An Introduction to Smooth Manifolds
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Lecture 46
Tensors and Differential Forms 2

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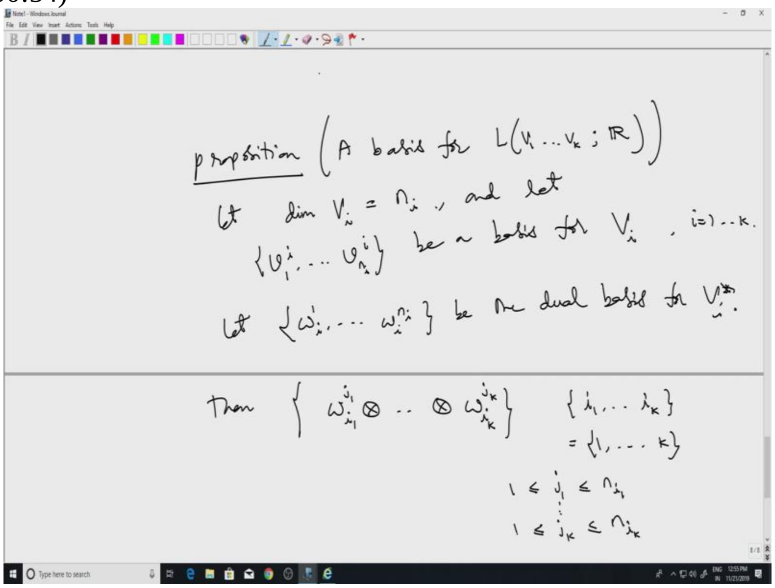

Welcome to the 46th lecture in our series. So, last time I had started discussion, started discussing multilinear maps on vector spaces and I had introduced the operation of tensor product of 2 multilinear maps. Now, using the tensor product operation, I want to get a nice basis for this. I start with some finite dimensional vector spaces V1 up to Vk. And I look at multi linear maps of V1 cross the Cartesian product of these Vi's, which we have denoted by this L, L of V1, V2, Vk all the way up to, ya, and to the target being R.

Now, the building blocks for this basis will be these 1 forms. In other words, the basis of 1 forms, namely the dual basis of dual basis. So, of course, when I say dual basis, what I am already assuming that, I start with a basis for vi then get a dual basis for vi step. So let us, I stopped at this point.

The claim is then, as we have seen before, if I take the tensor product of all these 1 forms, I get an element of L V1, V2, Vk. So, the claim is that all those tensor products will give me a basis. So, I look at the collection of all omega i1 for each of this. So, this i1 is just keeping track of which vector space I am at and then within that, there is a whole bunch of possibilities. So, let me use j1 to denote that tensor omega ik and then jk.

This can be a bit confusing but it becomes clear if we explicitly mention the range of these indices. For instance, this i1 here all the i, i1 up to ik, this set is essentially keeping track of, this is just a, 1 all the way up to k. It is just a permutation of this 1 up to k. So, nothing much going on here. As for this, now there is more freedom in this superscript. And this j, j1 for instance will vary from 1 to n1, no, sorry, the vector space here is i1, so ni 1, jk will vary from ni k.

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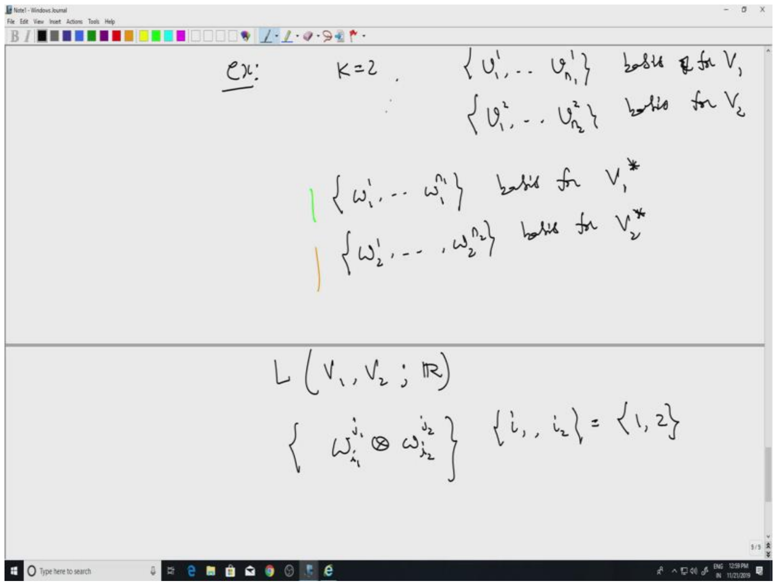

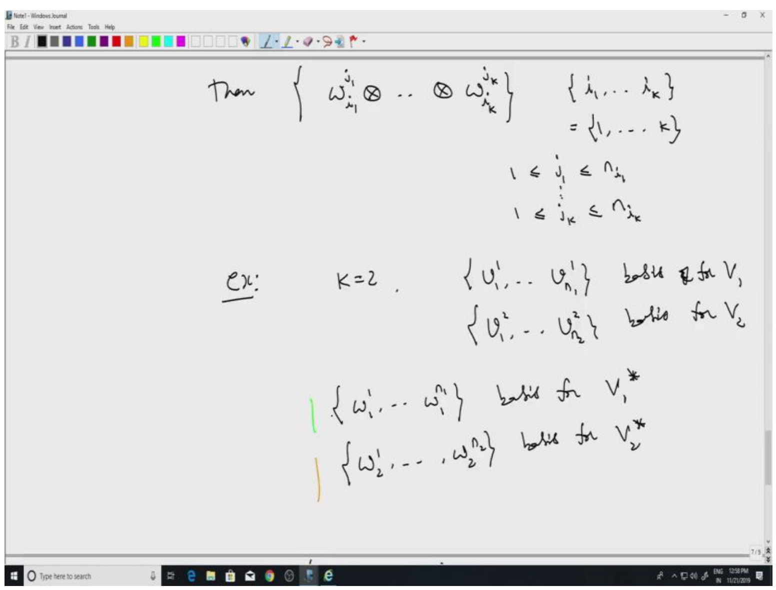

So, perhaps as an example, let us just deal with the simplest case. I have 2 vector spaces. K equals just 2 let us say. So, I start with the basis v with our notation v1 1, v1 n1. So, this is a basis for, basis for V1. Then v2 1, v2 into basis for V2. And then we get a dual basis. It is, the index comes below omega i omega 1 n1, oops, 1 n1, basis for V1 star. Omega2 1, omega2 n2 basis for V2 star.

And then, what we are, ya, now what we are claiming is that one wants, so one wants a basis for L V1, V2 R. So, what I claim is that, look at all the collection of all tensor products of every element in this first set here with every element in the second set. So, in other words one look at omega, so just to be consistent, so I have used i and j. So i, so i, let us use the same notation. i1 j1 tensor omega i2 j2. So, this and i1 i2, i1 comma i2, this is just a permutation of 1 comma 2. Actually, right. So here one has to be, in the case of different vector spaces, one has to be slightly careful. Notice that when I have 2 different vector spaces, you have to be bit more cautious here.

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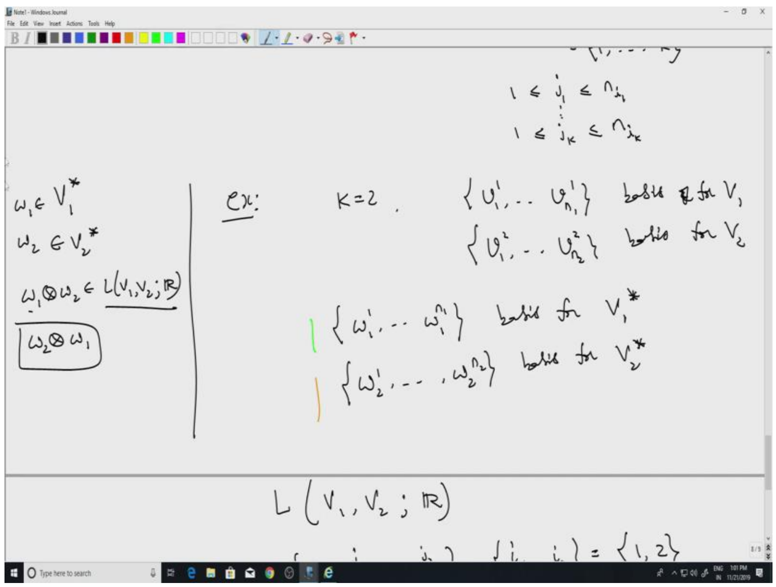

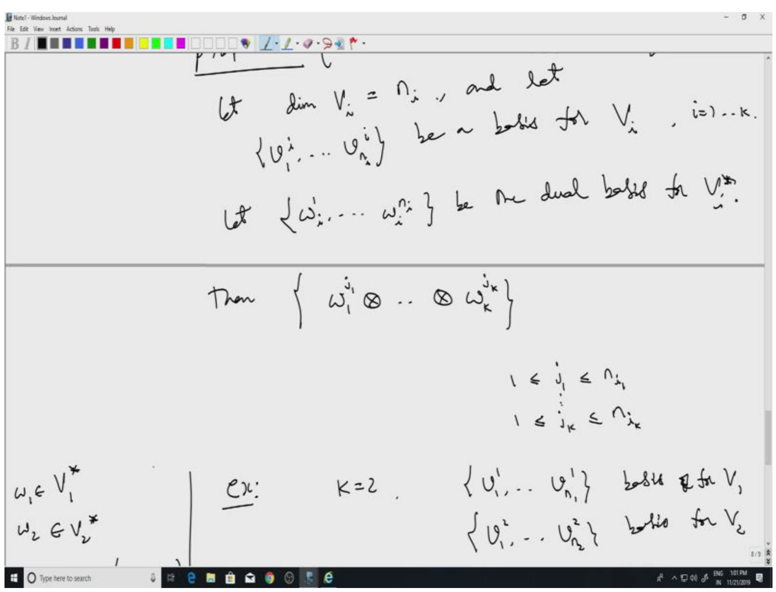

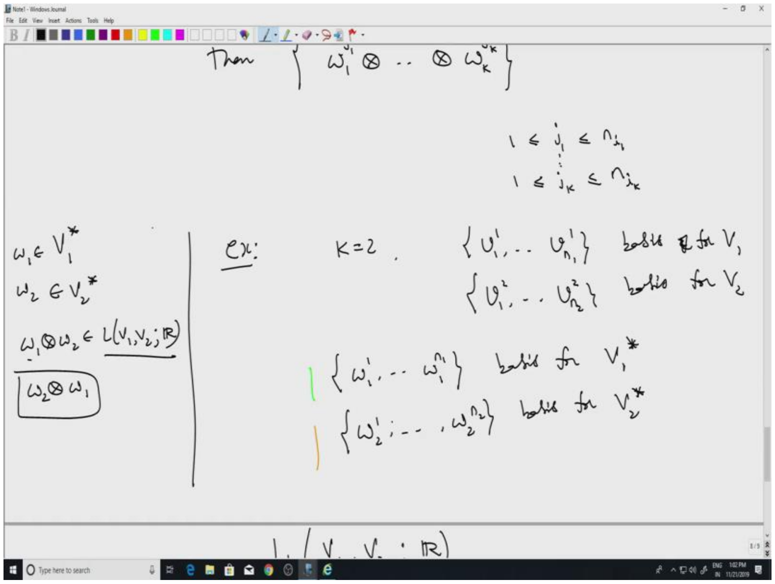

So, when I have 2 different vector spaces and if I have, so with my notation omega 1 is an element of V1 star, omega 2 is an element of V2 star, then omega 1 tensor omega 2 an element of L V1 V2 R. But, right. So, this makes perfect sense the way I have defined it. However, if I look at omega 2 tensor omega 1, this is not quite an element of L V1 comma V2 R. So, the reason being of course that, the way we defined it, omega 1 tensor omega 2 is going to be a mapping from V1 cross V2.

So, the first variable necessarily is from V1. While if I write it like this, the first variable will have to be from V2. So, this is not inside this, in this space, this one is. Now if V1 equals V2, then both of these will be inside this. It does not, both of them are well defined and both of them belong to that. So, here for instance when I write i1 up to ik, so, it is not, I have to be actually, I cannot even permute them in general. So, let me just erase this. So, I will necessarily have to deal with this stuff. So j1, the rest remains the same. So, when k equals 2, so, right. So, this is a basis for this and this is a basis for this.

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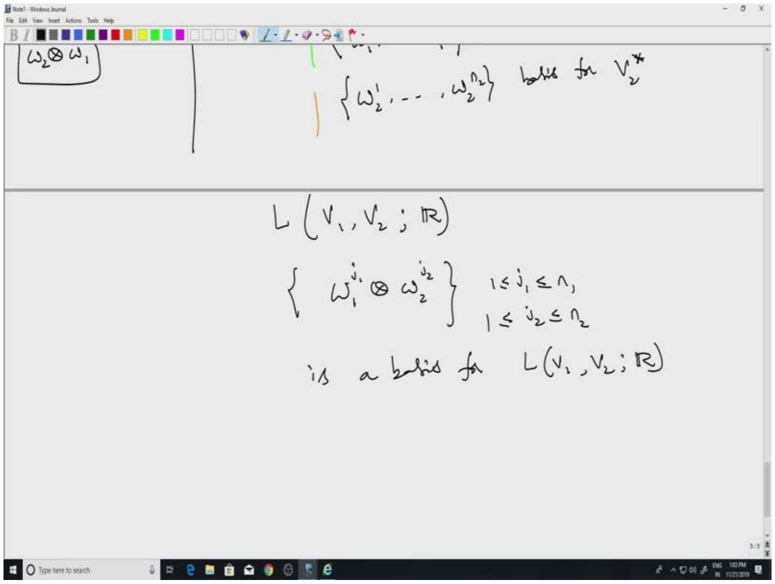

So here too, I have to make a small change. So, this is, I just, in fact it is simpler. It is omega 1 omega 2. And then I no longer have to mention this. So j1 is going to vary from 1 to n1, j2 is vary, going to vary from 1 to n2. The collection of all these tensor products is a basis, this is the claim basis for L V1 comma V2 R. And notice that if we count how many elements there are, well j1 can vary from 1 to n1 and j2 can vary from 1 to n2. So, there are exactly n1 n2 elements here. And that is going to be, in the general setup also that is going to be the case.

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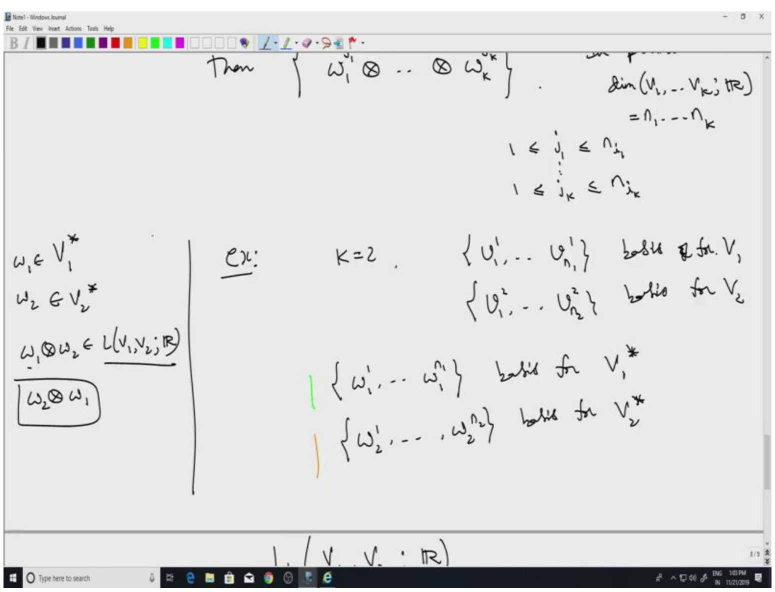

So, let me mention in particular, dimension of V1, Vk R is n1 the product of all the dimensions of the underlying vector spaces.

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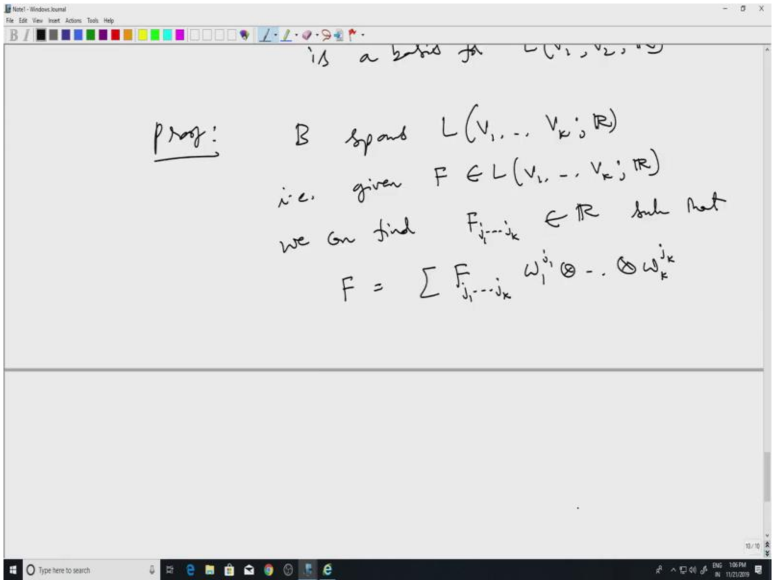

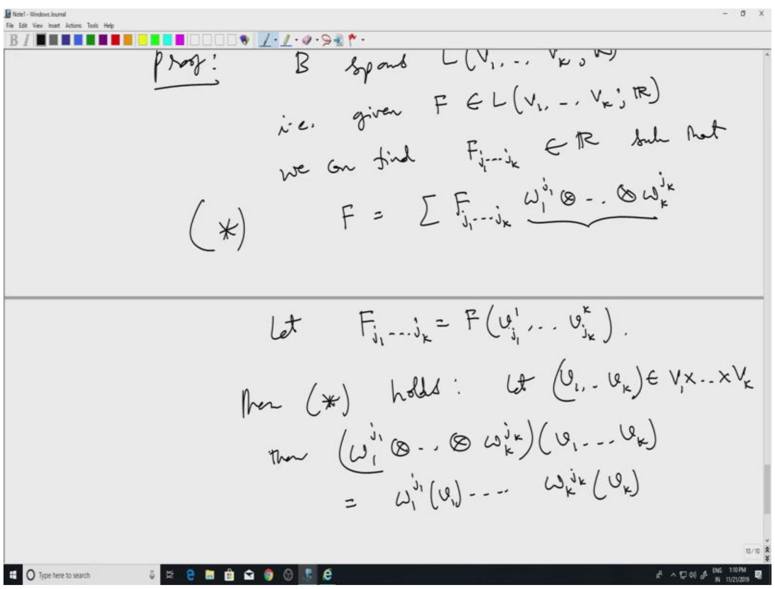

So, let us prove this. The proof is simple enough. So, proof. So, we want to show that these elements, these multilinear forms, multilinear maps actually form a span this L V1, V2, Vk and they are linearly independent. So, let us just check that they span. So, in other words, any multi linear map can be expressed as a linear combination of the special one multilinear maps. So, let us check that. So, let us give it a name actually, this set, the what we want to be the basis, so let B, let us call this so set B. B spans L V1, Vk, R, that is the first thing. And i.e. given F in L Vk R, we can find F. Now j1 up to jk. Fj1 dot, dot, dot jk in R such that, of course, as usual, it is not a single number, so this ji is vary from 1 to ni.

So, we can find all these numbers such that F can be written as a linear combination Fj1 and jk and then this basis elements, which I had here omega, i1 to the power omega i1 j1 etc. Omega i1 j1 tensor omega k jk. So, this is the claim that we can.

So, in fact, so we have to come up with these constants, Fj1 jk, for all possible values of j1 and jk. And it is clear how to find these coefficients. The idea is very simple. You just evaluate F on basis vectors, then these special forms, multi linear maps will all be 0 except the ones with the right index.

So, in fact, let us take, let Fj1 jk be equal to F of vj1, oh, I have to be again careful with whether I use a superscript when I go between 1 forms and vectors, the superscript changes.

So, for this it is, ya, okay. v v1, right v1 j1, vk jk. So, these are the, some basis vectors that we started with. So, I evaluate the given multilinear map on certain basis vectors and I get a number. I call that as Fj1, jk and this j1 jk keeps track of which basis vector I am plugging in here. The same index which is occurring here.

So, let us take these coefficients with, the claim is that, with the choice of these coefficients, then this equation, then star holds. Well, when we say star holds, remember that both sides are multi linear maps. So, we have an equation of 2 multi linear maps. In other words, if I plug in, any K, if, if I evaluate the multilinear map on any K vectors, I should get the same thing.

So, let us see that we do get the same thing. So, let v1 all the way to vk the domain of this map F is V1, belong to V1 times Vk. Then, here on the right side, I have a linear combination of multilinear maps. So, let us evaluate each term separately. In particular let us just evaluate and this is just this F, I, F is just a constant here. I evaluate this tensor product. Then omega i1 j1 tensor omega ik jk acting on even vk equals this entire thing acting on this equals, so well by definition of tensor product, this is omega 1 j1 v1 dot, dot, omega k jk vk.

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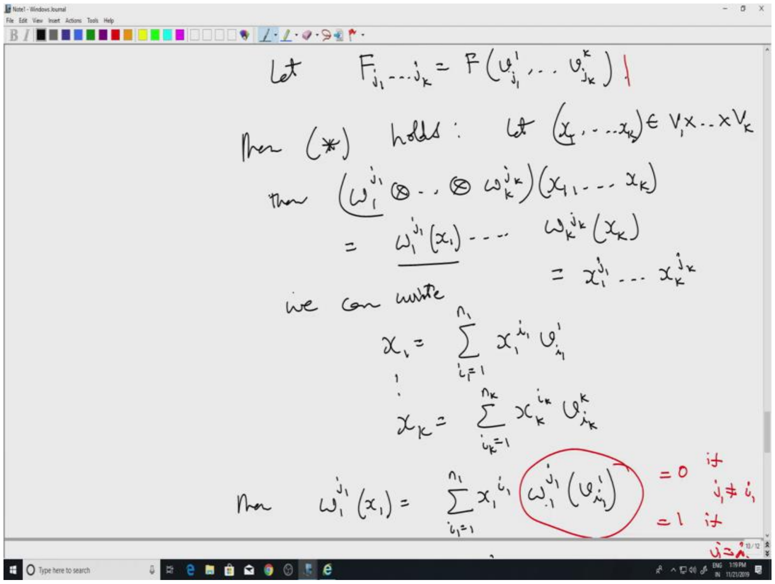

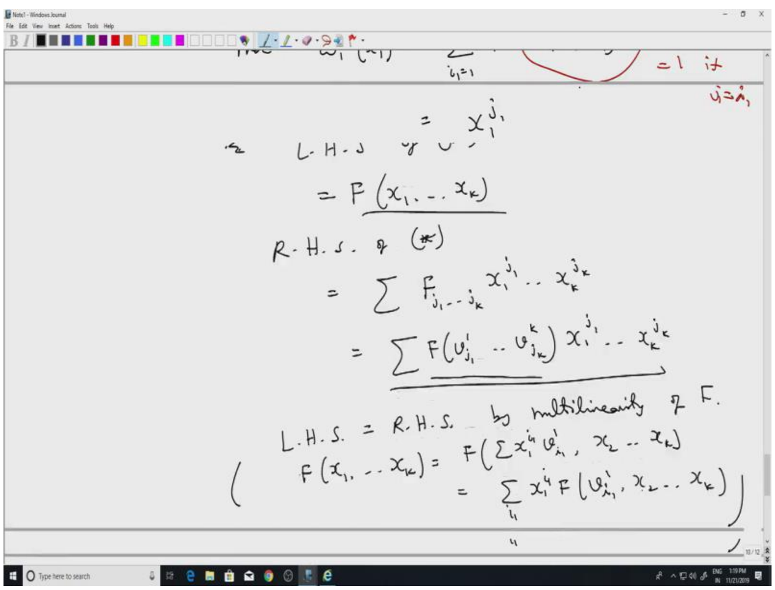

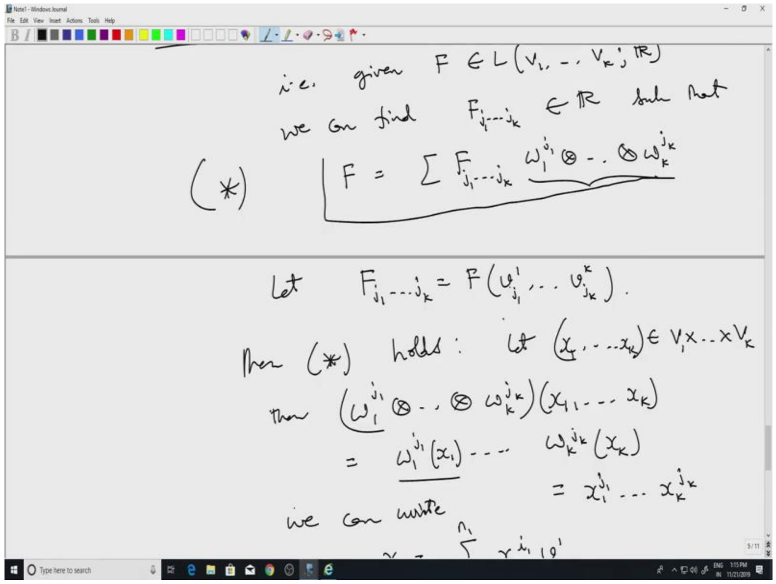

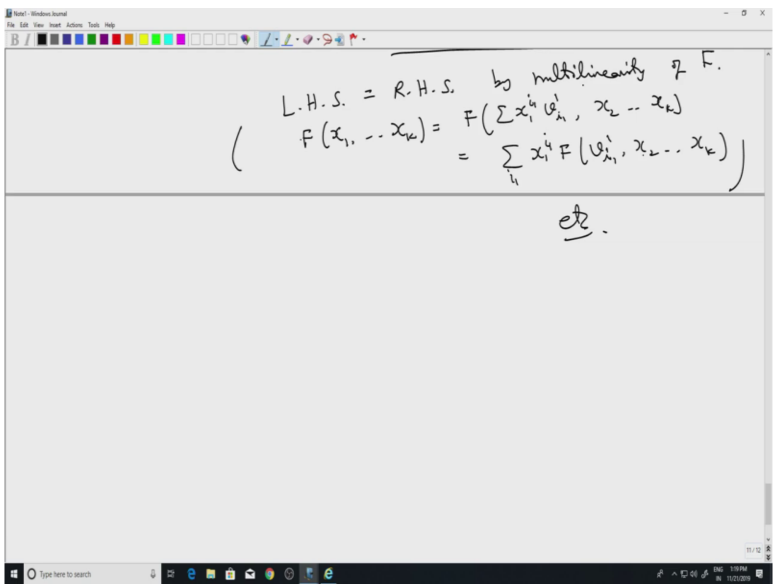

Now, we can simplify this further. So, vi, after all, each of these vector, these are arbitrary vectors in this vector spaces. So v1 itself and I already have basis for all of these vector spaces, so let us write v1 as, in terms of the basis vectors. Perhaps it is better instead of, if v has already been used, let us use x. Let x1, that will also make it clear that these are variables. And so x1 xk and then x1 xk. And so, let us express this x, xi in terms of, we can write x1 equals summation x1 i1 v. So i1 equals is a index running from 1 to n1. Similarly, all the way up to xk. xk equals ik running from 1 to nk x. I put a k here ik vik and then k here.

Once I do that, then omega 1, so now I want to look at this. Omega 1 j1 x1 will be i1 equal to 1 to n1 x1 i1 omega 1 j1 acting on vi1 1. Now, by the very definition of a dual basis, the only way, if j1 is not equal to i1, I get this term here is equal to 0 if j1 not equal to i1 equal to 1 if J 1 equal to i1. So, when it is 1, so the only term which will survive is i1 equals j1. And when i1 equals j1, I just get this coefficient. So, this is x1 j1. Right. So, and the same thing happens for all of them. So, finally this thing here can be written as x1 j1 xk jk.

And therefore, remember that we were in the process of evaluating, so we wanted to check that this equation holds. So, we evaluated both sides on this k (())(22:01) of vectors x1 up to xk. On the left-hand side, I get, so this is not, therefore LHS of the equation star is F of x1 up to xk. RHS of star is, that is what all these simplifications show, RHS of star is summation of this, well, summation of F j1 jk, F j1 all the way to jk and then x1 j1, xk jk. So, we have to show that this LHS this equal to this.

But we have not used the definition of F j1 jk. This is F evaluated on, this is the definition. So, what we have here, is the definition of F j1 jk, v1 j1, vk jk, v1 j1, vk jk this times Fj1 xk jk. So, we have to show that this quantity here is the same as this and that follows immediately from multi linearity. So, by multi linearity of F. In other words, I will just write, so, for example, the first step in proving this would be, I just, instead of x1, I use the, I will not do the, complete the whole calculation. Let me just indicate how it goes. F x1 all the way up to xk. Instead of x1, I plug in the expansion of x1 in terms of the basis as I have defined it here xi1 v1 i1 and the other stuff I keep as they are, then I get a summation over i1 and then x this i1. Then F of v i1 1, x2, xk. in the next step, so this is a summation if over the index i1.

For each i1, I can, instead of x2, I can plug in the expansion of x2 and again expand. So finally, I will get a sum which involves only these terms that I have here. And these coefficients come out as they are, as expected. So, I will just write etc. So that completes the statement that these special one, special tensor, multilinear maps obtained by tensor product of 1 forms actually span the full space of multilinear forms.

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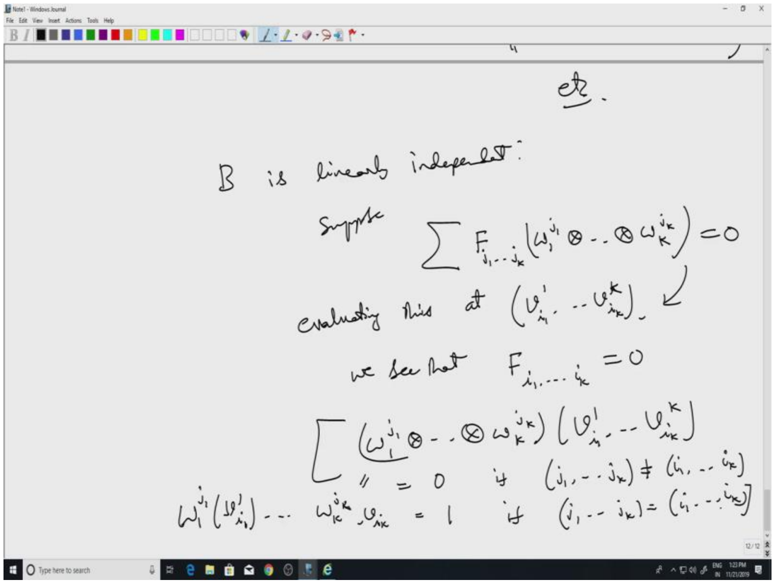

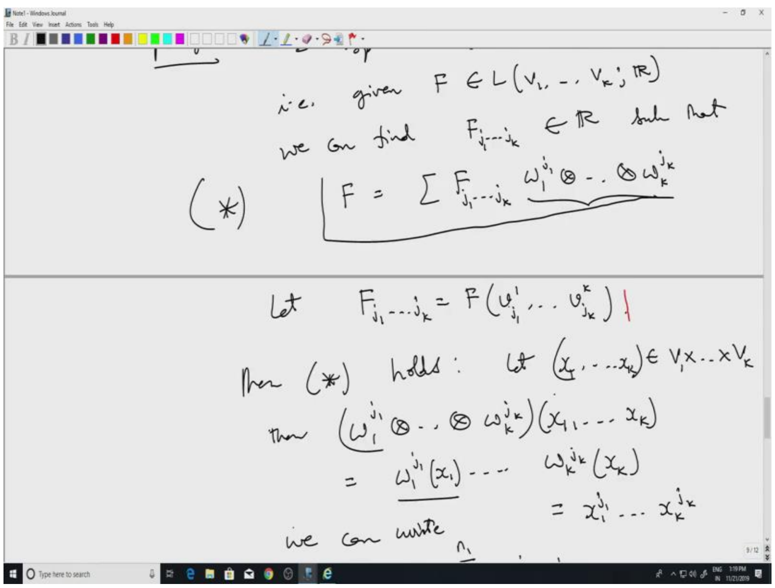

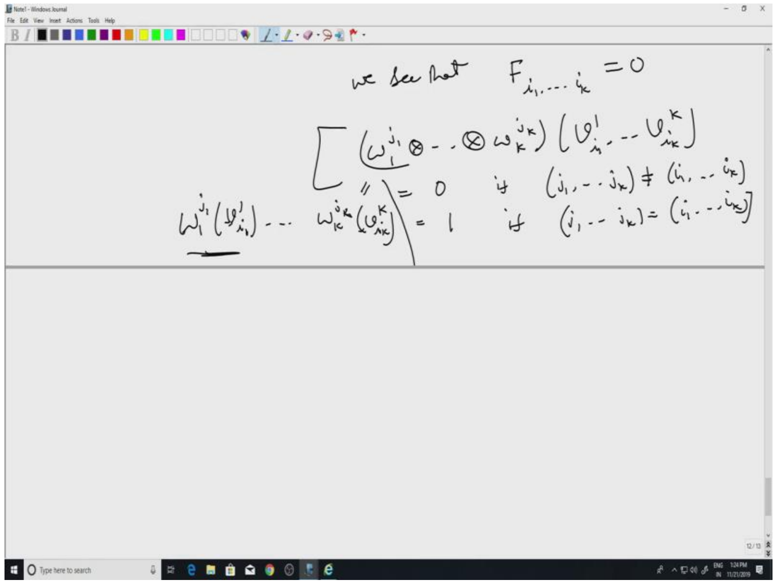

Now, let us prove that this B is linearly independent. Again this, the same calculation can be used. So here, suppose some linear combination is 0. Suppose in linear combination, now the linear combination as I, right, so this the right-hand side. Let us again say, here we do not have a multilinear map. But I will just use the symbol capital F for the coefficients. I need some coefficients. I will use this capital F and omega i1 j1 omega 1 j1 tensor product all this omega k jk equal to 0. So, the fact that this is 0 means that it is a 0 as a multi linear map evaluating this at, so I just take a base, basis all the inputs are basis vectors just like I did here.

So, i1 v1 i1, vk vk ik, when I evaluate this the same calculation as earlier shows that this thing here evaluated on this evaluated on this will be 0 unless these indices i1, i2, ik coincide with j1, j2 jk. We see that and when they coincide, I just get 1. So, the only term which will survive in this big sum is when the index j1, j2, jk is i1, i2, ik, so that will give me this constant here F i1 ik equal to 0 and this is true for all such i1 up to ik.

So, therefore all coefficients are 0. So, one sees that they are linearly independent as well. So just to be clear what we have used is F j1 omega k jk evaluated on v1 i1 comma v ik k equals 0 if this j1 jk is not equal to i1 ik. So even if one index here is different the whole product (())(29:17) otherwise it is 1 if j1 jk equal to i1 ik. It is immediate from the definition of the tensor product. After all, this number here is, we have already done that calculation, but let me just write it. Omega 1 j1 thing v1 i1 omega k jk evaluated on v ik, v ik and here it is a k.

So, this thing here is exactly equal to this. And here the moment one of these indices ji is not equal to well jp is not equal to ip, then this thing will be 0. The corresponding term will be 0. So, we get the required.

So, this is a good point to stop here. And in the next class we will talk about special kinds of multi linear maps and then move on to differential forms. Okay, thank you.