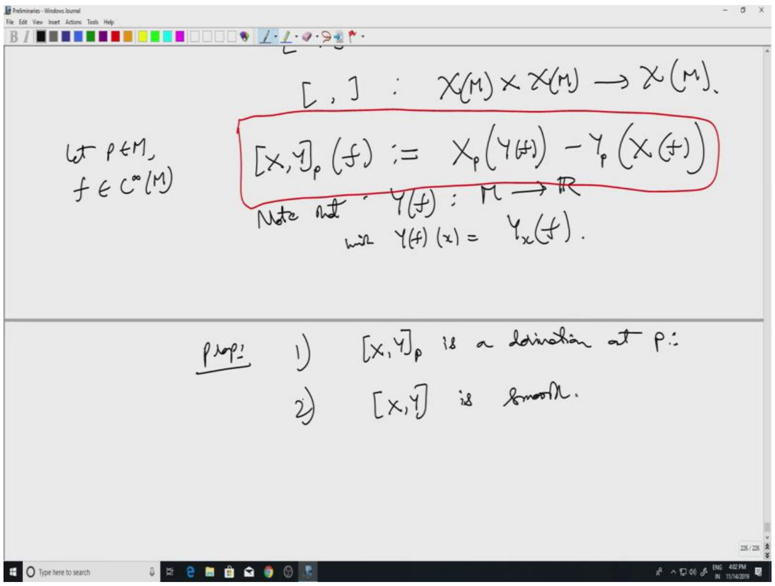
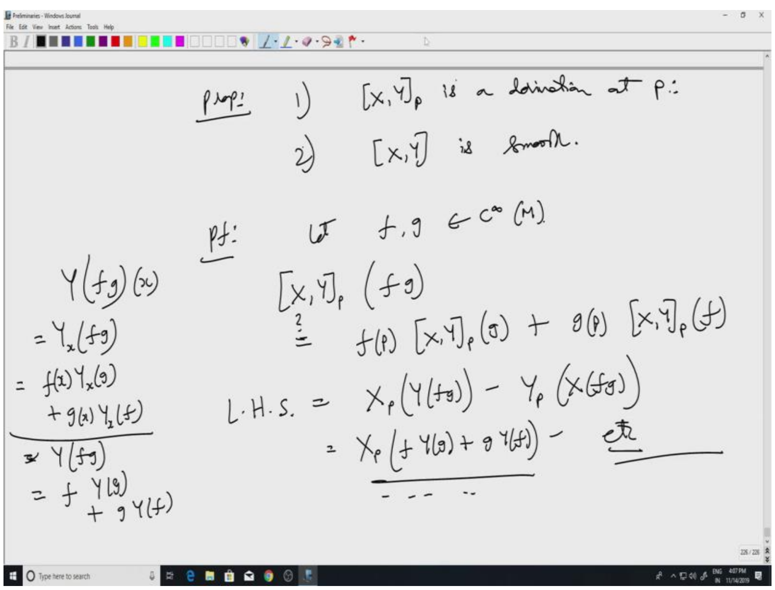
An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru
Lie Brackets (Part 2 of 2)

Hello and welcome to this discussion on vector fields and in particular, Lie brackets.

(Refer Slide Time: 0:40)





So, last time I defined this operation starting with two vector fields. So, the main definition is here. Notice that one actually needs vector fields, tangent vectors will not do because I need to get a function here on the manifold in order to act Xp on that and in order to get a function, well, I have to have Y defined at all points on the manifold. So, now let us check, let me quickly start the checking the derivation property of this. I will not complete this, it is fairly straightforward.

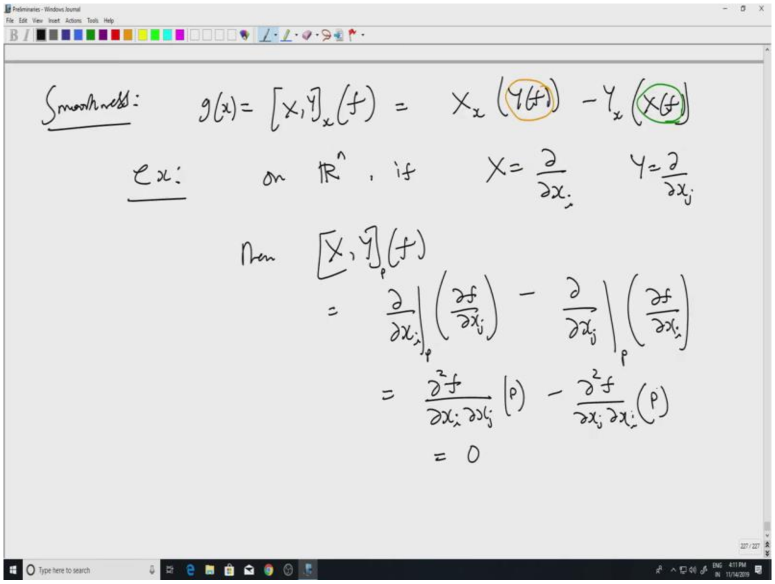
So what does it, yeah, so what do we have to check? Now before I start checking I, I should write the second part. Second thing is that X Y is smooth, in other words it is indeed a (vec) what we called a vector field, as p changes smoothly, this X Y changes. So, to check let f g belong to C infinity M. So, what I want to do is, I want the Leibniz's rule to hold. So, I want X Y p acting on fg, we want to verify whether this is f of p X Y p acting on g plus g of p X Y p acting on f.

So, this is, so let us say that, I will put a question mark here is this equal to the right-hand side. So it is just a matter of going through the definition, so for instance, the left-hand side, the L.H.S., L.H.S. equal to Xp Y now the function here is fg minus Yp X fg. Now here, when Y acting on fg, so let us look at this Y acting on fg, now the point is this is a function on the manifold. So, let us see its its value at any point X is Yx acting on fg, by definition.

And now this is a, once I take an X, this is a tangent vector and then I use the derivation property for this, so this would be f of x Yx g plus g of x Yx f which I can write as, now, if I do not want to mention X at all, this (eq), the above equation amounts to same that Y of fg equal to f, this function f multiplied by Y of g plus g multiplied by the the function Y of f. So when, we can use this without mentioning X, so instead of Y f g I put f times Y g plus g times Y f, f times Y g plus g times Y f.

And similar thing here, so I will, I will not complete the calculations, it is fairly straightforward, so you do a similar thing here then, you again use the derivation property of Xp, so here that will give me, two terms here two, so altogether there will be four terms coming from this one, this, and another four from this. One combines the appropriate terms and writes it in this form that is it. Notice that some cancellations will also take place and then one gets what one wants.

(Refer Slide Time: 6:06)



So now, as for smoothness, if one uses the the definition of (smoo) instead of using local coordinates, one just goes by definition of smoothness, one has to look at this, X Y so the function, this function for any f in C infinity M. So this is what I have, sometimes called g X equal to this. Smoothness, one just looks at this and this by definition is X at x Y of f minus Y at x X of f.

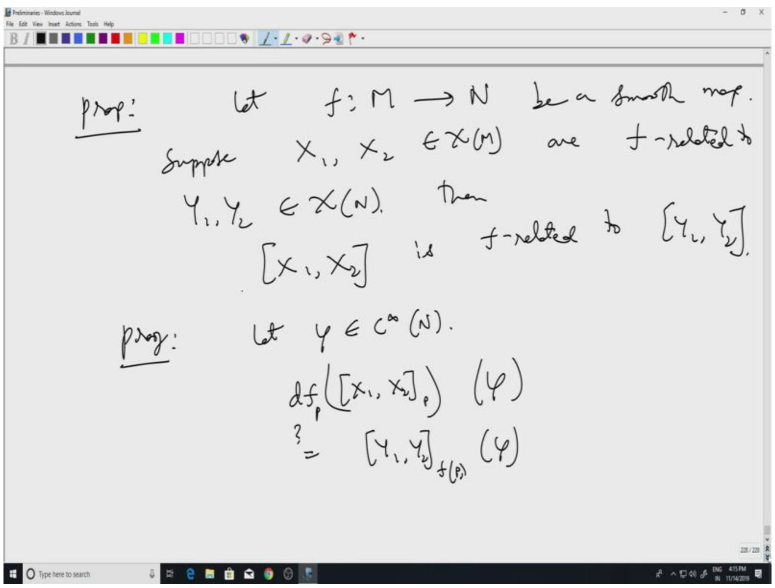
Now, since individually X and Y are smooth vector fields, this thing here is a smooth function and so is this thing, Y f and X f are smooth functions again the fact that X is smooth will say that this is also a smooth function where the variable is X, likewise this, so, overall you will get a smooth function. So, it is quite straight forward. So let us look at some examples of lie bracket. Oh, when I was, one brief remark, when I I just checked the Leibniz's rule property but linearity is even more clear.

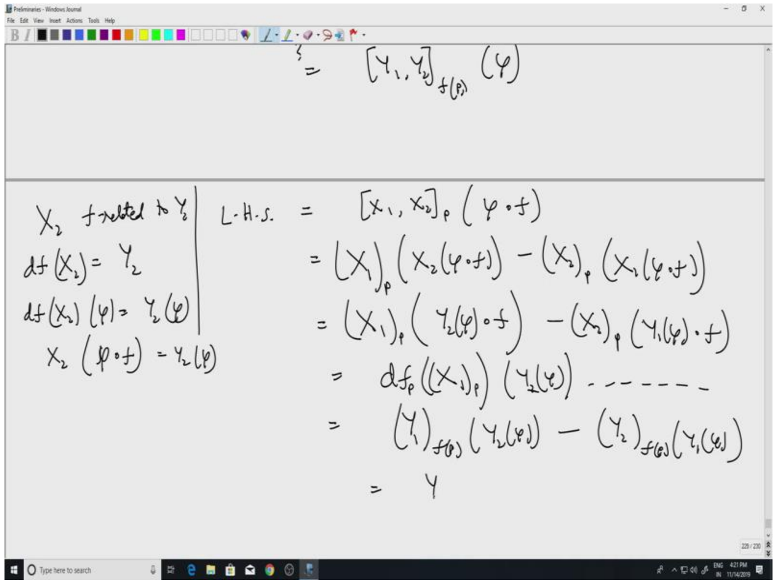
So here linearity with respect to the function f, if I take two functions, add them up f and g then it is clear that the right-hand side is linear as well. So the Leibniz's rule is the only thing to check there. Now, as an example let us just take a trivial thing but very important example. So, let U contained an Rn, actually you do not even need the open set let me take the whole manifold on Rn if X equal to del by del xi, Y equal to del by del xj are the two basis vector fields, then, let us see what X Y is. X Y acting on f would be, X is del by del xi at a point p.

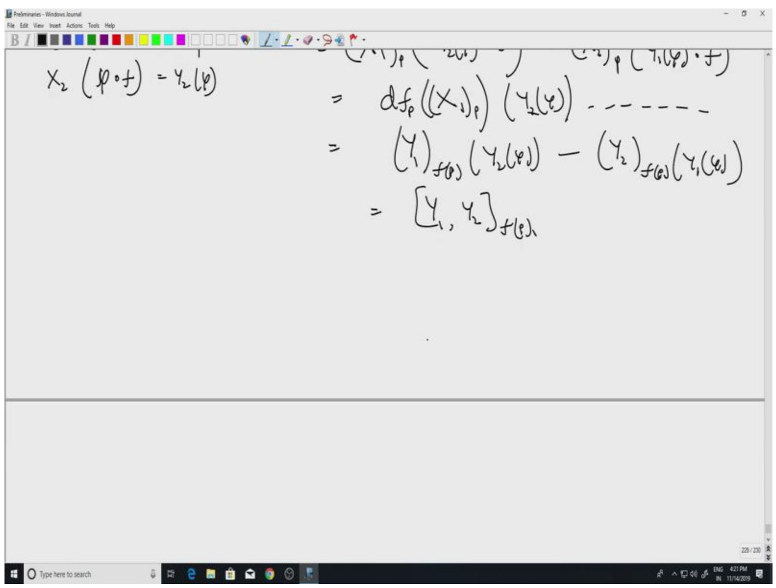
So, let us say del by del xi at p acting on Y of f, now Y of f is just del f by del xj minus del by del xi del by del xj at p acting on del f by del xi. But this thing, both these are what one usually denotes by mixed, what one usually means by mixed partial derivatives, and they are denoted like this del squared f by del xi del xj at p minus del squared f by del xj del xi at p.

And we know from multi variable calculus that mixed partial derivative for a smooth function, mixed partial derivatives are equal, so therefore this is 0. So, if we start with any two elements any two vector fields, the standard vector fields namely partial derivative vector fields the lie bracket is 0.

(Refer Slide Time: 10:37)







Well, the significance of this I will come back to later but for now I need one proposition relating so, let f from M to N be a smooth map. Suppose, X1 X2 are two vector fields f related to Y1,Y2 on N then, the lie bracket of X1 X2 is f related to Y1,Y2. So this, the notion of f related ties in nicely with the this lie bracket property and proof is again a straight forward computation, so here so what, it just a matter of knowing what one has to check to say that this is f related to Y1 Y2 amounts to saying checking the following.

Let phi be a C infinity function on N. Then we are asking the question if df, dfp of X1 X2 at the point p act. So this acting on phi is the same as Y1 Y2 at the point. Now dfp of this f related amounts to same that this is the same as f of p acting on phi. So again a question mark here, so this is what we are trying to verify, so let us see how it goes. Now this, the left-hand side, the left-hand side is equal to, by definition of the differential of a smooth map this is the same as X1 X2 at p acting on well, it is phi composed with f.

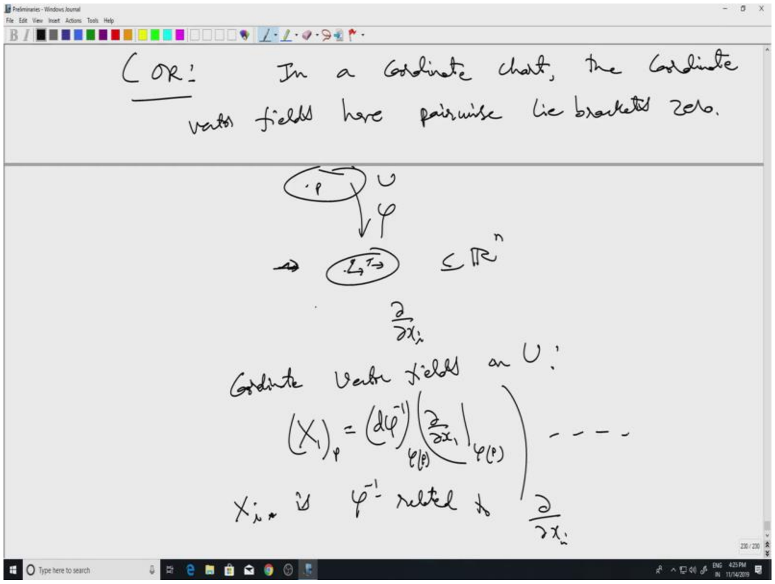
Now I use the definition of the lie bracket, so this is X1 at p. Actually, yeah, somewhat inconvenient to retain this p. X1 at p X2 of phi composed with f minus X2 at p X1 of phi composed with f and then this brackets. So, now I know that it is just a matter of now unravelling the various hypothesis so for instance, X2 is f related to, X2 f related to Y2, this just means that df of X2 at the appropriate point is equal to Y2 at the appropriate point.

So, if I actually act it on some function, so in in particular, so X2 df of X2 so right, if I act it on a function which I call the function phi, df X2 acting on phi equal to Y2 acting on phi and this would be the same as, well it is X2 acting on actually f composed with, rather phi composed with f. So in short X2 of phi composed with f is just Y2 of phi, everything has to be considered at the appropriate points so, for instance, right, so if X2 is at p then Y2 would be at f of p.

So, here this would be Y2 phi at composed with f. Y2 phi is a function, and I am evaluating it at f of something, so this is like this – X2 of p it is the same thing with X1 and Y1 will give me this. Again, go back to the definition of differential of a smooth map this would be, df, df of, well, X1p acting on the function Y2 phi, similar term here, so let us write, actually here just to be clear I will just put dot, dot, dot. So you will get a corresponding term on the other thing.

So let me just focus on one term, simplifying one term so that will be, and finally, well what is, yeah, again I use the fact that, right, so this is actually at p. So this would be Y1, the fact Y1 at f of p Y2 phi and the second term will turn out to be Y2 at f of p Y1 at phi. Just by definition, the lie bracket of Y1 Y2 at the point f of p which is what I wanted. So this is what I wanted and I get that. So, one has this very important and useful fact that f related vector fields go to f related vector fields under the operation of lie bracket.

(Refer Slide Time: 18:58)

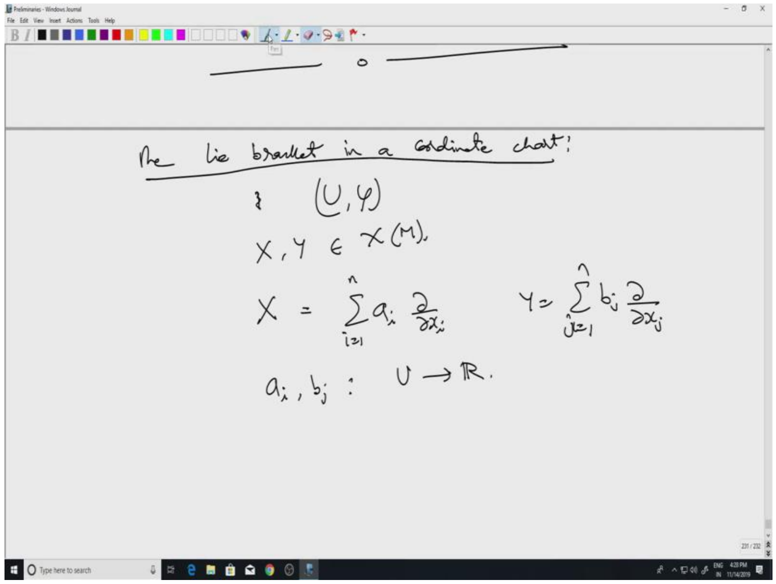


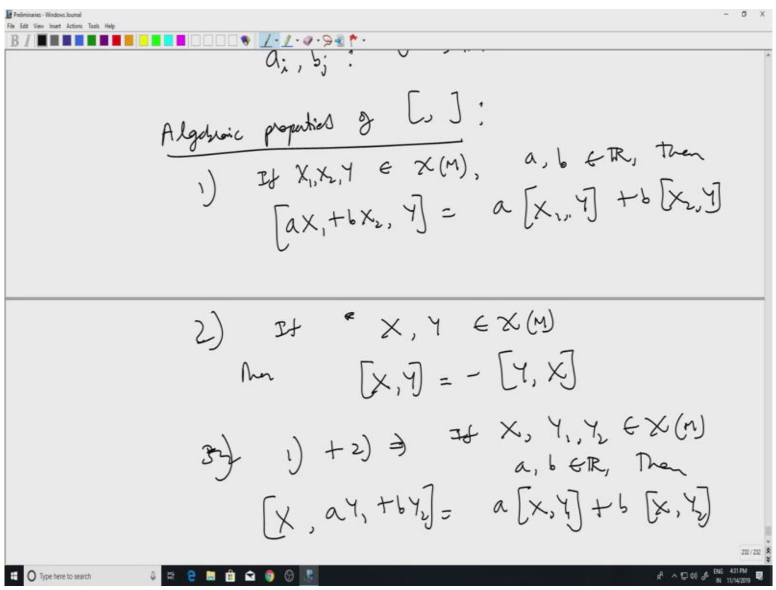
So in particular, one corollary of this is that in a coordinate chart, the coordinate vector fields have pair wise lie brackets zero. So remember that for us, coordinate vector fields so, this is a coordinate chart to something in Rn. On Rn, at each point we have this del by del xi. We have a local, we have a frame on Rn so, we have del by del xi here at each point and when we pull this back to the manifold why had this using the diffeomorphism phi, we get vector fields on this open set U which we called coordinate vector fields.

So the coordinate vector fields are by definition, coordinate vector fields on U are basically at the point p if I start with a point p, I just look at, so let us see if I, X1 equal to, so I look at the corresponding point, del by del xi del by del x1 at the point phi of p. This is the here, and then use the phi inverse, derivative of phi inverse, d phi inverse at the point phi of p. So this is X1 at p, what I have written here, so we have seen this many times before, usual.

So, here the thing is that this X1 X2 et cetera are phi inverse related to del by del xi. Xi, Xi is phi inverse related to del by del xi. I just wrote the X1 equation here but, so this is this is just, the very definition of this coordinate vector fields shows that they are related like this. Since we have already seen that the coordinate vector (fie) the standard partial derivate vector fields have lie bracket 0, the same result holds for this xis as well.

(Refer Slide Time: 22:35)





Now let us see what the lie bracket looks like in a, the lie bracket in a coordinate chart. I will fix a chart U phi and then let, so I will just take two vector fields X and Y on M. We know that, as just talked about in the last page that, on U we have this frame, in other words n vector fields X1, X2, Xn which form a basis for the tangent space at each point of U. Therefore, I can write, as we have done earlier X equal to ai. Ah, I should mention one thing, so, the U, I will follow the usual convention. So, instead of writing this X1 et cetera. I will just use del by del xi, here as well.

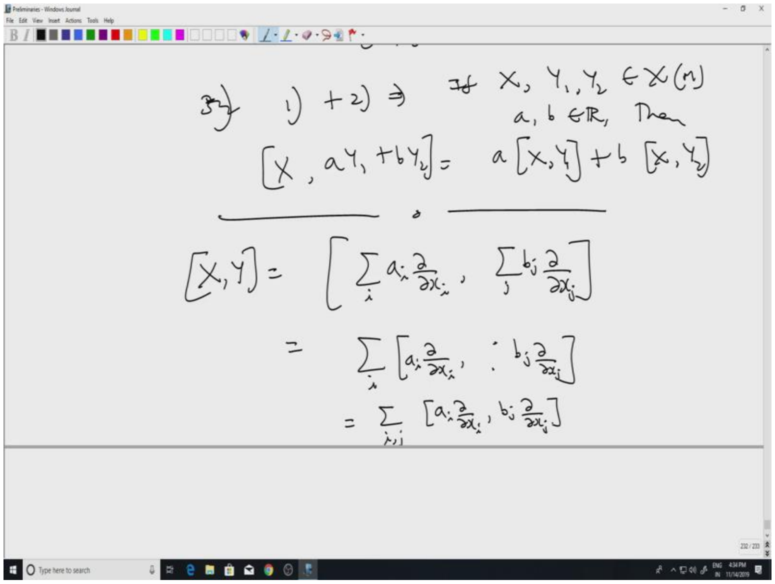
So, del by del xi is used for on Rn as well as on the manifold. On the manifold what we actually mean is d phi inverse of del by del xi. So I will write it as del by del xi and then Y equal to some other function, like this, so here i equal to 1 to n, j equal to 1 to n again and this ai and bj are both functions from, C infinity functions from U to R.

So let us see what this is, so X Y, so before I proceed, let us make a few general comments about the lie (brac) algebraic properties of lie brackets. So, let algebraic properties of the lie bracket, the (fir) what I need here is that, it is what is called bilinear in both input variable. So in other words, if X Y are two vector fields on M and if you have two real numbers a b. Actually I need, right, let me take three vector fields.

If X1 X2 Y and then a b are two real numbers, lie bracket of a X1 plus b X2 Y, b X2 with Y, this is same as a X1 Y plus b X2 Y. Similarly, X Y are two vector fields then the lie bracket of X Y is the negative of the lie bracket of Y X, so this will automatically, if I combine this and this one and two, I see that the third property even in the last second slot as well one has linearity like this.

So 1 plus 2 implies if X Y1 Y2 are two three vector fields and you have two real numbers then X a Y1 plus b Y2 is equal to a X Y1 plus b X Y2. Now there is a, actually these are the trivial properties there is something more unexpected which happens here which has to do with three vector fields which I will, I will not mention it now, for now let us just use these and simplify our calculation, go back to this coordinate chart.

(Refer Slide Time: 28:01)



So, in the coordinate chart I had this, so this I can write as lie bracket of summation of ai del by del xi summation bj del by del xj. I, now I can use the linearity in the. So, if I keep, if I have two sum of two vector fields and I keep the second, the I think fixed have a splitting like this so that is what I will do here. First I will write it as summation over i lie bracket of ai del by del xi, then this is kept fixed, bj del by del xj j.

Then I will keep this del by del xi fixed and then expand this over that. So summation over i, j ai bj and then, oh sorry, yeah well. These are not for constants, this this holds only for constants so here I cannot take out this ai, so I have to keep it inside the, it is in. So here, what one gets is, right. So summation and then the ai will stick with this and similarly here, this, one cannot take it out so, here what one gets is ai del by del xi, bj del by del xj.

Both within brackets. And, in my next lecture I will simplify this, and we will see it is yeah, it is (jus) mainly for computational purposes it has no special significance, the expression that we are going to get. And, then I will talk a bit about lie bracket of, lie brackets of left invariant vector fields. Okay, thank you.