An Introduction to Smooth Manifolds
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Lecture No 36
Vector Fields on Manifolds

Hello and welcome to the our discussion of Vector Fields on Manifolds.

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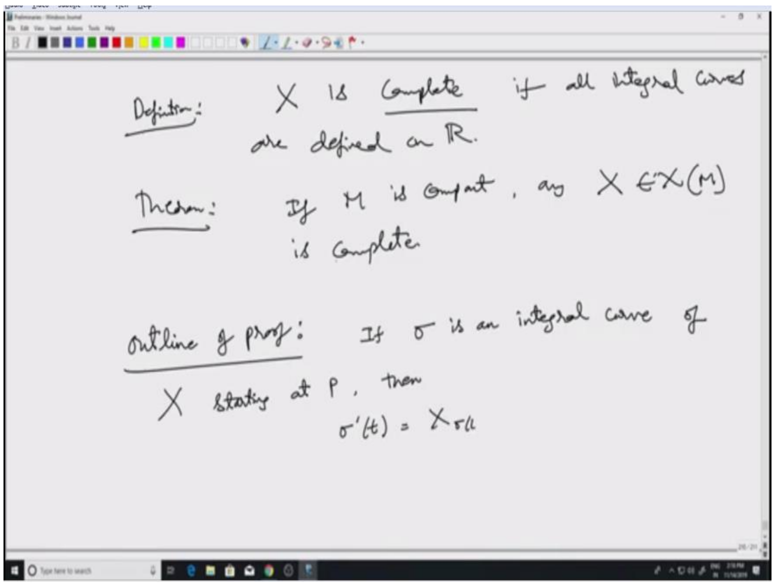

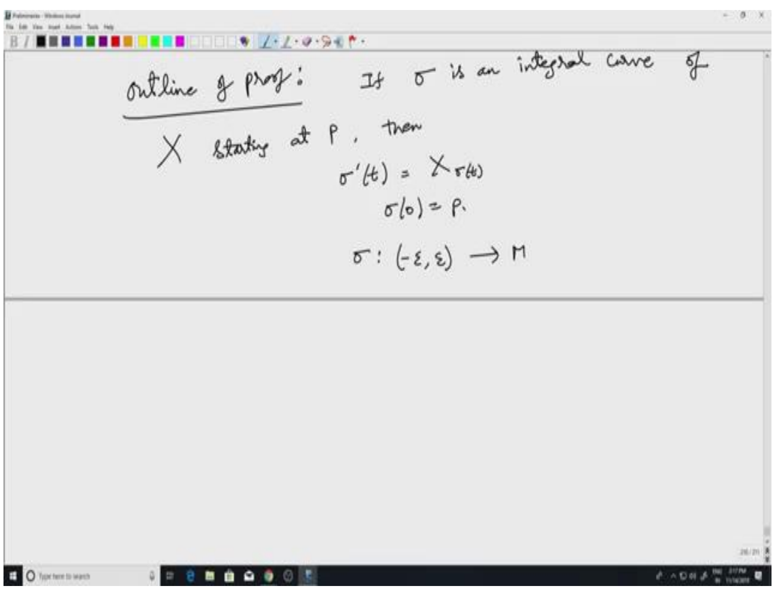

Now towards the end of last lecture I had stated this result that if you have a compact Manifold, then any Vector Field on the Manifold is complete. And before that I defined what a complete

Vector Field means, it is just that the condition that all integral curves of X are defined on R. Now, the, let me briefly outline the proof of this theorem.

So let us take this and then this outline of proof. First let us note that we have the notion of given. So let us consider this differential, the equation for an integral curve. If sigma is an integral curve of X starting at P, then by definition, sigma satisfies the following equation, which is sigma prime t equals X at sigma t and sigma 0 equals p. So, and here sigma will be defined on some interval containing 0.

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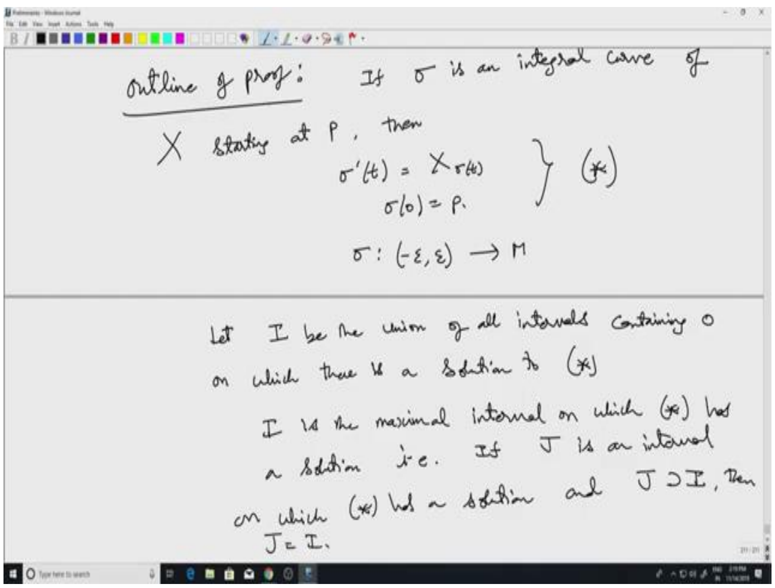

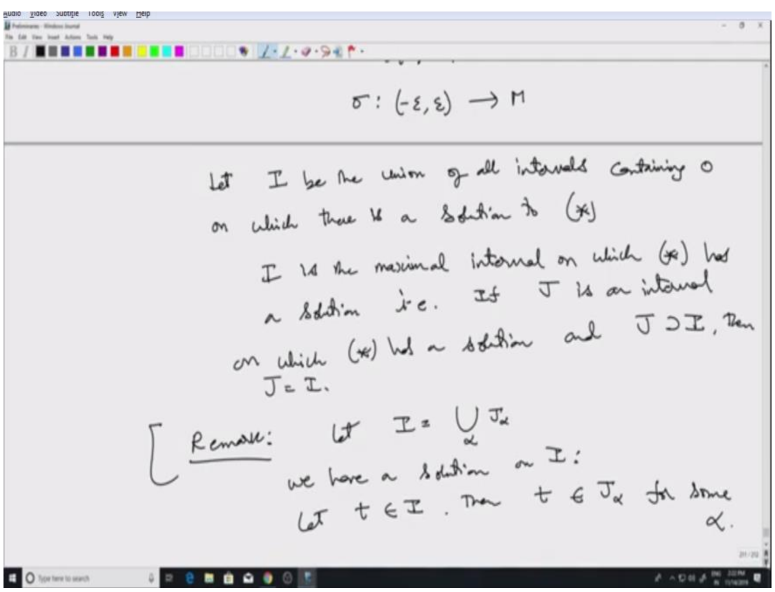

Now what one can do is let I be the union of all intervals containing 0 on which there is a solution to Star, Star is the system, this initial value problem. So the point here is that I have taken a specific in, in when I wrote this I took a specific integral curve defined on some interval. But it is conceivable that there is a larger interval on which there is a solution to this Star problem.

So what we do is, we take the union of all intervals on which there is a solution to this. And when we do that, by uniqueness, if I we know that there are two intervals on which there is a two intervals containing 0. Then on which there are solutions to Star, then on the common intersection interval, these two solutions must agree. This enables us to talk about the maximal interval which is which I have called I.

I is the maximal interval on which Star as a solution i.e, maximal in the following sense, i.e, if J is another interval, J is an interval on which Star has a solution and J contains I then J equals I. So you cannot make go past the interval I and still get a hope to get a solution. Now, the point is that the very unique I made a remark about uniqueness of solutions on common intervals and so on.

That is used to say that on I itself there is a solution. The, the way I was defined was that I is a union of intervals on which there is a solution. But we did not assume that there is a solution on I, but the fact that on each member of the union there is a solution enables us to and they agree on the common and the intersection potion enables us to define a solution on I.

So, let me just say that remark, if let I equal to union over some index at alpha, J alpha and J alpha are solution intervals on which there is a solution. We have a solution on I. So how do we define the solution? Well, we just take let I have to just specify what the value of the solution is at any point let t belongs to I, then t will belong to some J alpha for some alpha.

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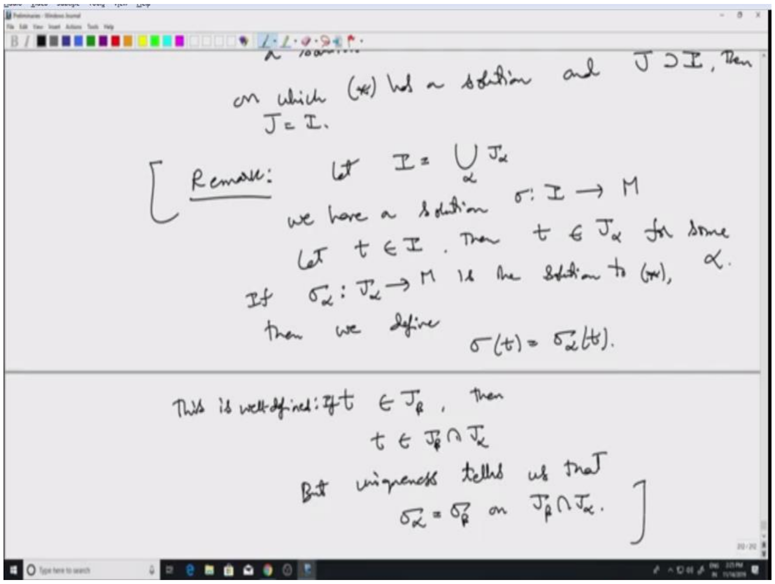

Then what I do is I just use the solution on J alpha to define. So, we have a solution. So, here let me denote the solution by sigma. Sigma from I to M, we have a solution sigma from I to M and that is defined as follows. So, let t belong to I and then t belongs to some J alpha. If sigma alpha is the solution on J alpha, then we define this sigma t to be sigma alpha t. So, here what can happen is that I took a point t in I, then I said that this t belongs to J alpha well it can happen and then I use the solution sigma alpha on J alpha to define the sigma t.

Now, it can happen that this t can if t belongs to J beta for some other index beta then this is actually what one would like to claim is that this definition is well defined. In other words, there is no ambiguity in writing it like this and that happens because if this is well-defined if t also happened to belong to J beta, then on the common so then t would belong to J beta intersection J alpha.

And we know that but uniqueness tells us that sigma alpha equal to sigma beta on J beta intersection J alpha hence, you get, I might as well put this sigma t equal to sigma alpha t or sigma beta t. So, when even when we take a union of intervals on which there is a solution on the union itself we can that is the solution is defined. So, once one has that one has a maximal interval maximal in the sense, there is no larger interval on which there is a solution.

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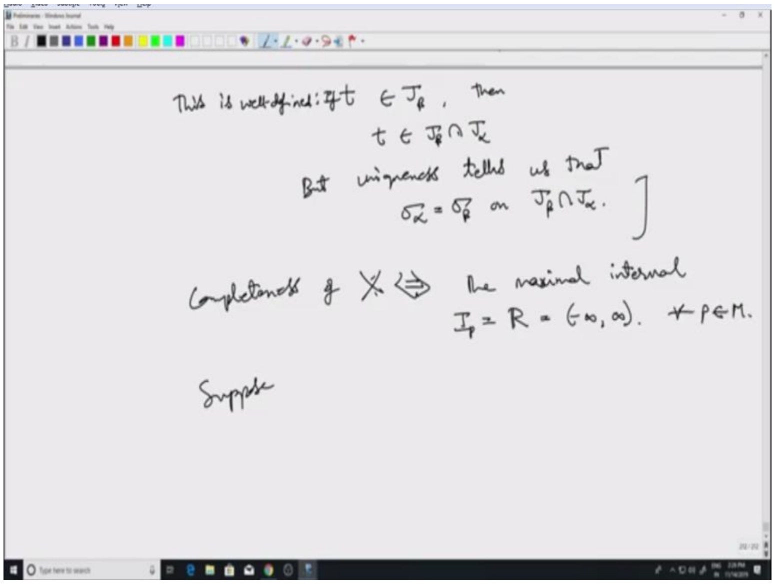

So, one would like to claim. So, what one is aiming for completeness of X is equivalent to saying that the maximal interval I equal to the full realign minus infinity to infinity. So, this is the, what one is aiming for. So, now, so far I mean compactness of X did not play a role, all this remarks that I made about was just to define the maximal interval. Now, coming to compactness suppose, now we have fixed some point P and suppose I, in fact, I should put Ip here for all t in m.

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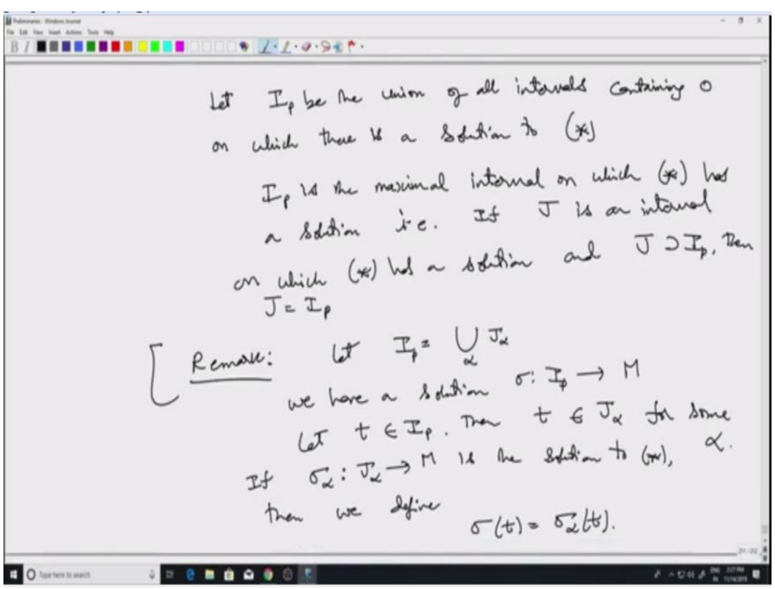

So Ip is this let us use Ip, let Ip, Ip and Ip.

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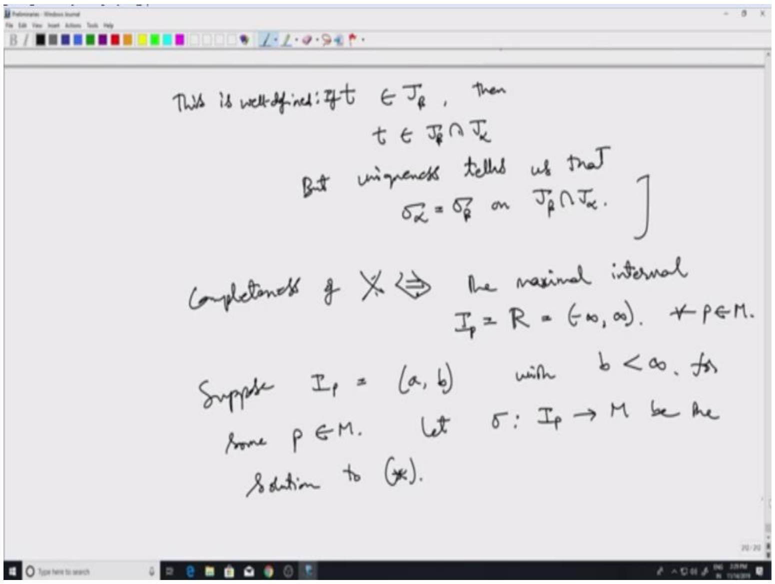

So suppose Ip equal to, now, if it is not the full realign two things can happen. Either the right endpoint is finite or the left endpoint is finite. So let us further assume the right endpoint is finite. Suppose Ip is equal to a, b with b less than infinity for some P, for some P in M. So, in other words, the integral curve starting at this specific P cannot be defined on the full realign, you have to stop at some positive, it goes all the way up to some positive time, finite positive time which I called b.

Now, what I would like to say is that the idea of the proof is quite simple. So, you just want to say that and we already have a solution here. Let sigma be the solution to Star. Now, idea of the proof is just to this Ip is an open interval, what I would like to do is that I would like to define sigma all the way up to b.

So, in other words, I want to define sigma of b and one in a smooth way and once I do that, then I can again start this solution to the find a solution to the differential equation to the integral curve problem starting at this point b. So, and then combine the two solutions and get a new solution which is defined on a larger intervals.

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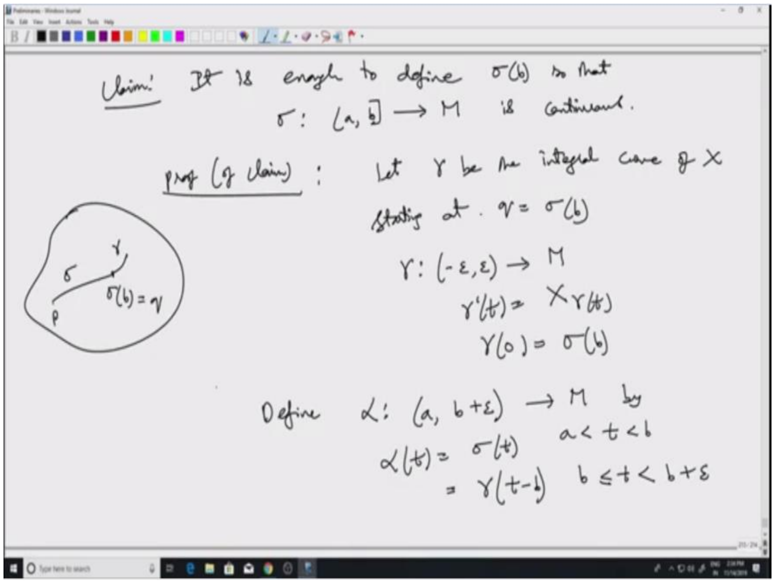

So, let me just so, make this elaborate on this, it is enough to define sigma of b. So that so this map now that I have included this is, in fact, it is enough to just say that this is actually continuous. So, suppose I know this claim. So this is a claim. It is that it is enough to do this, because let us see why this claim would imply the main result. So we are supposed to get a contradiction since we assume that b is less than infinity.

So, the contradiction arises as follows. It is enough to define this. So that is a proof of claim, if not, so yeah. So, if suppose we have such a sigma, let gamma be the integral curve of X starting at b, so this is all manifold. So, the point P was here, so, I taken the sigma and sigma b some point. Now, what I am looking at gamma is the integral curve it starts at oops, not at b but rather sigma b.

Let us give it some name at q equal to sigma b. So, this is q and this part is gamma this part is sigma. Now, as one can expect, after all this is an integral curve, so it is tangent to the Vector Field throughout and one can expect that the sigma and gamma combined patch up smoothly to give an integral curve starting at P. So, the claim is that so first of all let us denote the interval. So, gamma is defined on some interval minus epsilon, epsilon.

And it goes to the manifold and by assumption, gamma prime t equal to X at gamma t and gamma 0 equals sigma of b then. So, what we can, once we have this gamma, I would like to say that I get the Sigma combined with gamma, I will say what exactly what combined means. Actually gives a smooth integral curve, let define.

So, define alpha from curve start on the interval a to b plus epsilon and to M by alpha of t is just sigma of t. For t less than b greater than a and then alpha of t equals gamma of t minus b if t is greater than or equal to b less than b plus epsilon. Now, the I would like to claim that alpha is actually an integral curve of x.

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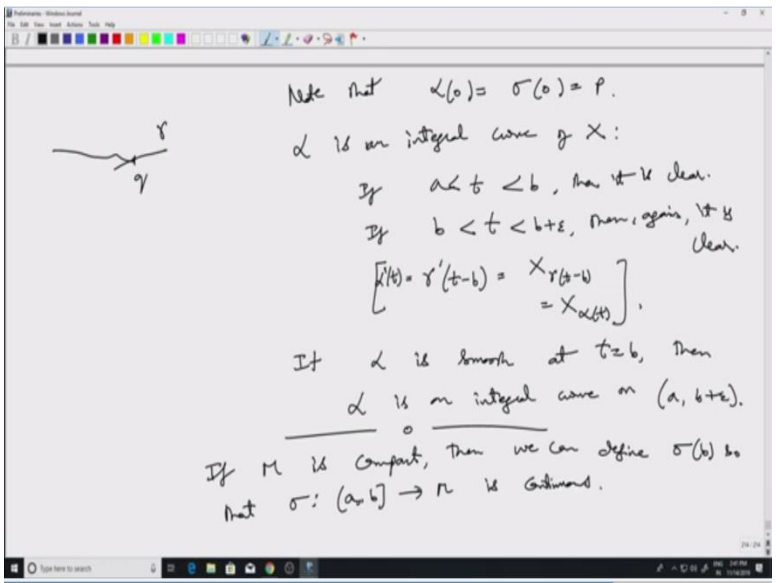

And note that alpha actually starts at note that since the alpha 0 as well as long as t is less than b, alpha is just sigma, so this is same as sigma 0 is P. And also alpha is continuous everywhere the only issues is at b, but as we have assume that as t goes to b sigma t goes to specific number with a specific point of which we are called sigma b, I mean we have defined sigma b like that. So, the only thing which, yeah I would like to claim that alpha is actually an integral curve, alpha is an integral curve of X.

As long as if t is in this open interval there is no issue then alpha is the same as sigma which is an integral curve then it is clear and similarly, if t is strictly greater than b and less than b plus epsilon as well. Then, again, it is clear. So the only issue is where what happens at, oh, by the way, notice that attain the parameter by this constant, this t minus b does not change anything, because if I take the derivative of this gamma prime t minus b is supposed to be well X at gamma of t minus b.

So, this is alpha prime t and this is x of alpha of t. So for even in this interval it is fine. The only issue is that t equals b. So, the potentially one problem can arise, this sigma or this gamma is actually defined, so this is my point q, gamma and this is gamma, gamma is an integral curve, so therefore a smooth curve which goes both to the left of q and right of q, I mean left of defined on left of 0 and right of 0 so that is okay.

But we are on the left when t is less than b, we are actually not using gamma, we are using sigma. So, sigma is coming here. So there is, it can be it is the smoothness at q is not entirely clear. So, that is the only thing one has to check that it is actually smooth. If alpha is smooth at t equals b, then alpha prime t exists. And I can just use the right hand side for that. So alpha prime t equal to gamma prime t equal to, well, it is actually gamma of t minus gamma prime of t minus b.

And just like what I wrote here, I mean I do not have to repeat this, in fact, the same calculations as in this brackets applies then as an integral curve on a to b plus epsilon because the tangent vector at q, I can just use either the right curve or the left curve, either sigma or gamma can be used to define the tangent vector, to calculate the tangent vector t equals b. So, in both cases I get what I want.

So, the smoothness is the only issue. So, how do we see that this is smooth? Now before I incidentally this I will, I will not address this instead this, the this part the proof of the claim is actually again does not use compactness as such. I have already assumed that sigma is it is enough to define sigma b, so that this is continuous. I have already assumed this. Once we have that we can do all this but the issue is, so here it does not require compactness of M.

So where compactness does play role is in precisely is improving that one can define sigma B. If M is compact then we can define sigma b so that sigma from a to b, is continuous and in fact, one can do something more actually one can make it differentiable from the left but let us just see where compactness has.

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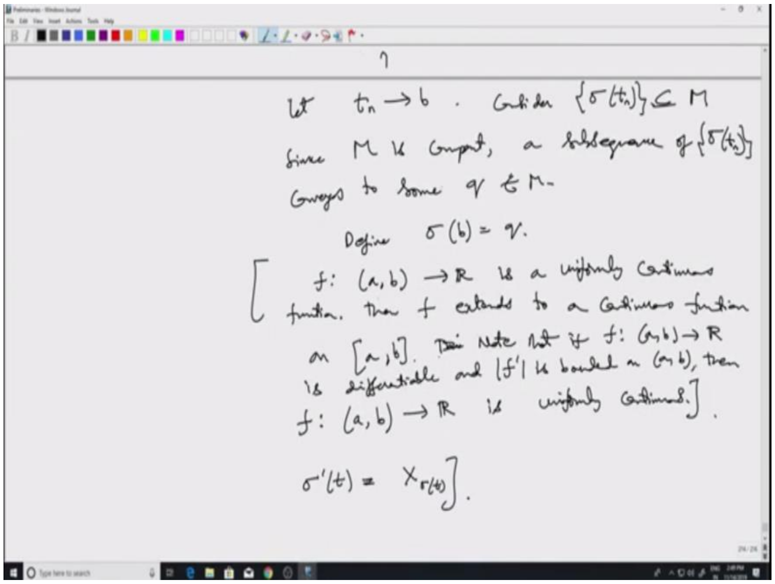

So, the role to play what, what we can do is let tn be a sequence going to b. Now we look at consider sigma tn this has a since M is compact. Now, this is just a subset of the manifold, since M is compact subsequence of sigma tn converges to some q and M. Define sigma b to b, this q. Well, this is a one, one can start this way, but it is not at all clear that this way of defining sigma b will actually is well defined.

I mean for instance, that different subsequence might converge to some other point, moreover, if I start with a different sequence going to be it might end up with completely different points and so on. So one has to show that just, this choosing one subsequence is good enough and in fact, you can define q, just catch hold of q, we are just using once and the reason is, it is just one no one notices that.

One recall the classical if setting if f from a, b to R is a continuous function. Rather actually what I need is not just continuous, uniformly continuous function, uniformly continuous function then f extends to a continuous function on a, b. This is the basic simple idea behind this why this works.

So, what one can do is claim that, so that sigma from a, b to M is uniformly continuous. Well to talk about uniform continuity, one has to put a metric, I mean a distance function regard M as a metric space and then do the calculations. The thing is that, yeah, why would we expect that it is uniform, the uniformly continuous and again go back to the one variable calculus setting.

So, here what the small results that I wrote here, this is implied, this condition rather note that if f from a, b to R is differentiable and note that f, and f prime is bounded on a, b then f from a, b to R is uniformly continuous. So in short, if we have a derivative bound, then we can assume a, ensure uniform continuity on the open interval. And that will enable us to extend the function all the way till the end points.

That is essentially the idea even here. The sigma is after all, the derivative of sigma is just given by the value of the vector field at that point. Now, the fact that M is compact again will help us here to say that the derivative in some senses bounded. Now, to talk about derivative being bounded, derivative is a of the sigma is after all a tangent vector. And to talk about its normal, one would need a, what is called a Romanian metric.

But there are ways around that you, do not need actually need Romanian metric. You can still work in local charts, one can get around that, but I will not go into details but essentially the idea is that the derivative is the Vector Field and the Vector Field on the whole manifold, since the manifold is compact is a Vector Field is in some sense, a bounded quantity. And that some sense on as option of using metric Romanian metric or a local chart.

And then use that to prove this and extend sigma all the way till the end point and then next thing So, there are a couple of things left unsaid which I have not proved here which is this here the fact that the derivative what how to make sense of this bounded derivative and how to prove that sigma actually extends and this is, this part uses compactness crucially.

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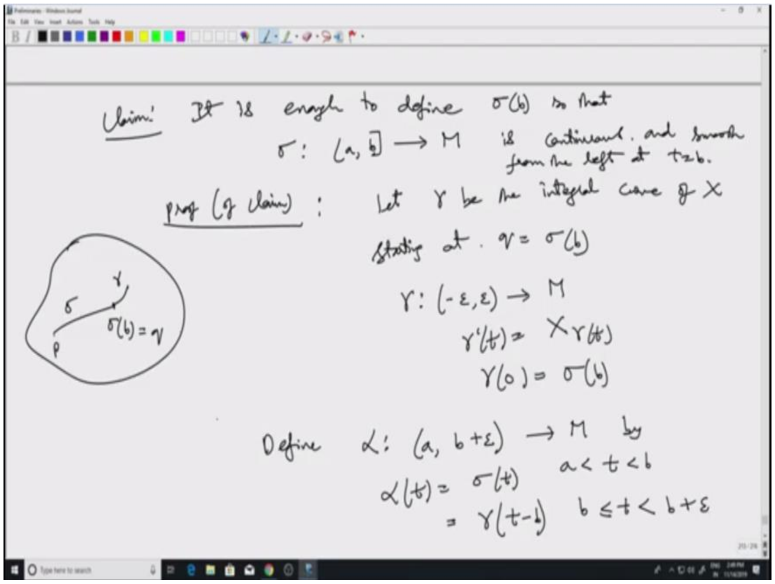

And then, once we have this then we have to show that this. So here actually this process gives not only that it is continuous, smooth, continuous and smooth from the left at t equals b.

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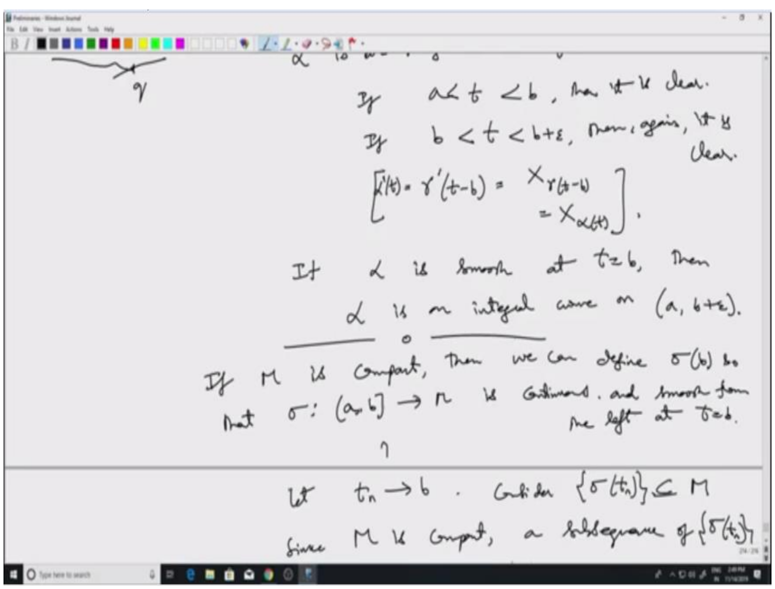

So, sigma b is continuous and smooth from the left. All right, so we will stop here. And we will resume our discussion of Vector Fields in the next lecture. Thank you.