**An Introduction to Smooth Manifolds**
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**Lecture No 37**
**vector fields on Manifolds**

Hello and welcome to the continuing series of lectures on vector fields on manifolds. Last time I sort of outlined brief sketch of why it is a vector field on a compact manifold is complete. Now, this outline I gave is more for an intuitive understanding of why one would expect this. The actual proof is somewhat. This can be made into a precise proof, but there are more, the better ways of stating this, in fact, there is a common proof for this result and the next results that I am going to state.

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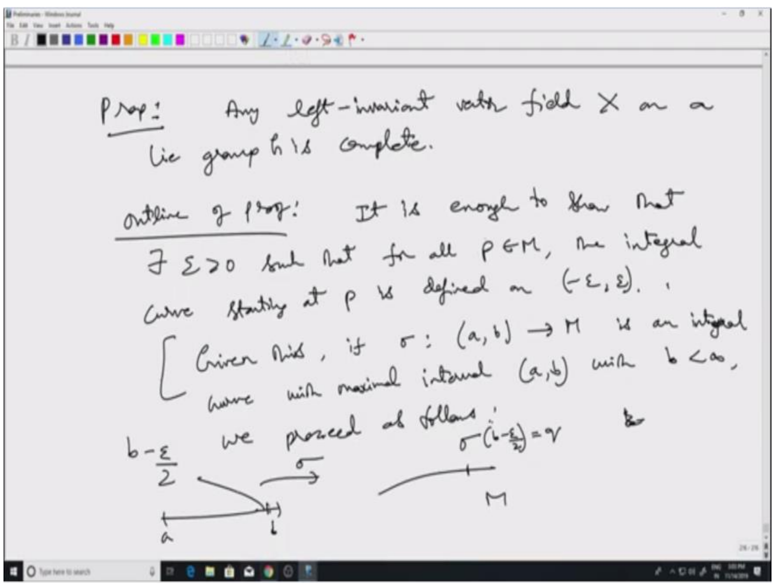


Another class, another situation where you get complete vector fields. So, the proposition is, is that the proposition is that any left-in variant vector field, vector field X on a lie group is complete lie group G. So here we do not assume the G is compact G can be non compact. And, but the assumption is the, the vector field is not an arbitrary vector field but f left-in variant vector field.

And outline of proof as I was saying the, it is possible to give a proof which will deal with this lie group case and the compact manifold case more or less simultaneously. The essential ingredient turns out to be the fact that both on a compact manifold and on a lie group we can ensure that any if I take care vector field X in the lie group case it has to be left-in variant. Any integral curve started integral curve starting at any point is at least defined on a common interval minus epsilon, epsilon.

In other words, it is enough to show that there exists epsilon greater than 0 such that for all P in M. The integral curve starting at P is defined on minus epsilon, epsilon. In other words, the same epsilon should work for all the points in the manifold. So, integrals is defined on this. So, the maximal interval of definition can be something larger. In fact, what we are trying to prove is that the maximal interval is the full real line.

But what we demand is that, at least it should be defined on minus epsilon, epsilon and the same epsilon should work for all points in the manifold. Now, it turns out that this condition is satisfied for both left-in variant vector fields on a lie group as well as arbitrary vector fields on a compact manifold.

So, for arbitrary vector fields on a compact manifold, one can just cover the manifold after all the ODE Existence and Uniqueness theorem tells us that if you are close enough to a given point, then this, the one can choose an epsilon like in this, what I have written here. Locally, one can always choose this epsilon, and that is part of the theorem, part of Picard's Theorem and its refinements.

So, you cover the manifold by these open sets on which one has a uniform epsilon, then you get a finite sub-cover and take the minimum of those epsilons and that will do the job. So that is for a compact manifold, but for a lie group, one proceeds as follows and actually if given this what if sigma is an integral curve, is an integral curve with maximal interval a, b with b less than infinity.

So this should lead to a contradiction. So, let so we do the same thing as we did here in the previous proof except that I do not have to worry about defining the sigma b looking at the limit of sigma tn as tn goes to b. I do not quite do that rather what I do is slightly different. So, here I will say let is an integral curve with maximal, we proceed as follows. Well, let us look at the sigma.

Now, the thing is that I know that by assumption, I have assumed that this epsilon exists at all points, we can do this and so on. Now, let us just take b minus let us look at in this real line. So after all, this is a and this is b, sigma is mapping it to the manifold. So what I do is I look at this point, this point is b minus epsilon by 2 on the real line now, the corresponding point is sigma of b minus epsilon by 2.

This is what I will call might q. So unlike the previous sketch where, where I said we go all the way up to b, I just stopped slightly earlier at this b minus epsilon by 2 and call that q. Now, I am going to continue the find a new solution with starts at q, and then patch up these two just like I did in the last proof. So the rest is this same thing, but this simplifies this having this common epsilon will help us.

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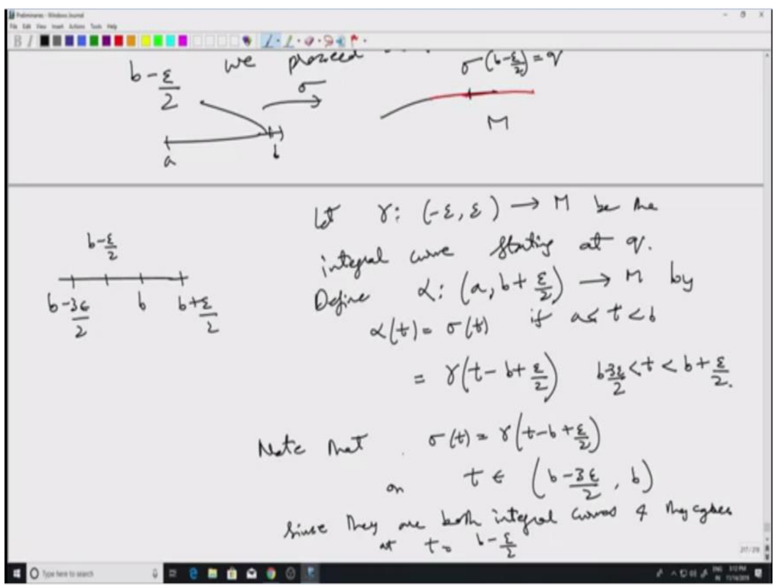


So, given this, we proceed as follows. So now, let gamma be the, so gamma I know is already defined that at least on the whole point is that the same epsilon will work. So, gamma is defined on minus epsilon, epsilon to M be the integral curve starting at q. So, I will draw a gamma. So, I will use a different color it starts at q. So that means it goes all the way here, but it will go slightly past wherever sigma was sort of ending.

So define alpha. So now I would like to patch up these two define alpha from a to b plus epsilon by 2 to M by alpha of t equals a to b rather alpha t equal to sigma of t from sigma of t, if t is in between a and b and equals gamma of t minus. Now, when, essentially when, so I want to start at so where was I? So, I want to start I want to start at b sigma of b minus epsilon by 2. So this I will make t plus b minus epsilon by 2.

If t is in between b minus epsilon and b plus epsilon by 2. So, this, so this is been, so I want to start here. So if I go rather no, not quite this, so this goes all the way up to minus 3 epsilon by 2, b minus 3 by 2 epsilon. And then when I add epsilon this part is okay, yeah, it is fine. So, t plus this. So, note that I had to change the sign a bit here. So this should be negative and this should be minus, this should be plus rather than the other way around.

So, I changed it a bit. So with this in hand, now notice that around the relevant points on the real line here, b is here, b plus epsilon by 2, b minus epsilon by 2, and b minus 3 epsilon by 2. Well, the, this thing here, this alpha, this gamma is defined the all the way starting from b minus 3 epsilon by 2, while sigma is defined all the way up to b. So, the common region of intersection is this note that alpha rather sigma, sigma equals sigma t equals gamma of t minus this on the common region, common open interval of intersection.

And that is all the way from on t belonging to b minus 3 epsilon by 2 all the way up to b, and the reason that they agree here, since they are both integral curves and they agree at t equals b minus epsilon by 2. I mean, a t equals b minus t equals b minus epsilon by 2 the left hand side a sigma b minus epsilon by 2, and the right hand side is gamma 0. But the way we chose gamma, gamma of 0 is I mean, starting at q, and the q was this point sigma of b minus epsilon by 2.

So the way we set up things they automatically agree. And by uniqueness as usual on the common interval, we have two solutions. So they agree on the common interval. Well, the, but they agree on the common interval, but the advantage of gamma is while sigma was defined only up till b, gamma is defined all the way up to b plus epsilon by 2. So therefore, that enables us to extend the solution.

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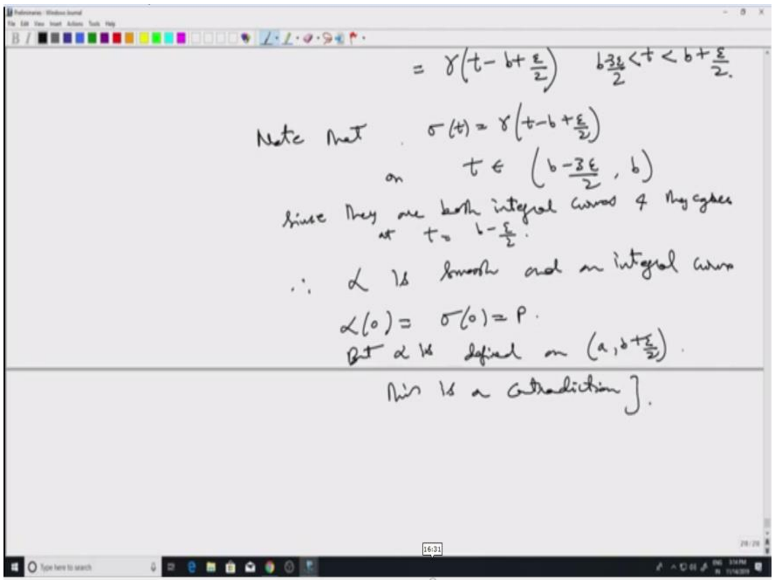

So the fact that they agree on the common intersection, therefore, alpha is smooth and an integral curve, you can directly differentiate alpha and it is basically sort of combination of two integral curves and it is smooth. So this is an integral curve and then one is done. The point is that alpha 0 as in the previous case is sigma of 0 equals P. But alpha is defined on a larger interval contradicting the, this is a contradiction.

So, this is the main idea actually if, if one wants to, if one wants to prove this lie group case and the compact manifold case, the main idea is to get a single epsilon which will work for all points. Now, I briefly mentioned how one can do it in the compact manifold case, now let us see how to do it in the for a left-in variant vector field.

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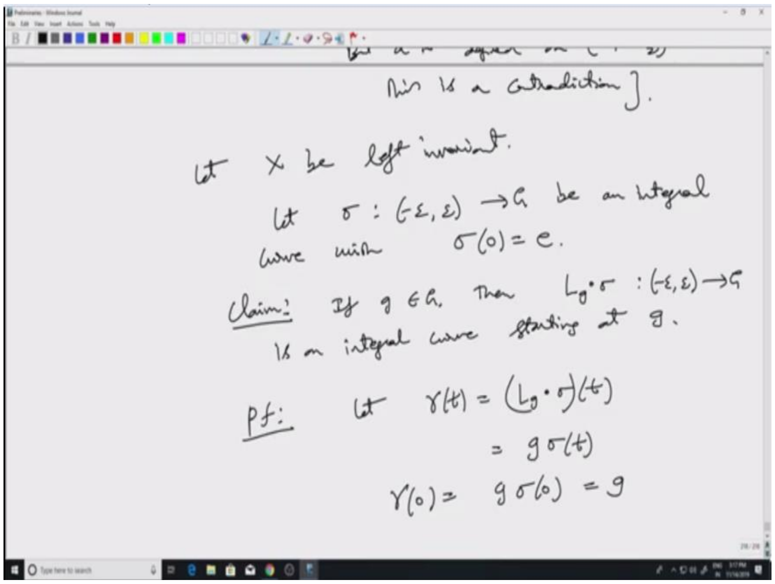


So, let X be left-in variant. And what I want is, let us, let us just look at let sigma from minus epsilon, epsilon to G be an integral curve with sigma 0 at equals e, the identity element. So, as we have seen, everything in a lie group sort of boils down to at least one, if one is considering left-in variants objects, everything boils down to considering what happens at the identity.

Here too, if I get hold of an integral curve starting at identity, I can use this to get define integral curves at all other points for this vector field X. So, the main claim here is that and that will do the job because at the same epsilon will hold. So, claim Lg, the left translation. So here let us see, if g belongs to g then Lg composed with sigma again from minus epsilon to epsilon to g is an integral curve starting at g.

What we have to write? So let gamma t is equal to Lg compose with sigma t. This is by definition the same thing as g times multiplied by sigma t, g sigma t. Now, the starting at g is trivial, gamma 0 equals g sigma 0 and sigma starts at identity. So this starts at g. The thing to check here is that this is actually an integral curve.

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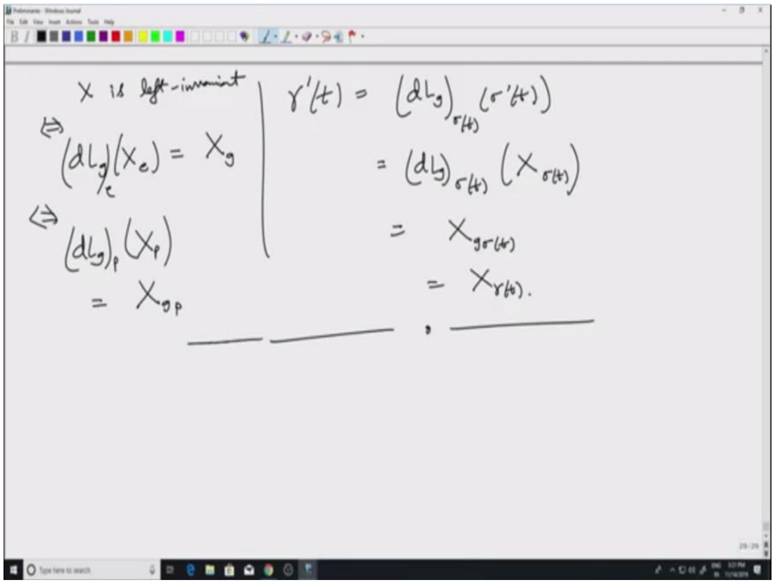

So, in other words gamma prime t, so let us say that it is an integral curve. So, now gamma prime t is equal to, I use chain rule dLg at sigma t acting and then acting on sigma prime t and sigma prime t is since sigma is a integral curve. This is X at sigma t. Now, the if we recall the definition of a left-in variant vector field was that X is left-in variant is equal the way we defined it, d Lg of Xe equal to X Xg, dLg of course, evaluated at identity, this at identity, Xe equal to, but we saw that this is also equivalent, instead of doing everything at identity suppose I start at some point p.

So, Xp dLg at the point P Xp equal to X at gp. So, let us use this form here. This one here, and this will tell us that this is the same thing as X at g times sigma t, which is the same thing as X at gamma t. So therefore, gamma is an integral curve. Now, so if one has an integral curve starting at identity, it is very easy to and if the vector field is left-in variant, it is very easy to get all of the integral curves.

All one has to do is just multiply this one, one starting at identity with g and you will get the digital curve starting at g. Now, here notice that because this new curve is also defined on minus epsilon, epsilon. So therefore, one has a common epsilon for all points in the lie group. And one can use this argument of patching up curves to show that the vector field is. Yeah, actually we already proved that we just the moment you get a common epsilon, you are done. So, and that concludes the statement that every left-in variant vector field is complete.

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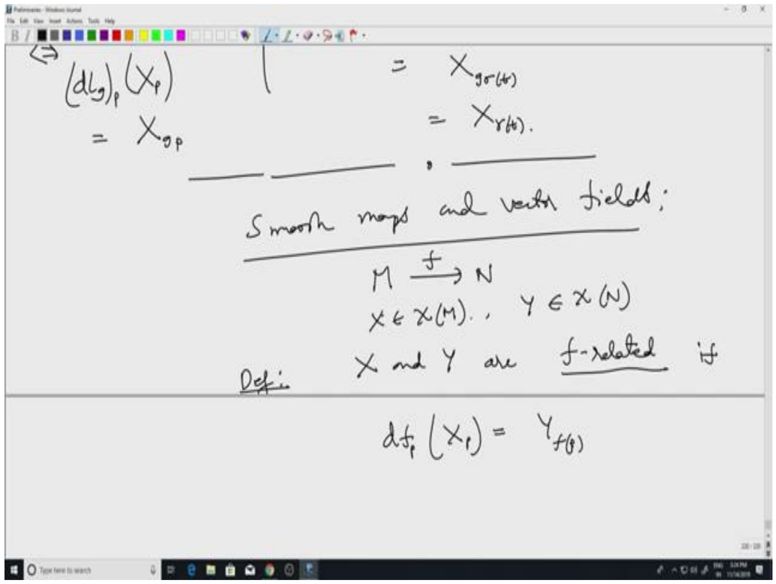


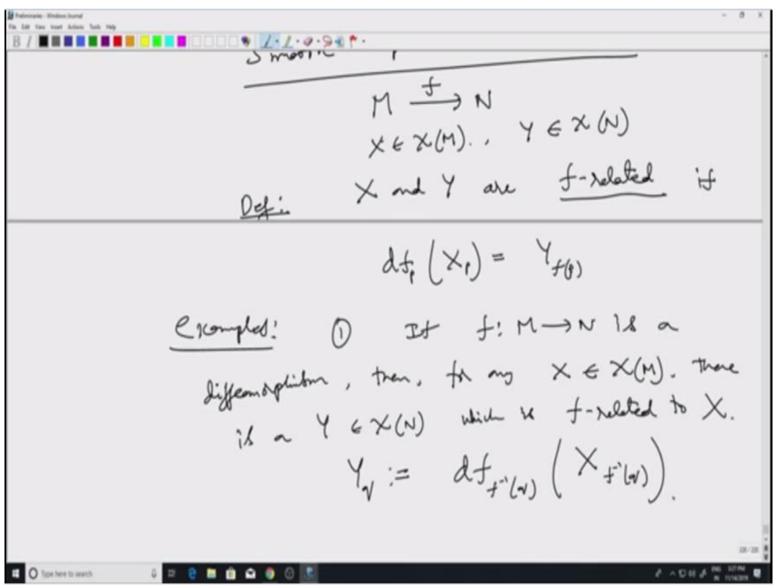


So, I want to say a few words about Smooth maps and vector fields. I have already remarked that if, if you have too smooth manifolds and a smooth map f. If I have a vector field on X, it may there is no natural way of defining a vector field on N using the derivative of f. So unlike the case of a tangent, if I have a single tangent vector, I got a corresponding tangent vector here. But that is not the case for a vector field.

So but there is a notion of when the smooth map can be brought into play, but for that we already have to start with two vector fields on M and N. We cannot use something in that domain to push it to the range or vice versa. So here the definition is, so let us start with two vector fields. One on M and the other one on N. X and Y are f related if well, you, we just use the derivative of f at a point p.

At that p I have Xp the vector field at that point, then well the derivative will take Xp to a tangent vector on N based at f of p, but at f of p I have, I can use Y so this is what we want. It is a very natural condition, it is just that wherever I land in the target, the value of the vector field should be by, so this, I am not assuming anything about f other than that it is smooth. For instance, f did not be one to one, in which case several points p in the domain can go to the same f of p.

The part of this definition is that df of all those Xps will actually be the same in the target. That is inbuilt into this definition. So let us look at some examples. Let us start with the diffeomorphism, if f from M to N is at diffeomorphism then for any X in a vector field on the domain, there is a vector field on the target which is f related to X. So, this is a special case in which one of the few cases were just given the domain vector field I can actually find a target vector field which is f related to this.

So, and we have seen this before. So, we have to define a vector field on N using this X and f so let us do it like this Y at q defined to be just use the derivative of f to come back to M, no not quite. So, you look at the corresponding point corresponding to q on M, which is f inverse p, f inverse q, look at the vector field X at that point use the derivative to go back to df at f inverse q and then X at f inverse q. And we already seen that this construction gives rise to a Smooth vector field on and so, but this is a very special case.

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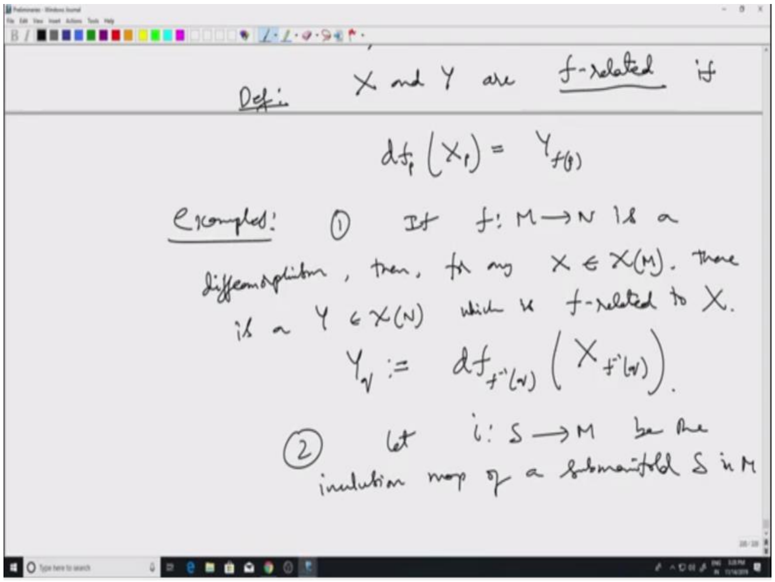


And the here is an example which is not, which is not related to a diffeomorphism. So, let i be the inclusion map of a sub-manifold be the inclusion map of a sub-manifold S in M. So, I want to start with the vector field on the big manifold and ask when is it f related to something in S? And conversely and so on. Conversely, I can start with a vector field on the sub-manifold and ask when it is i related, actually f here is i. So, I will talk a talk about this in the next lecture. So we will stop here. Thanks