An Introduction to Smooth Manifolds
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Lecture 43
Exponential Map

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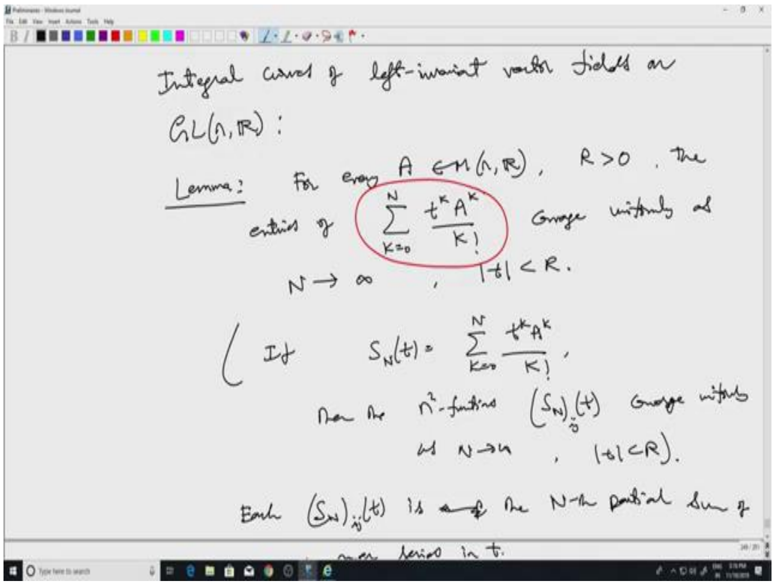

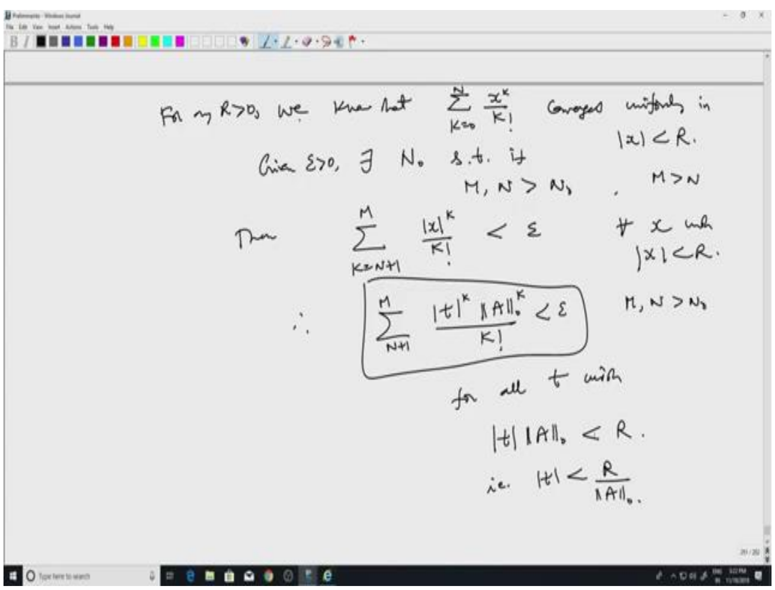

Hello and welcome to the 43rd lecture in the series. So let me, I will start by completing the calculation that I was doing last time. So I wanted to describe integral curves of left invariant vector fields in GL (n,R) and for that I need the motion of the exponential of a matrix. So I was in the process of proving that for start with any matrix in M (n,R) and then look at this power series in t N squared actually the entries of this the N squared entries of this matrix will claim is they converge uniformly.

So I wanted to apply the Cauchy criterion then I looked at the difference in partial sums between the ijth entry of a SN and SN, by using the operator norm I was able to reduce it to this thing here the actually I do not need to call it anything. So let me just say that, we know that the exponentials series, we know that sigma x to the power K over K factorial K equals 0 to N converges uniformly in for any R greater than 0 we know that this converges uniformly in mod x less than R.

So because this can add, this limit is what we call the exponential function. Well this is what I have here is the difference in partial sums of these of particular series for the specific choice of x. So the fact that this converges uniformly implies that again the Cauchy condition will imply that there exist N naught such that if M and N bigger than N naught than the difference in partial sums which will be and again I will assume M bigger than N, K equal to N plus 1 to M mod x to the power K by K factorial this will be given, so I should say given epsilon, given epsilon greater than 0 there exist N naught such that if this happens then this thing here as less than epsilon for all x with mod x less than R.

So this is f a fact from one variable calculus, so we know this and therefore now what we have here is exactly this where x has been replaced by t times norm A. But the condition on x is mod x less than R so what we want is so we know that this thing here K factorial less than epsilon again M and N for any M and N because in N naught but and for all t with modulus of t norm A naught less than R i.e., mod t less than R by A naught. So in this open interval this one has this estimate here of course this R was arbitrary so I get, I could have taken R as large as I want.

So I can take if I start with an R I can take a new R. R times norm A naught and get the required R here, that is not a big thing there is no restriction on R. So the statement is true, so as I said the point about proving uniform convergences that I know that it will converge to a (func) each entry SN ij will converge to Sij which is infinitely differentiable and more over the derivatives of SN will converge to the derivative of SN ij will converge to derivative of S ij.

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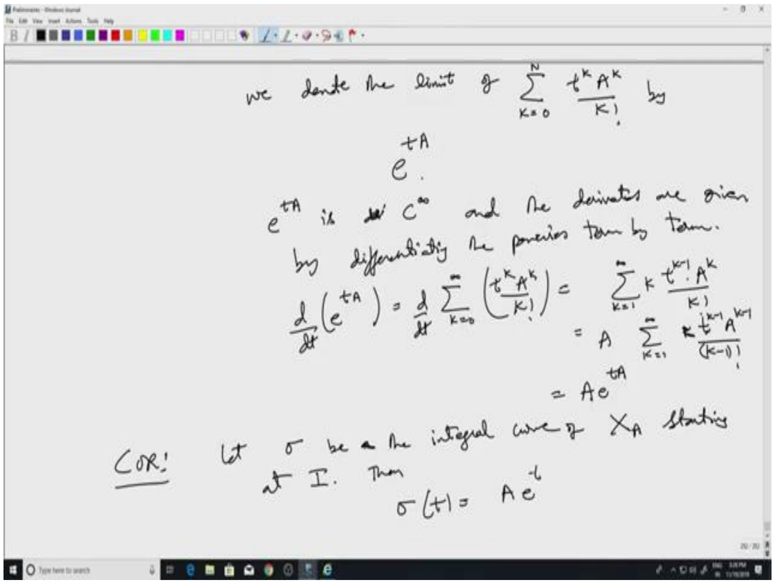

We denote the limit of all the entries will converge therefore the matrix itself will converge and so we denoted by e to the power tA. So e to the power tA is differentiable is C infinity as a function of t and of course this is a matrix when I say C infinity I mean that all the entries are C infinity and the derivatives are given by differentiating the power series term by term. So in particular if I want the first derivative d by dt of e to the power tA.

I just have to differentiate this series expansion term by term this will be d by dt of K equals 0 to infinity t to the power K by A to the power K by K factorial and this is K t to the power K minus 1 A to the power K by K factorial. Then, now of course K starts from 1 to infinity, so I can get out this I can bring out this A 1 outside and write it as K equal to 1 to infinity K t to the power K minus 1 A to the power K minus 1 this K cancels off and then K minus 1 factorial.

Which is just A e to the power tA again by changing indices. So this is what we need derivative of e to the power tA is A e to the power tA and this immediately tells us the following thing that corollary of this derivative from less that, let sigma be the integral curve of XA starting at the identity than sigma t is given by A e to the tA.

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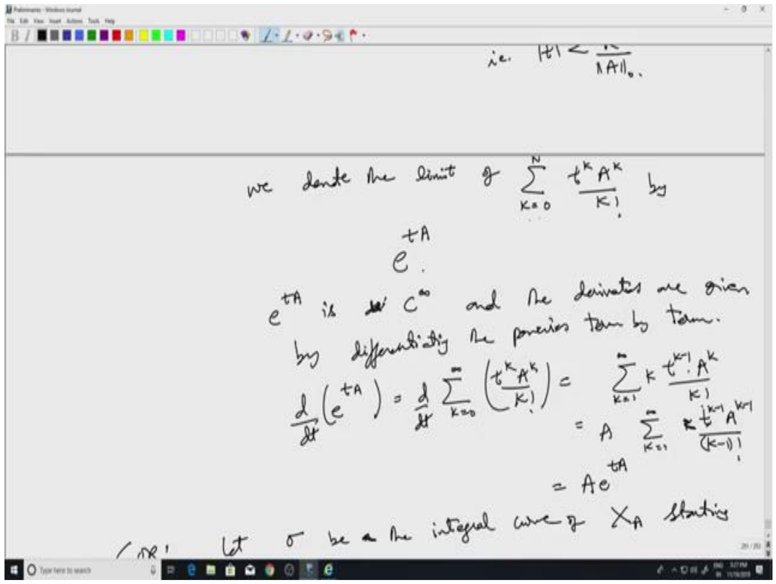

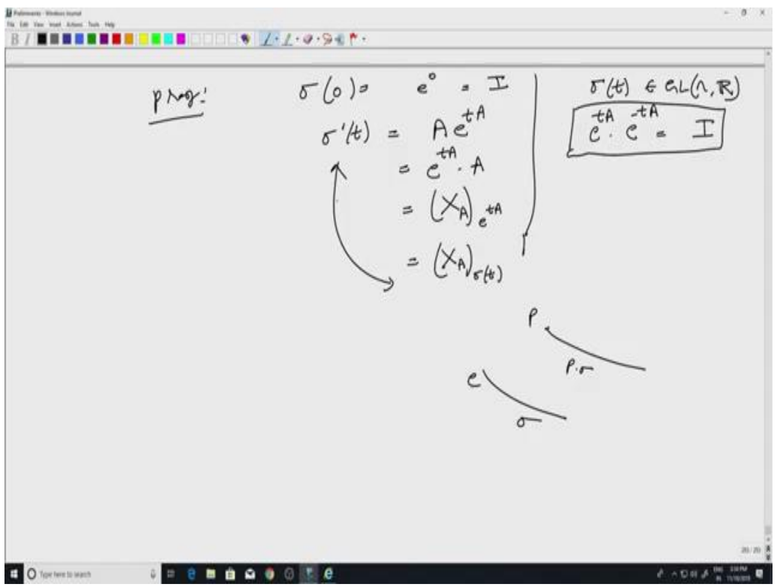

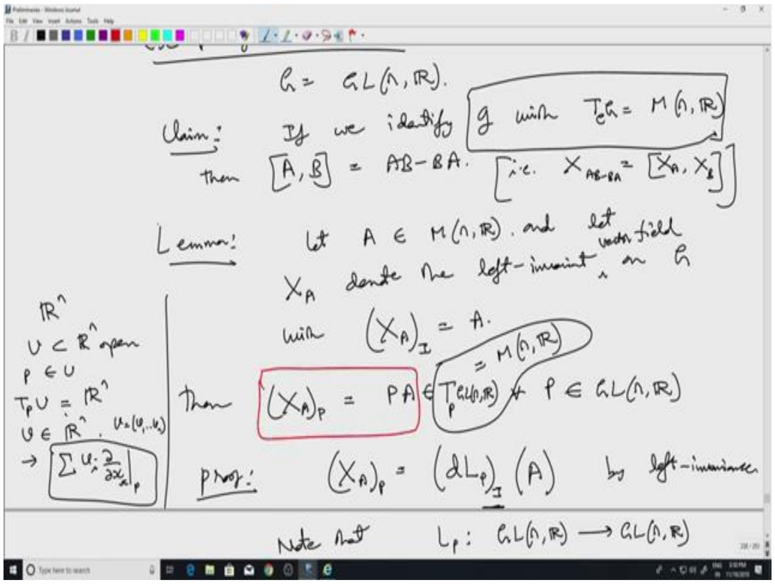

The proof is quite short well of course it starts at identity because sigma is 0 I need to check first that its start at t equal 0 and notice that this is define for all t, t and R. So sigma 0 is A e to the power 0 well if I plug in 0 in the for t equal 0, the only term which will survive is the 0th K equals 0th term which is just I. So A times I so this is A, sorry this is not, I should not put an A here. The A will come after differentiating so this is just I.

Now the main thing is the integral curve condition so we want to check that sigma prime t is equal to the value of the left invariant vector field xA at sigma t. So let us take the derivative of sigma prime t and we have seen that this is the same as A e to the power tA and this we know that this is the same as, well first of all I want to write it, I can write it as e to the power tA times A and I can do this because this e to the power tA is a power series in A.

It is a limit as N goes to infinity so when I multiply by A essentially I just have to multiply all this powers of A by A. So I can do this on left or right so this commutation duration holds here and what one has is right, so this is the same as the value on the left invariant vector field at the point e to the power tA times, no sorry I should put A here and then the value of the left invariant vector field given by A at e to the power tA.

So I am just using this what I have proved last time that the left invariant (vector) on GL n, R. On GL n, R this lemma, no even yeah this lemma that on if start with any A then XA at any point P is given by PA, this is the thing which I need and so I use that here XA at this point e to the power tA, notice that so here I actually need to make a small remark that I am working in I am trying to find integral curves in GL n, R.

So I should first of all even before I do this calculation here I should perhaps remark that sigma t actually belongs to GL n, R we know that the exponential exists, I mean the power series converges for every t but it may give rise to M matrix and just MNR which may not be invertible. So I want ensure that I stay in GL n, R, so and that is because I can explicitly write the inverse e to the tA, e to the power tA times e to the power minus tA equals identity.

And the way one sees this is that the fact that this is equal to identity is just again goes back to general properties of this power series, when you multiply it to power series, you can just multiply the partial sums and then, then take the limit if they both converge so I would not do this fact so perhaps this can be topic further a live session or the tutorial. So assuming this, I mean I am assured that sigma t belongs to GL n, R for every t, A can be anything no restriction on A Whatever A is this always belongs to GL n, R so this a first thing.

So with that in hand then this one has this and this is the same thing as XA and e to the power tA is the same as sigma t so we have the integral curve equation, sigma prime t equals to XA at sigma t. So that gives so integral curve starting at identity are easy describe and it is also easy to describe integral curves starting at any other point as well, so we already seen this that we just can just use left translation to take the integral curves starting at identity.

So this is the identity this is true in any lie group if I have an integral curve starting at identity If I want an integral curve starting at any point P, all I have to do is multiply this integral curve by the group element P and that will give me the integral curve starting at P. So that conclude my description of the various quantities that we have been talking about various object that we have been talking about for the specific, in the specific case in GL n, R and its subgroups.

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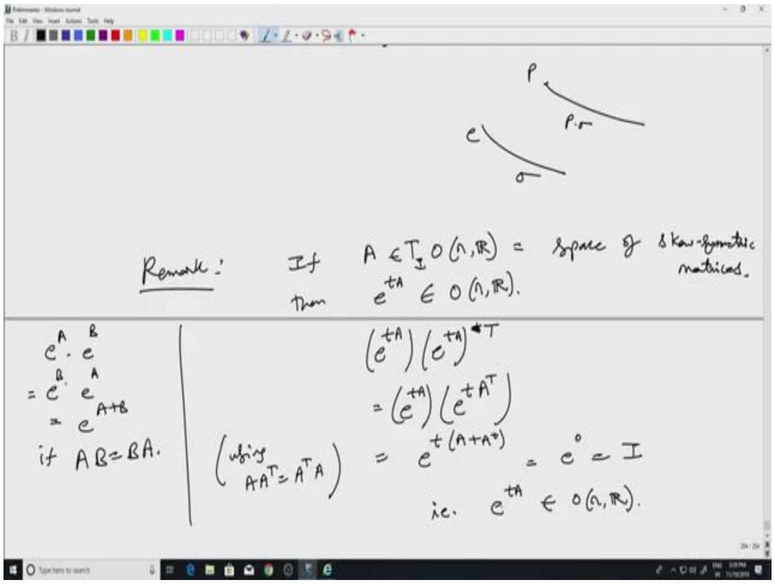

Incidentally so while before I finish this I should just remark that if A belongs to, now let us take the tangent space to the other two subgroups of GL n, R that we had were O n, R. Let us start with O n, R the tangent space at identity to O n, R and we had already seen what this is. This is the space of serially denoted by small o but with my notation it is kind of hard to distinguish so I just write space of skew symmetric matrices.

Then again I can if I want to find integral curves for the left invariant vector field on O n, R, I could not have very well tried I mean I could try to just use the same integral curve, after all I know that this left invariant vector field we already seen that on entire GL n, R its values on O n, R will be all tangent to O n, R. So in other words left invariant vector field on O n, R generated by is the restriction of the big left invariant vector field.

Since the, since that is the case we know that e to the power tA which is an integral curve will have to lie in sub manifold because its derivative is tangent to the but one can also see that directly e to the power tA belongs to O n, R and the same thing with SL n, R and the reason is well I can directly check e to the power tA times e to the power tA transpose. I cannot use small t anymore I will use T for transpose.

Now the way the exponential has been defined I can take the transpose inside to the exponent when I get e to the A transpose and again using the fact that A and A transpose commute using A

A transpose equal to transpose A what I get is this is e to the power t A plus A transpose and that will be since A skew symmetric this will be 0.

So e to the 0 which is identity, in other words this e to the power tA is an element of ON because it multiplied by its transposes identity. So I should remark that it is a general fact which again follows just from power series is e to the power A times e to the power B equals e to the power B times e to the power A equals e to the power A plus B if AB equals BA. Otherwise we cannot switch this around or neither can be claimed that we can add the A and B and get it equal to this.

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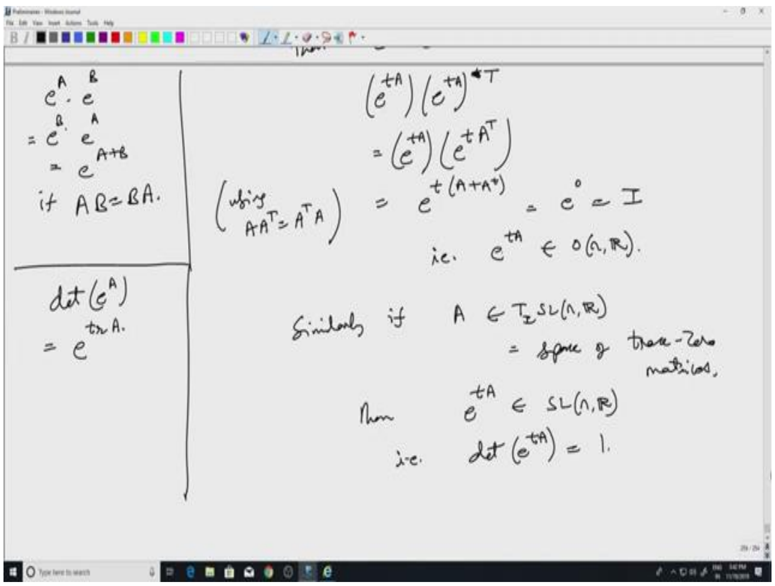

Similarly, if A belongs to the tangent space at identity of SL n, R this is space of trace 0 matrices then e to the tA belongs to SL n, R, i.e., that of e to the power tA equals 1 and this again can be seen by the fact that determinant of, so this is another separate small fact that determinant of any matrix is equal to, rather determinant of exponential of any matrix that of e to the power A is exponential of traces of A.

Now to see that this is not a purely power series thing. So there are some several ways of seeing this and one is to use the fact that this equation is obvious if A is a diagonal matrix and then generalized is it to diagonalizable matrices and then prove it for all matrices by using density of diagonalizable matrices in all matrices.

So one can do that as I said the fact that e to the power tA belongs to SL n, R again follows both of these statement that I have here follow from more general fact that since the left invariant vector field on the big group GL n, R is tangent to the subgroup at point of the, so the integral curve will have to line the subgroup. One can prove that as a separate fact and then proceed then one would not need to this special matrix fact.

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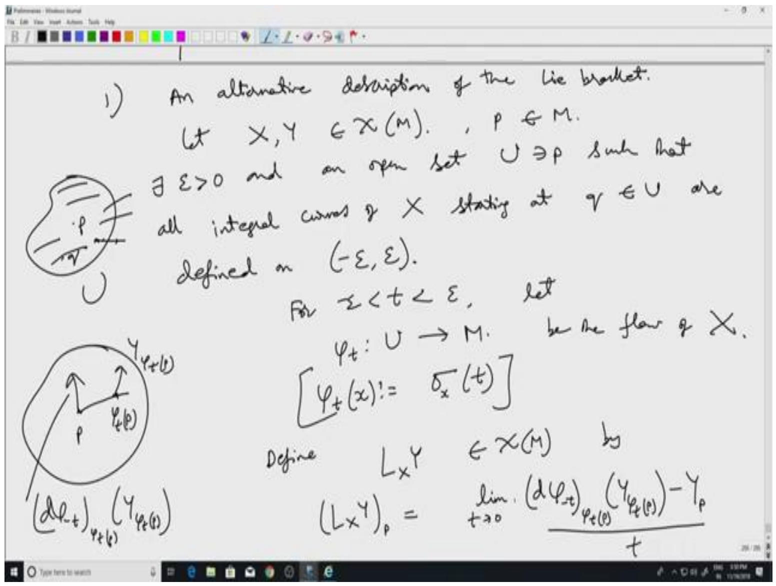

Before I move on to the next topic of differential forms so I would like to say few things about the lie derivative again. So the first topic is, not the lie derivative sorry the lie bracket. An alternative description of the lie bracket. So for this what I would need is, yeah I proceed as follows. Let X, Y be vector fields on a smooth manifold.

Let us take a point P in the manifold, we know that by Frobenius existence and uniqueness theorem there exist epsilon greater than 0 and an open set U which contains P such that all integral curves of, all integral curves of X starting at q in U are defined on minus epsilon, epsilon. So this is U this is P what I am saying is that as long as I start I take a point inside this U the integral curves which start at q are defined on a definite size interval minus epsilon, epsilon if they cannot shrink to 0 of course as the integral curve may exit U.

I am not claiming that they stay inside U just that the interval of definition is minus at least minus epsilon, epsilon. So for t in this for t, let phi t be the flow, to be the flow map. So be the flow of X, I had earlier talked about flows when I assumed that all integral curves are defined on the entire real line but that is really not necessary I can just wherever we just need a fixed size interval and I have this.

The only thing is that I cannot quite say that U goes back to U. So it is not a diffeomorphism of U back to itself but rather a map from U to M and its definition is the same phi t x is you look at the integral curve which starts at X and move along that for time t. So this is the definition of, now we will define the so called lie derivative, define the lie derivative of Y with respect to x this is going to be another vector field on M by LxY at the point P equals limit, so the idea is as follows I would basically what I want to do is, well I just so here is my point P.

I travel along the integral curve for time t so in other words I see where the flow phi t takes this point to at this point I have the vector field Y. So this is, if this is phi t P and at this point I have Y phi t P. Now what I will do is I will use phi minus t, so move back along the flow starting from here and end up back here. So that and you take derivative of that, that derivative will push this particular vector to something here.

So this is this vector is d phi minus t with the point phi t P Y phi t P. I start with some vector here and use the flow derivative of the flow to get back a vector at P. I can do that and then I can just take the usual difference quotient so then take the usual derivative basically. So I do not even have to write limit so I just say, no perhaps let me retain so I just take the usual derivative so in other words I will take d phi minus t of phi t P Y phi t P minus Y P and then, so this Y P corresponds to t being equal to 0.

sSo I am going to take limit t, when t is 0 I know that the flow map is just the identity map in this case it is a inclusion map. So the integral curve sigma x X 0 is just X itself, so when I put t equals to 0 I get this is just Y phi of 0 and this is just identity map so I get Y P then by t. So in short it is just a derivative of the first term which I have written out like this. We will stop here in my next lecture I will make a few more remarks about this lie derivative, of course the main claim is the lie derivative equals the lie bracket. So I will just say few words about that and then I move on to another topic, thank you.