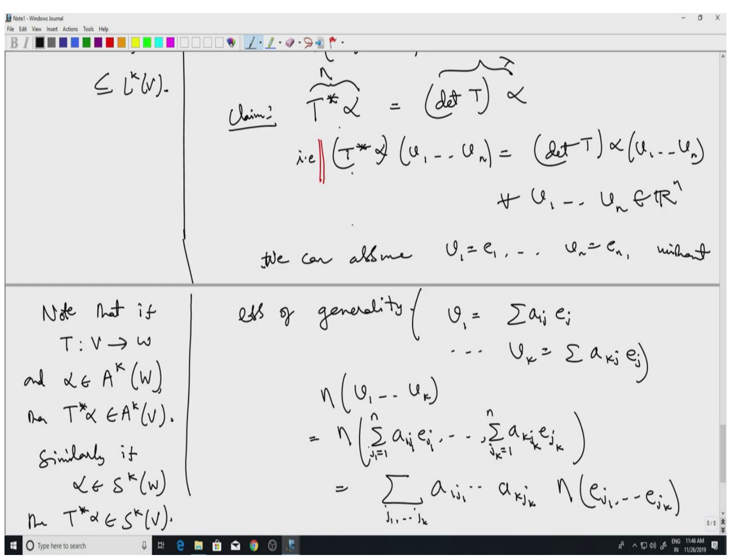




Hello and welcome to the 50th lecture in this series. In the, towards the end of the last class, I was discussing an example involving, seeing how pullbacks look like when we take pullbacks of (symet), certain very specific symmetric and alternating tensors. So, we did the case of the inner product and then, now we are looking at the determinant.

So, here is, and I want, the claim was that when I take the determinant on the form, the tensor given by the determinant, then the pullback and by the linear transformation, you will get back the same form or tensor, except that it will be multiplied by this constant, det T. And I was in the process of proving this. So, let me resume with that. So, I claimed that this equation that I had have, wanted to check, this one, instead of working with all vectors in Rn, it is enough to assume that v1 is e1, vn is en etcetera, a1j ej.

(Refer Slide Time: 02:14)



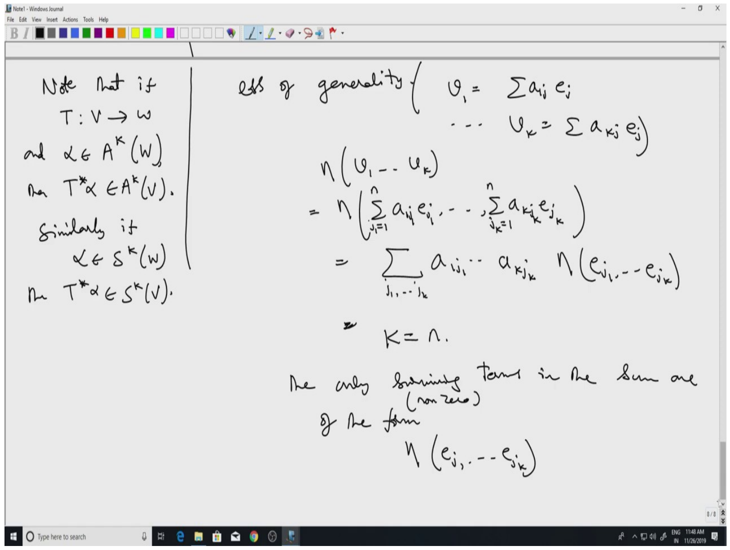
Right, so, without loss of generality, we can assume that these are this because in general we have this, any vi can be expanded like this. Now, note that T star alpha acting on v1, vn. Actually, let me write it like this. So, here is an separate statement that I want to make here which is that, note that if T is from V to W and alpha is a alternating k tensor on W then T star alpha, the pullback, is an alternating k tensor on V. This is again something which is immediate from the definitions, so I will not prove this. It is quite straight forward.

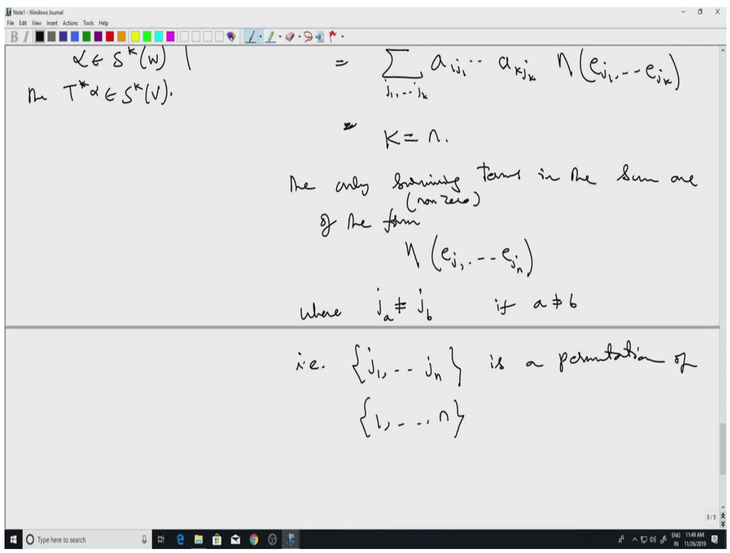
Similarly, if alpha is an symmetric k tensor which I denoted by Sk W, then T star alpha is in Sk V. So in short, the property of alternating or being symmetric, is preserved under taking pullbacks. Now, keeping that in mind, this T star alpha what I have here, this is alternating. And the right hand side is just alpha multiplied by a constant. So this is certainly alternating by assumption, no, not by assumption, because alpha is the determinant. So both sides are alternating tensors. And we have to check that they are equal.

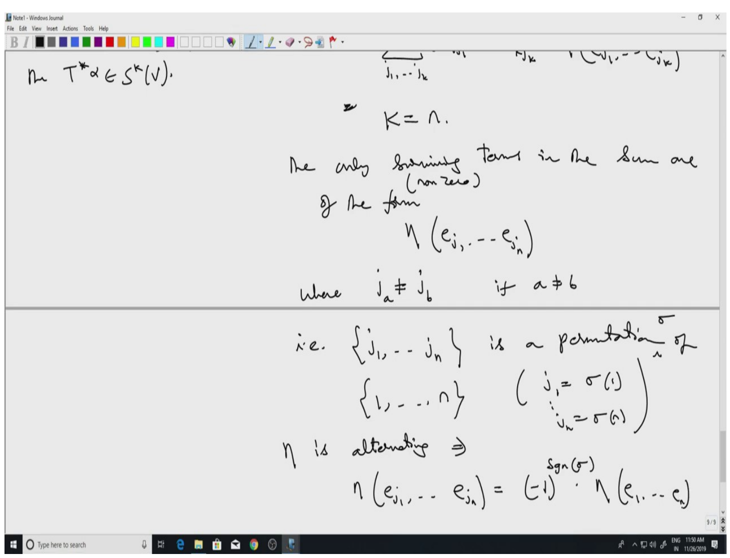
So, it is enough. So in that case, so it is immaterial actually, whether that I am dealing with T star alpha, determinant and so on. I might have as well, let us call it, let us call it eta and let us call this whole thing as lambda. The specific nature of these tensors is not that important. I just have two alternating k tensors and I want to check that they are equal. So what I do is, if I take either eta or lambda and act it on v1 to vk, then I will have this, then I plug in this expansions in terms of the basis vector, standard basis akj ej.

Now I use the, well, at this point I have to, the notation becomes a bit messy, because I cannot use the same j for all of these sums. So let me call the first one as j1. So j1 itself runs from 1 to n. And here I have, I have to call it jk equals 1 to n as well. So here I would have to say jk. And here I would have to put k as well, and with, the point is that after, that is not the main thing. The main thing is that, using multilinearity, I will get a sum over j1 all the way up to jk. I club all these coefficients together, product of this akj then eta of ej1 ejk. By the way, here, I am working with k, even though ultimately I am going to take k equals n, that is going to matter.

(Refer Slide Time: 07:08)







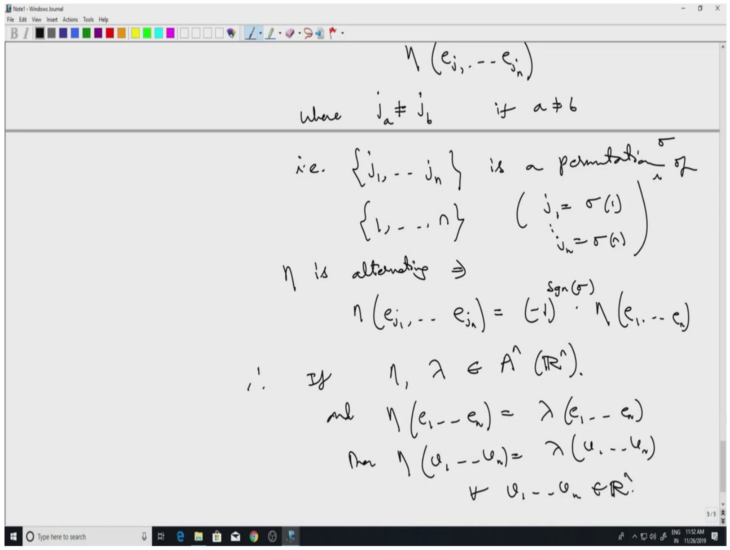
So now, this is what I have and then, in this expansion, in the sum, this jis, the indices jis can very well equal one another. But notice that, because of this assumption that if eta is alternating, the moment if one ji equal, ji1 equal to ji2, then this will be 0. So the only way this, the terms in the sum will survive, all, if all the j1, j2 all the way up to jk, they are all distinct. That is, in general, if k equals n, then what happens is that, the same remarks apply. So the only way this will survive if all this indices ji are distinct.

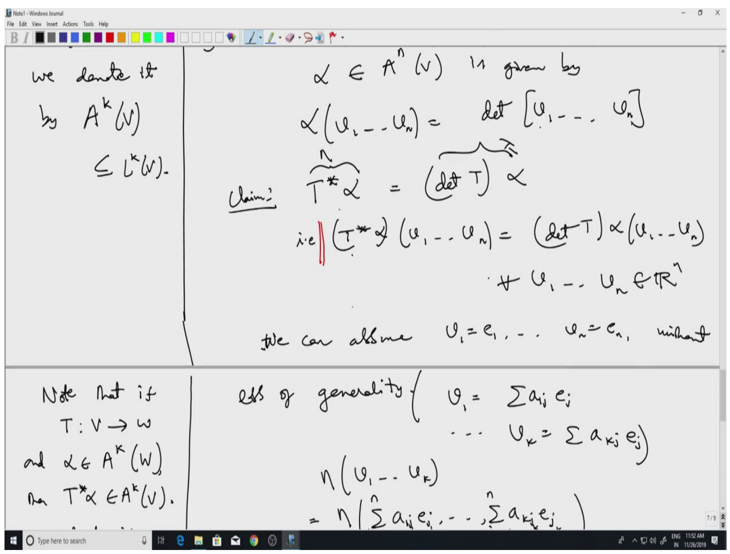
So, in other words, the only surviving terms in the sum are of the form, when I say surviving, I mean non zero, are of the form eta ej1 ejk, where ja is not equal to jb, if a not equal to b. And in other words, this would be i.e. j1, ejk oops, not e is immaterial, it says the indices which matter. So j1, jk is a permutation of 1. Now, let, I can drop, so I should no longer keep k. So, j1, jn, so here 1 up to n. So this index set is just a permutation of this 1 to n, because they are all distinct and they are exactly n of these. And each one is, lies between 1 and n. So, therefore, it is a permutation of this set.

So, in other words, so this is, but then if this is a permutation, and since we have assumed that eta is alternating permutation, let us call the permutation sigma. So, when I say this is a permutation sigma of this, what I mean is j1 equals sigma 1, jn equals sigma n. Eta is alternating, we have seen that. Then this thing here, ej1 ejn will be this, minus 1 to the power sign of this permutation times eta acting on e1 up to en.

So, in short, what happened, what we are saying is that this, we started with eta v1 up to vk, then we used multilinearity to break it down to an expression of this type, to get a sum here. And then we looked at each term in the sum, the coefficients a1 aj are not, they are fixed by the vectors, but what is of interest is these terms here, eta, ej1, etc.

(Refer Slide Time: 11:36)

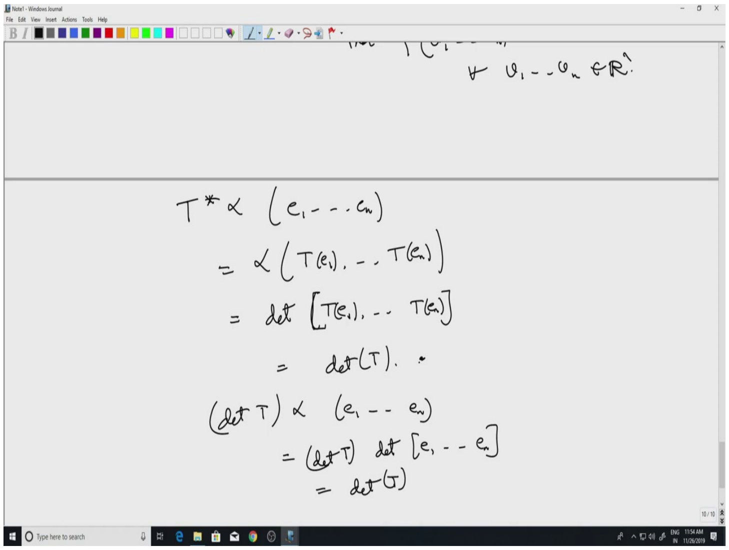




And what we have seen is that, each of these terms is essentially minus 1 to the sign sigma eta e1 en. So, in other words, therefore, the conclusion is that, therefore, if we have two, if eta and lambda are two n forms on Rn, actually Rn is not that important, but since I am working with, in this example, we are working with Rn. So, if eta and lambda both n, alternating n tensors on Rn, and eta of e1 up to en equal to lambda e1 up to en, then eta v1, vn equal to lambda v1, vn for all v1 up to vn in Rn.

So, which is what we wanted to claim, that, if they agree on this basis, basis vectors, then they agree for all vectors and this eta and lambda are what, T star alpha is eta, the right hand side is lambda. So, that has reduced it to this case of looking at a basis vector, but so let us see what happens for a basis vector now.

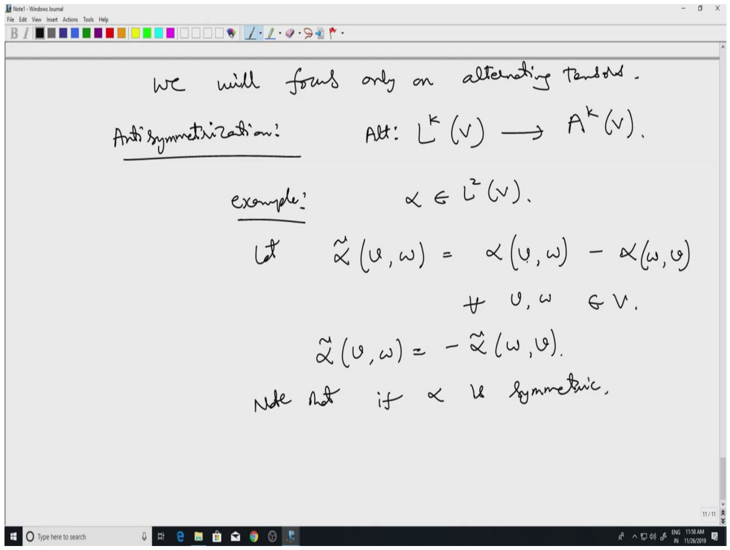
(Refer Slide Time: 13:10)



So, I want to look at T star alpha, alpha is the determinant on acting on e1 up to en, this is alpha acting on T e1 T en, which is just determinant of, because, well before that I should write it like this. Perhaps, determinant of T e1 square brackets T en. This is this determinant of T. After all, T e1 will give me the first column of the matrix of T etcetera.

So, I get determinant of T. And which is (equ) and the right hand side was determinant of T multiplied by alpha acting on e1 en, this is determinant of T times determinant of e1 en. This is just the identity matrix. So, this is also determinant T. So, both sides are equal to determinant det T and 1 is done. So, in short so, this, whatever we, the calculations we did in this are very useful for later on. And I will have occasion to refer back to this example subsequently.

(Refer Slide Time: 15:04)



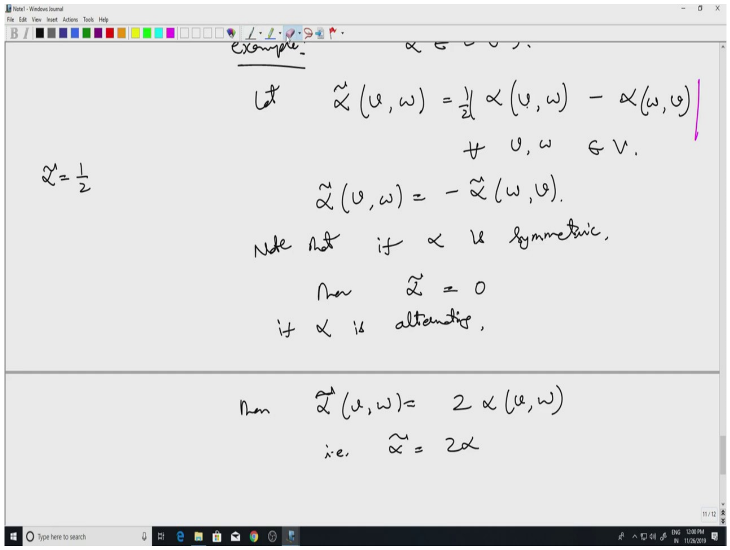
So, now I want to describe an important, so we will focus only on alternating forms. We will, in this course, symmetric form tensors will not play any subsequent role. So however, these alternating tensors are very important mainly because of the connection with determinant. Now, what I want to do first is, I want to describe a way of getting, it is sometimes called Anti Symmetrization, anti symmetrization.

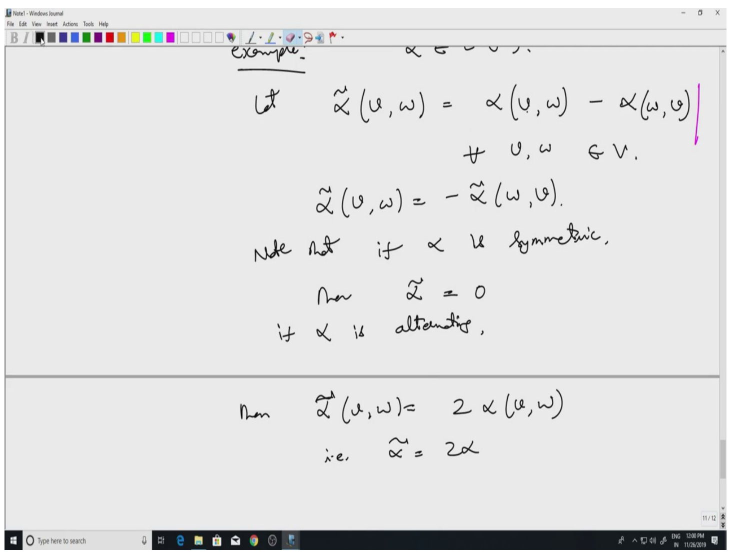
So, I want to start with any multilinear k tensor or form. And then, out of that I want to get an alternating one. This, let us call this map Alt. So let us see how to, I mean how one would go about this. For instance, let us say, right. So example, suppose I start with let us say alpha is a multilinear, it is a bilinear form, in other words, multi linear two form and I want to get an alternating two form out of this.

So what I do is, let us define alpha tilda of v, w to be, let alpha tilda of v, w to be alpha of v, w minus alpha of w, v for all v, w in V. Now if I do this, it is clear that this alpha tilda will still be a multilinear form. And the important new thing is that, alpha tilda will be alternating. Alpha tilde of v, w equal to negative of alpha tilde of w, v from the way it is defined.

And it is not the case that we will always get something interesting out of this. You can (ha) start with some alpha and end up getting 0, the trivial alternating form 0. In fact, that we can say, when exactly that happens, if the alpha that you started with, for example was symmetric. Then note that if alpha is symmetric, for example, if you started with the inner product, if we start with a inner product there is no, the inner product, so, if alpha is symmetric, then alpha tilda is the 0 form.

(Refer Slide Time: 18:42)

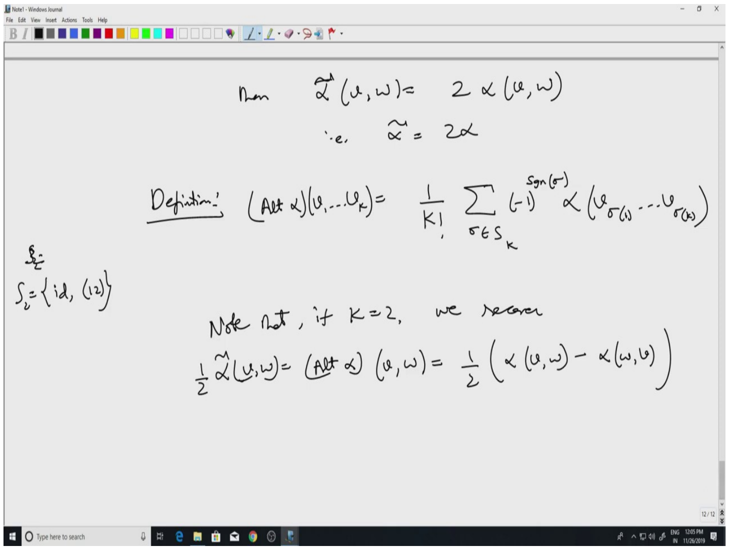




And moreover, alpha is symmetric, this happens. And if alpha that you started with is already alternating, then, well, so you would, alpha v, w alpha tilda of v, w would be equal to. So this, the second term here, would be the same as the first term, but get a minus sign, there is already a minus sign so it becomes plus. So I will end up getting two times alpha v, w. i.e. alpha tilde is 2 alpha. So that is why it is nicer to put, I should put a half here. If I put a half, if I defined alpha tilda as half, if I put a half like this, then I will be, I will end up getting alpha tilda equal to alpha if alpha is alternating.

So, this serves the purpose, the namely, we start, at least for 2 forms, it is quite easy. So, I just do this and I get what I want. So, ideally, I would like to get a, I would like to have a map, from, start with any multilinear k form and end up with an alternating k form and this should have the property that what, if whatever I start with is already alternating, then it should not change like this, except for this constant of 2. And there is a way of going about it. That is called anti symmetrization.

(Refer Slide Time: 21:05)

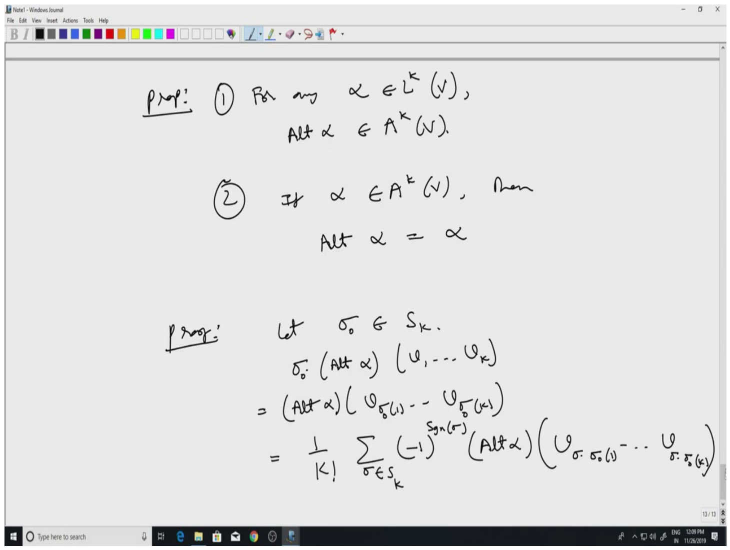


So, for any k, so, for 2 we have already seen what it is, for any k. So, we are going to, we can define this map. Alt alpha is 1 over k factorial, summation sigma in Sk. Oops. So I have to, the way it is defined, you define it by its (action) values, just like we did for k equals 2. So, I want to see Alt alpha acting on v1 up to vk. And this I defined to be 1 over k factorial alpha minus 1 to the power sign of sigma. And then I put a, then I see what alpha of v sigma 1 is, v sigma k is.

So, this should remind us of a, something that, that basically it should remind us of the definition of the determinant. The determinant is also exactly defined in like this, determinant of an n cross n matrix. Here the difference is that, this k need not equal to the dimension of the vector space. Even if k is less than that we can define it like this, we will see what happens, we will see that if k is bigger than n, then the right hand side is automatically 0.

And so, the only interesting ones are when k is less than or equal to n. So, this is what this is the definition, and note that if k equals 2, we recover Alt alpha, which I called alpha tilde, except for the factor of this k factorial, 1 over 2 and then so essentially there will be 2 S2. So I will be essentially looking at permutations. S2 is permutations on 2 letters, which is, is in, it consists of just two elements. One is identity. The other one is the transposition, which interchanges 1 and 2. If I, so for the identity permutation, of course, I will just get alpha of v, w. And for the transposition, I will get a minus sign and then w, v. So this is half alpha tilde.

(Refer Slide Time: 24:57)

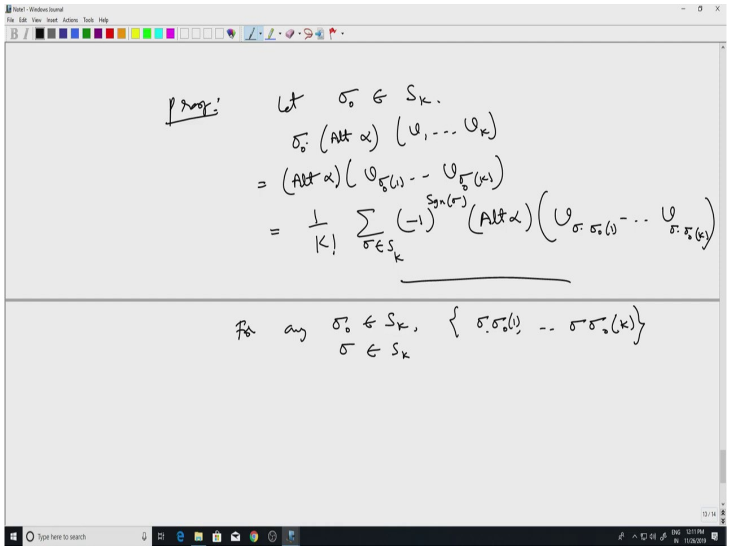


And in general, so the main properties of this are the following. Proposition, at least for now, let us know the goal was to actually get something which is in Ak V. So let us, that should be the basic property. So for any alpha in Lk V, Alt alpha belongs to Ak V. And the second thing we would like is, if alpha is already an alternating form, then Alt alpha should be equal to alpha. We do not want a process which will change something which is (al) already alternating. So, this is a natural thing to require and it is in fact satisfied by this definition of Alt alpha.

So, let me check the first one quickly. Right, so, I would like to know that Alt alpha. So I am going to use, remember we had various equivalent characterizations of, the property of being alternating. One of them was that if let start with the permutation, let sigma belong to Sk. Then we want to check if sigma times Alt alpha. We want to see what this is. If this Alt alpha is indeed alternating, then we would, we should end up getting minus 1 to the power sign sigma times, whatever we started with, which is Alt alpha.

So let us see if that happens. I am going to just act on k vectors. Now, this, by definition of thos sigma multiplication is Alt alpha acting on v sigma 1, v sigma k. And this, now I go to the definition of Alt alpha. This is 1 over K factorial. So I have already, so let us, let us call it sigma naught, sigma naught. So here we are fixed a specific permutation, sigma naught. In the definition of Alt alpha, I need to consider all permutations. So, index them by sigma and this would be, well, minus 1 to the power sign sigma and then essentially Alt alpha acting on v. Now here it is sigma multiplied by whatever index you have here, you just acted sigma of sigma on that, v sigma, sigma 0 k.

(Refer Slide Time: 29:08)



Now, here is the main point that, for any sigma not in Sk, if we look at these products, sigma sigma naught, sigma sigma naught. In other words, oops, so here I am evaluating that on sigma naught, this and sigma naught. So here, what I am doing is, evaluating this, sigma naught, sigma sigma naught on 1, 2 etcetera. Now the, this is anyway sigma multiplied by sigma naught is a (permute) again a permutation lying in Sk.

So, this set will be just a permutation again, and the question is, how do we rewrite this, the sum that I have here, so that I can take out the, I will get the right sign out, I need a minus 1 raised to sign sigma naught rather than minus 1 raised to sign sigma. So, I will, we will discuss that in the next lecture. So, we will stop here for now. Thank you.