An Introduction to Smooth Manifolds
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Lecture 49
Alternating Tensors 1

Welcome to the 49th lecture in the series. So I will continue talking about tensors and forms. In this lecture, I will, I will discuss a bit about alternating tensors, which are also called alternating, I mean, sometimes we say alternating forms or alternating tensors.

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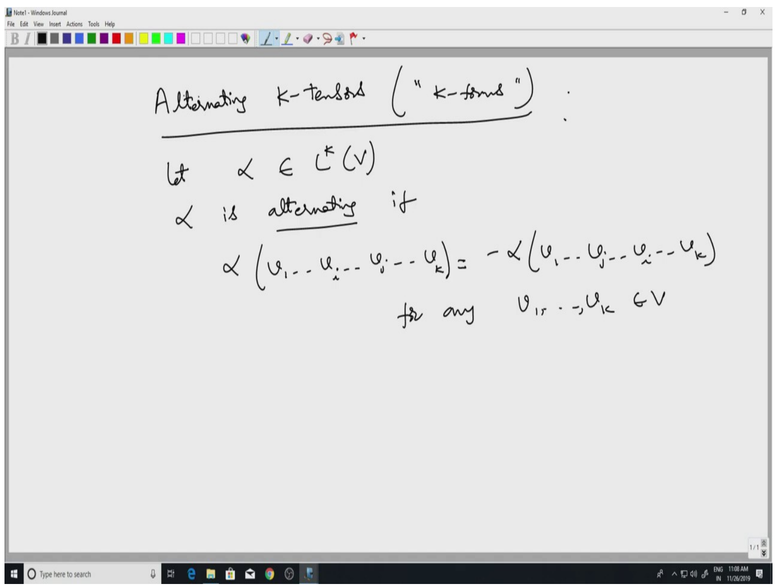

So, last time I talked about symmetric tensors, which is the, sort of the opposite of an alternating tensor. So let us say alternating. So we will look at, let, we look at a alpha and what we call Lk V, k multilinear maps on V. Alpha is alternating if, whenever we interchange two of the input vectors, this, we get a minus sign. for any.

So, recall that we called a tensor is symmetric, if, whenever we flipped two of the input vectors, the sign did not change, the value did not change, here the, the value acquires a negative sign. So, just like in the symmetric case, we have a nice characterization of alternating tensors in terms of the symmetric group action as on Lk V.

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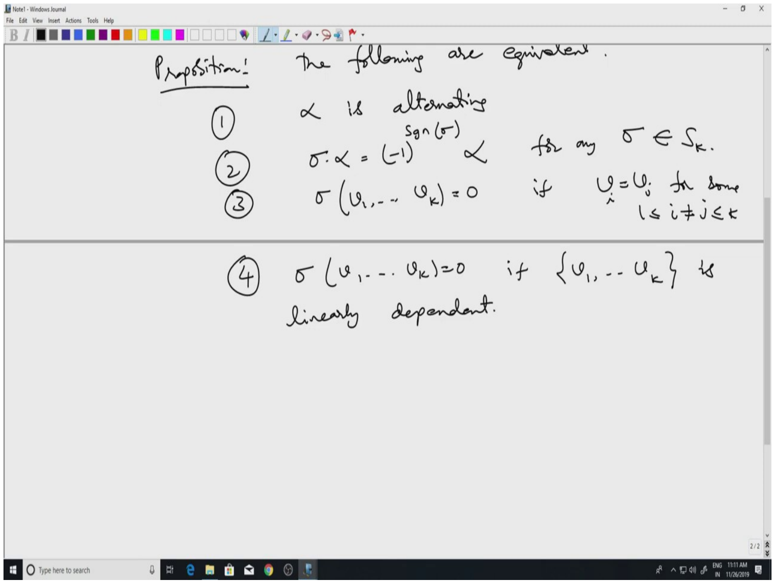

So, proposition, the following are equivalent. Alpha is alternating. The second thing is that sigma alpha, which I defined in my last lecture, sigma times alpha or sigma acting on alpha is equal to minus 1 to the power sign sigma alpha, for any sigma, the symmetric group on k letters. The third thing is sigma, if, whenever two input variables are equal. So here I want vi, vi equal to vj for some i not equal to j, greater than or equal to 1, less than or equal to k. The fourth thing is a similar thing. This is 0 if the set of vectors v1 up to vk is linearly dependent.

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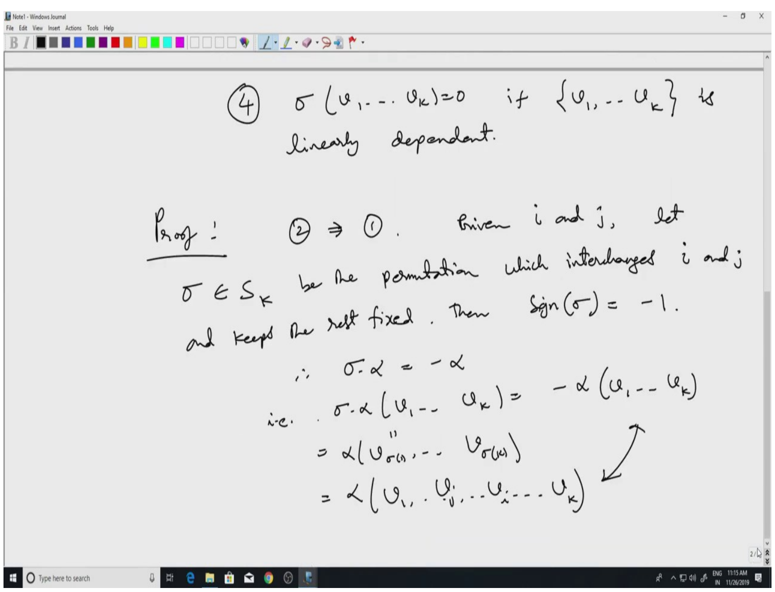

Right, so, let me, I would not go into all the details. I will briefly discuss some parts and let us start with the (equi) equivalence of 1 and 2. So it is where the proof is very similar to what we had for symmetric tensors, namely let us say, let us assume 1. Well, before I do that, maybe I should point out the easier one. 2 implies 1, is follows by taking given i and j, let sigma be the, sigma, the, take the permutation, be the permutation which takes, which interchanges i and j and keeps the rest fixed.

So, this is, this is what we call the transposition, sigma, take sigma to be the transposition which interchanges i and j and keeps the remaining fixed, rest fixed is a, right. Then sign sigma, the way we define the sign of a permutation was, you just had to write it as a product of transpositions. And then just count how many there are. They are even number, we say that the permutation has, the sign is 1, if they are odd number, then the sign is minus 1.

So here this exactly, sigma itself is a transposition. So the sign would be minus 1. It is exactly 1. Well, given this, then you just go back. So we are assuming 2. Therefore sigma dot alpha would be minus alpha, i.e. so if I act it on any vector, sigma dot alpha acting on v1 up to vk would be minus alpha acting on v1 up to vk. And this by definition is alpha v sigma 1, v sigma k, and since all sigma does is interchange i and j and keeps, keep the remaining numbers fixed, remaining indices fixed.

So what we get as 1, well, assuming that i and j are not 1 or k. So it will interchange, so wherever vi occurred, vj will occur and wherever vj occurred, vi will occur. Oops. So sigma k would be just k. Again, assuming that i and j are not 1 and k. Otherwise, we would have to do the swap in the first and last one of the two slots. So, the equality of this and this is precisely the alternating condition that we had started with.

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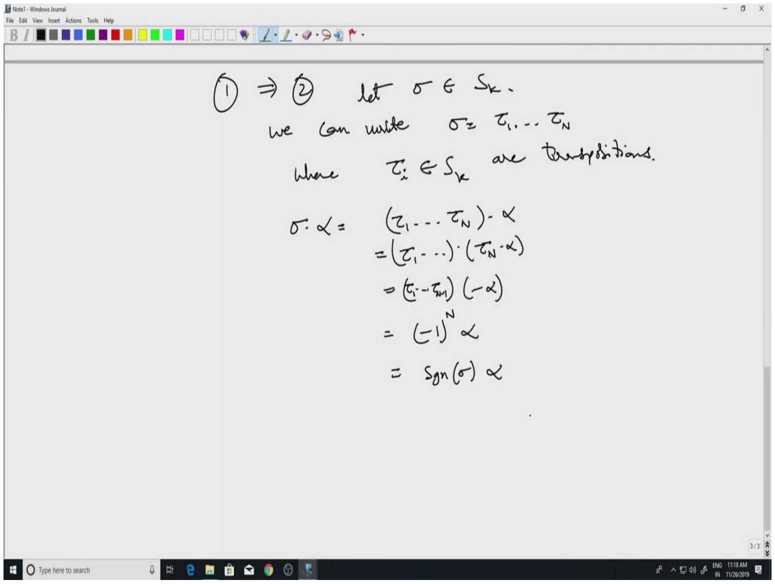

Well, and the, 1 implies 2 follows from again, just like in the symmetric case, we write, we write sigma, start with the sigma first. Let us start with, let sigma belong to Sk. We can write sigma equals tau 1 tau N, where tau i in Sk are transpositions. And once we have this, sigma dot alpha is tau 1 tau N act dot alpha, so we know that, first I can do tau N and then do the rest.

And we know that this, we are assuming that the form is alternating now. And the form being alternating amounts to, as the first step shows, it amounts to saying that the action of any transposition will reverse the value, sign, make the, keep the magnitude of the value the same, but the sign gets reversed.

So this would be, I will pick up a minus sign from each one of these. So after the first step, I will get minus alpha, and now there are tau N minus 1 transpositions here. So, and then now I do again, this tau N minus 1 of, so each time I will pick out minus 1 alpha minus 1. So it will be a product of N of these minus 1's and this is precisely what we called the sign of sigma times alpha.

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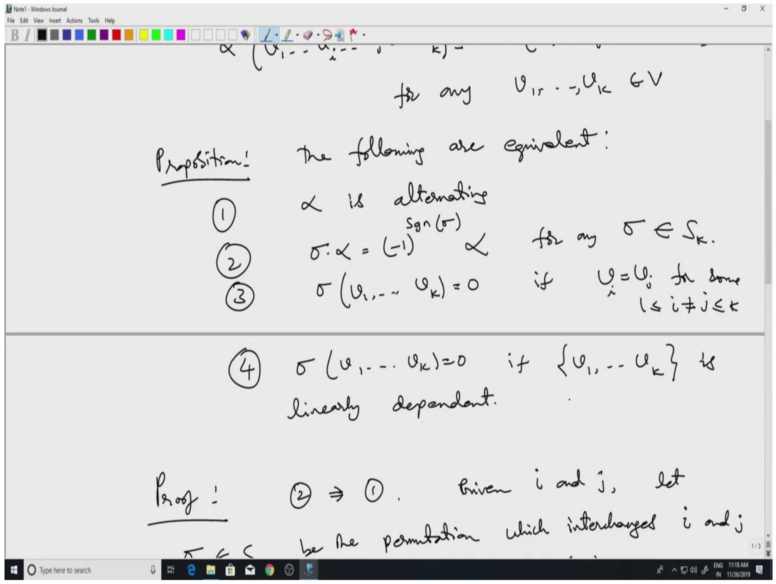

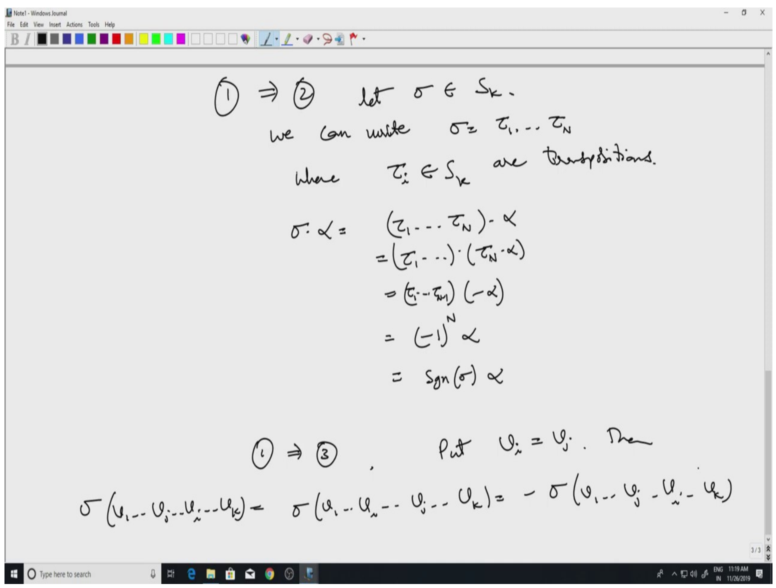

Now 3 and 4 again directly follow from 1. For instance, let us say 3 actually. So, if I have 1, 1 implies 3. That follows just by putting vi equal to vj. Put vi equal to vj. Then if I interchange these two, then sigma v1, vi, vj, vk equal to minus. This is from the alternating condition. But since vi and vj are the same, I can also write this as v1, vj, vi, vk. And, oh sorry, here I should make it, I have not interchanged the two. So this should be vj, vi. Notice that the left most term and the right most term are the same, same except for a negative sign. So, therefore each one is equal to zero.

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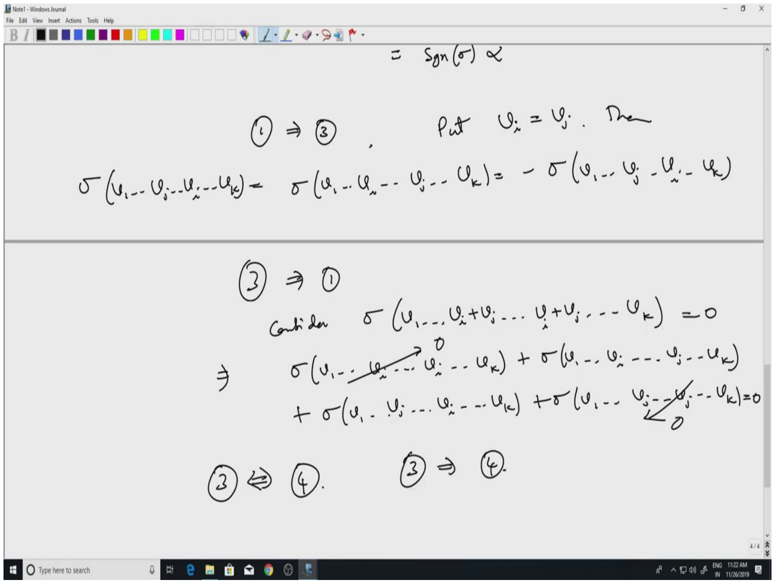

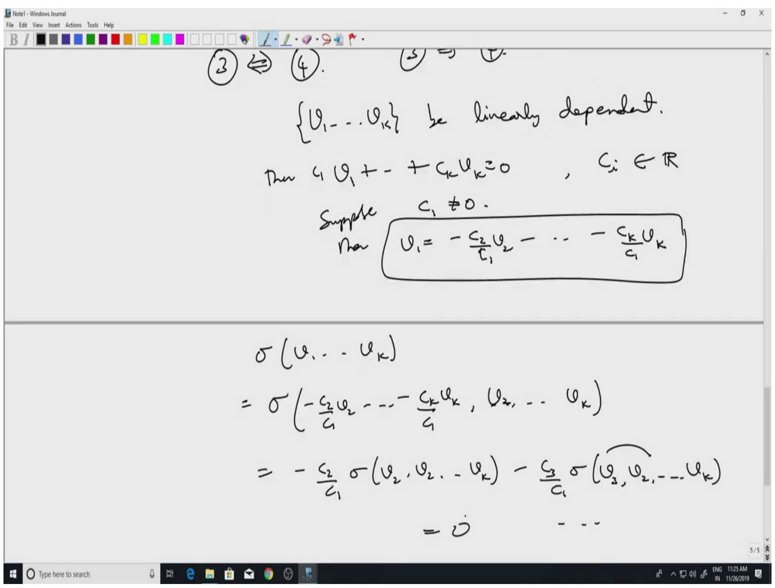

And as for 3 implies 1, if you know that, whenever the two variables are the same, the value is 0. And we were, if you know that, and we would like to see that whenever I interchange I get the required thing. The standard thing to do is consider sigma of v1, vi plus vj in the ith slot, vi plus vj, oops, not i plus j, vi plus vj in the jth slot and then vk.

Now, I know that this is 0 because these two input vectors, vi, I get, I have vi plus vj two slots. So I have this. So I expand it out using the multilinear property. So I will get four terms v1, vi, vi, vk plus sigma v1, vi, vj, vk plus sigma v1, vj, vi, vk plus sigma v1, vj, vj, vk equal to 0. And again, by hypothesis, this is 0 and this is 0 as well.

So, I am left with two terms. One involves vi first and then vj second. The other one is vj and vi. So these two have to be negatives of each other. So that proves that 1 and 3 are equivalent. And once you know 3, and it is easy to see that 3 and 4 are equivalent as well. For instance, if 3 implies 4, one can do the following. If I know that any two need input variables are 0 or, and I start with a linearly independent set.

Let be linearly (depend), linearly dependent, sorry, not independent, linearly dependent. Then, by definition I have this equal to 0, Ci, all the, so all the coefficients are just real numbers. Ci in R and not all of these 0. That is the part of being linearly dependent. So, let us say, suppose, just for convenience, let me take C1 not equal to 0. Then, I can write v1 equals, as a linear combination of the others minus C2 by C1 v2 minus Ck by C1 vk.

And then once I have this, I evaluate sigma v1, vk. Instead of v1, I plug in this, whatever I just had, this linear combination here. So minus C2 by C1 v2 minus Ck by C1 vk and here I would still have v2 and vk. Then I expand it out using the multilinear property. So I will get minus C2 by C1 sigma v2.

Well, the remaining stuff I keep as they are. So this would be v2 vk minus C3 by C1 sigma v3 v2 dot dot dot vk et cetera. So I will have k terms. Notice that each one of these terms, in each one of these terms, two of the input variables are repeated. First one, it is v2, v2 second one v3 and here another v3 would be there, that would repeat. So, all of these are 0. So, therefore I get this equal to 0.

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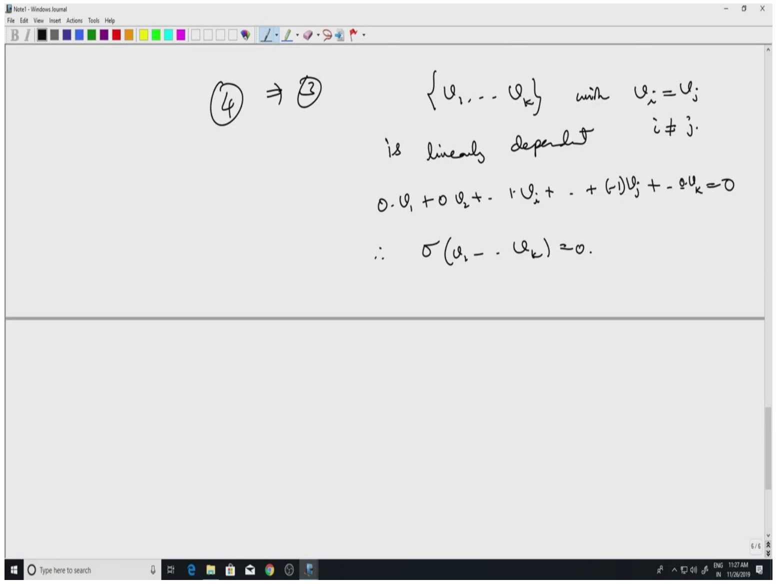

And 4 implies 3 is also straight forward. 4 implies 3 is the condition that, if you know that they are all linearly dependent, then inputting, if I, so let us say, let, I start with a set of vectors with vi equal to vj for some i not equal to j. Now the point is that, this is automatically linearly dependent, this, linearly dependent. Because I can put 0 times v1 plus 0 times v2 dot dot dot plus 1 times vi, I can go on putting zeros in everything, except when it comes to the jth slot, minus 1 times vj equal to 0. So this is linearly dependent. So, therefore I can conclude that sigma v1 vk equal to 0. Right. These are various ways of thinking about alternating forms.

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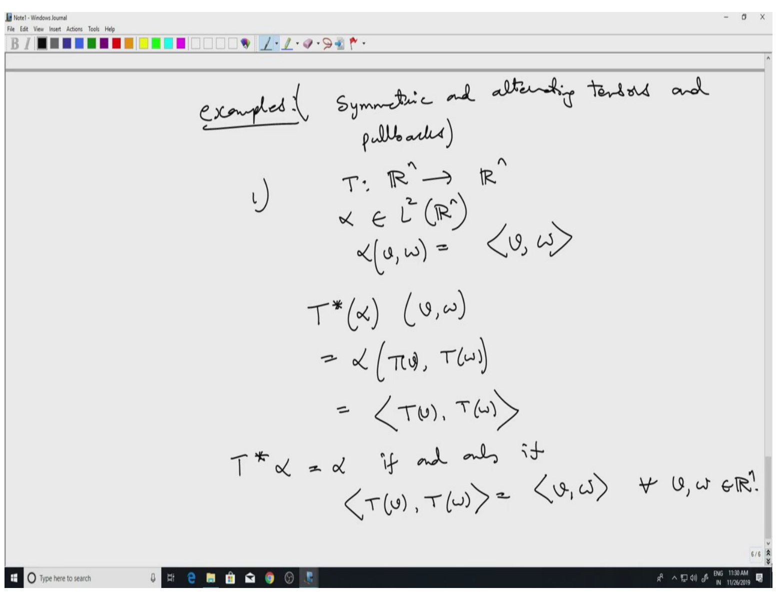

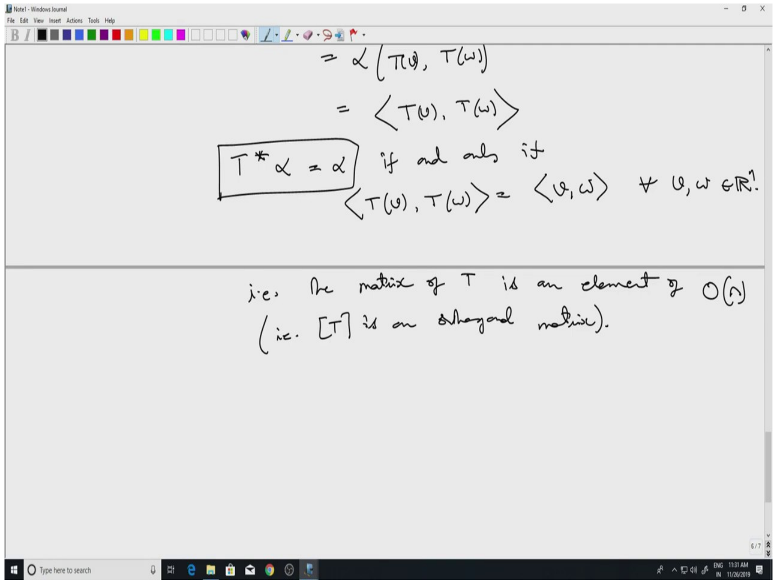

Now, I had briefly talked about pullbacks and the linear maps of a k tensor. So let us go to that and have a, in the, let us have a look at that in the context of symmetric and alternating tensors. So let us first look at the case of symmetric tensors. So these are examples involving symmetric and alternating tensors and pullbacks.

So, let us look at a linear map T from Rn to, let us look at this case, vector spaces are Rn, just one vector space Rn. And I want to look at the form alpha in L2 of Rn, the dot product of v, w is a product or dot product of v and w. Now so let us consider T star alpha. So, alpha can be regarded, of course this, the target and domain are the same. Alpha is present in both domain and target, but I am looking at alpha on the target.

So, let, since the pullback operation proceeds in, you have to start with a form, a tensor on the target practice space. So T star alpha acting on v, w would be, this is by definition, alpha of Tv, Tw which we know as Tv, Tw. So. so essentially the pullback operation of, acting on v, w is the inner product of, first you do T and then you take the inner product.

Now, if we ask the, if you want to know when, when is T star alpha equal to alpha. T star alpha equal to alpha if and only if inner product of Tv, Tw equal to inner product of v, w for all v, w and Rn. Well, we know exactly when this happens i.e. the matrix of T is an element of O n. i.e. (mat) the matrix of T is an orthogonal matrix. So, the, to say that T belongs to O n or the matrix of T belongs to O n, is the same thing as saying that the pullback of the inner product under T is the same as alpha. Now this is one example.

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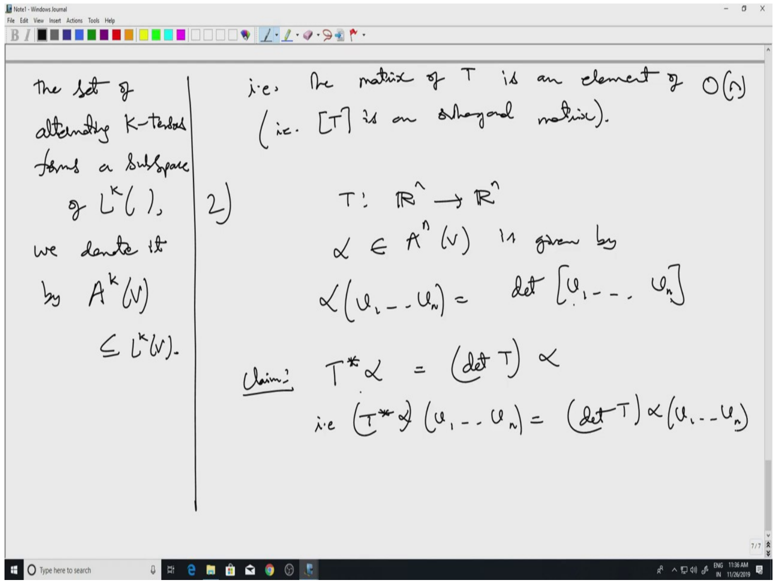

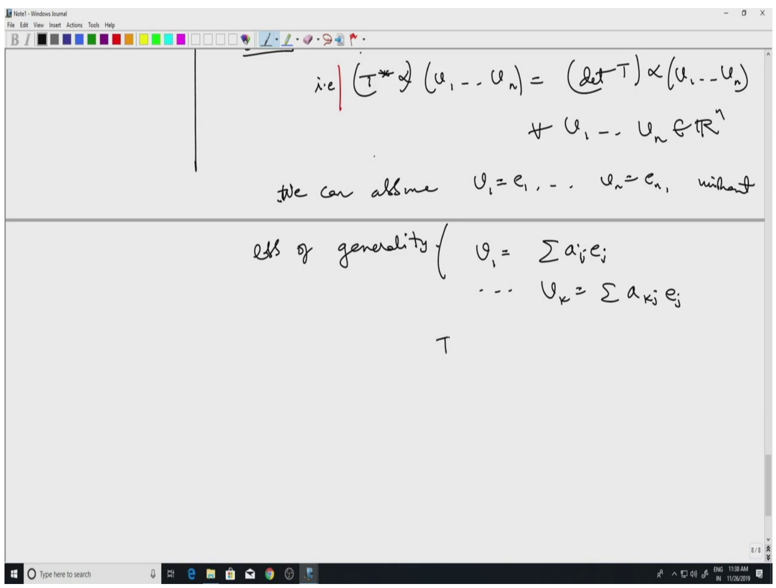

The second example is again, T from Rn to Rn. This time let us look at an alternating tensor. And the only one we have looked at so far. Well, we have looked at a couple. One of them was the determinant. So, alpha, ah, right. Before I proceed, perhaps I should point out that. But let me write it here. The set of alternating tensors, k tensors, forms a subspace of` Lk V denoted by V, well is not exactly standard notation. So let me denote it by, we denote it by Ak V. So this is subspace of Lk V. So it is easy to check that, if you add two alternating k tensors or you multiply it by scalar, again what you get is an alternating k tensor. So, it is a subspace of Lk V.

So, now let me start back to this problem, back to this example. So alpha in Ak V is given by alpha of v1 up to vn is determinant of v1 up to vn, where, as usual this vi's regarded as column vectors. Now we want to again look at the same thing, which is let us look at T stars alpha. The claim is that T star alpha is actually this, the determinant of T times alpha.

And this example is going to play a crucial role later on, when we talk about integration on manifolds. So, in fact, this is one of the main reasons, this specific formula is one of the main reasons why one even talks about alternating forms and so on. Now, the, the fact that determinant pops out like this, in this example, of course, we are starting with a determinant. So it is not that surprising that determinant is coming out.

But we will see that alternating forms are very, of any degree. So here, right, okay. So, here I should be a bit careful. This k is actually n. I am looking at, since I am putting n vectors, so this k is An, the same n as the dimension. So the point is that I will get a square matrix here. v1 up to vn will give me a square matrix.

Right, so, as I was saying, yeah, okay. So, now we want to, so in order to prove, first note that, to see this, well, this is an n form and the right hand side is n form as well. So we have to check i.e. T star alpha acting on v1 up to vn equal to determinant of T alpha acting on v1 up to vn, for all v1 vn in Rn. At this point, I would like to say that it is enough to check this, we can assume v1 equal to e1, vn equal to en without loss of generality.

In other words, if I want to check that this equation holds here for all v1 v up to vn, I do not lose anything by just taking it for a single choice of v1 up to vn. Namely v1 is e1, vn is en and why is this, it just follows from the fact that if I start with any arbitrary v1 up to vn, I can write v1 as summation a1j vj no sorry, ej. In other words, I expand it out in terms of the basis vectors, standard basis vector dot dot dot vk equal to sigma akj ej. Well, right, and then once we have this, then if we evaluate, right, and if we evaluate T star alpha. On this we will see, well, rather than go and finish the calculation now, let me stop here and resume with this in my next lecture. Okay thanks.