An Introduction to Smooth Manifolds

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Lecture 34

Integral curve and flows 2

Hello and welcome to the 34th lecture in this series. So in the towards end of last class we had started talking about integral curves and ordinary differential equations in the presence of a vector field. So let me quickly recall what we were trying to do?

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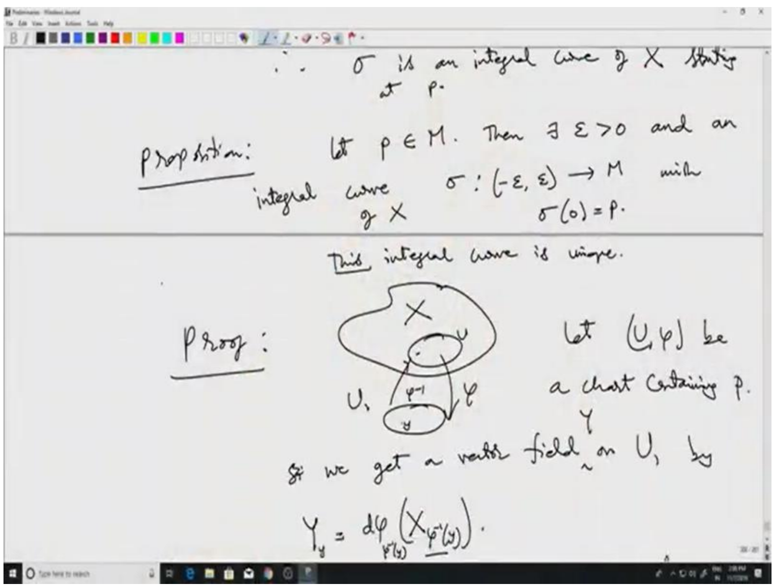

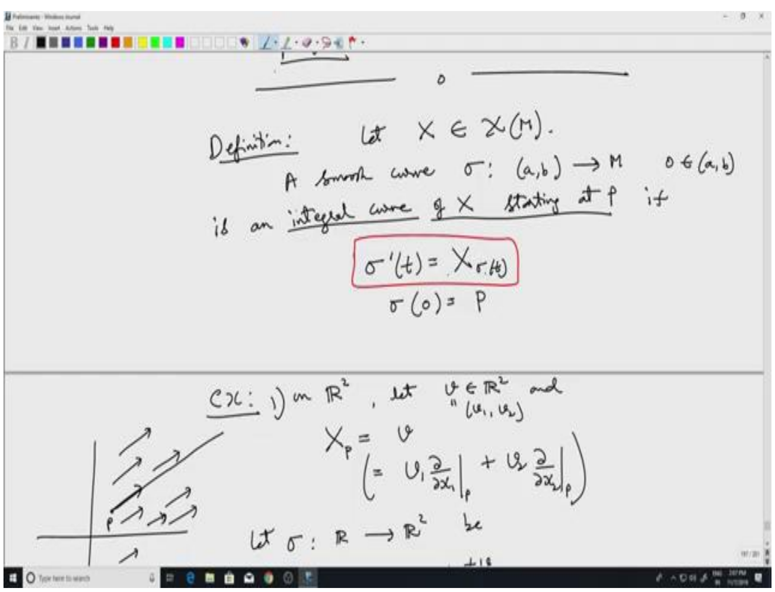

Well the defining property is over them. Let us say we start with a vector field on a manifold M. The smooth curve in M is said to be an integral curve of X starting at P. If it is velocity vector at time T happens to be the vector field. The value of the vector field at that point and the starting at P just refers to the fact that at time T equals 0. I have Sigma 0 equals peak right.

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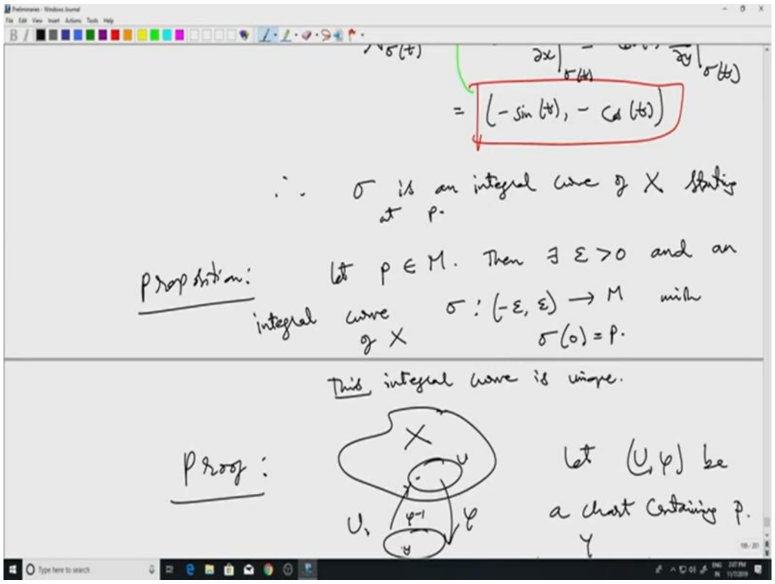

And the fundamental theorem is that for any point there exists epsilon greater than 0 and an integral curve of X which starts at a peak. This integral curve is unique in fact something slightly stronger holds which I will state in this lecture. So but in any case to prove this the idea is to first transfer the set up to Euclidian space to an open set in Euclidian space. And see what exactly the integral curve equation means so here we took a chart on P.

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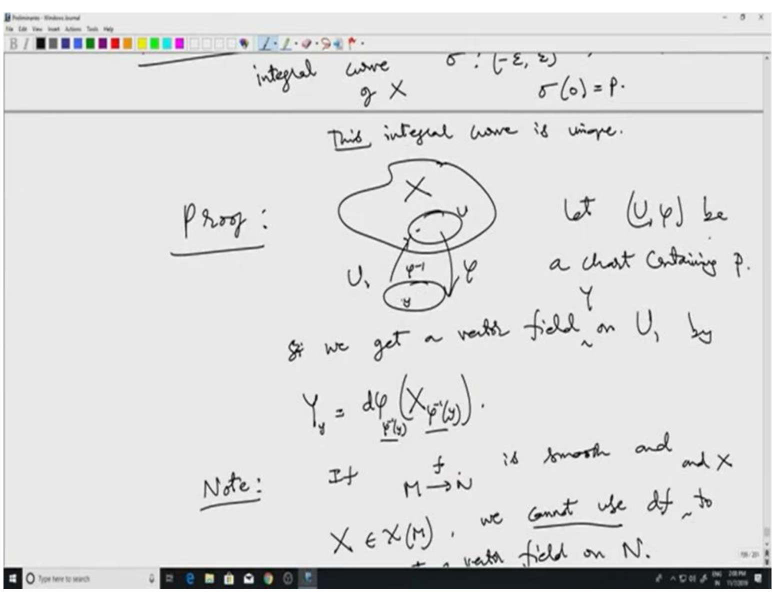

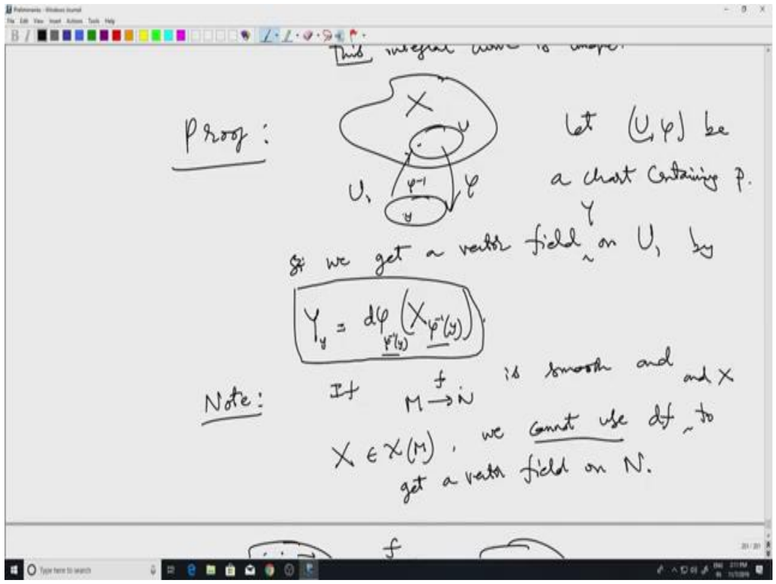

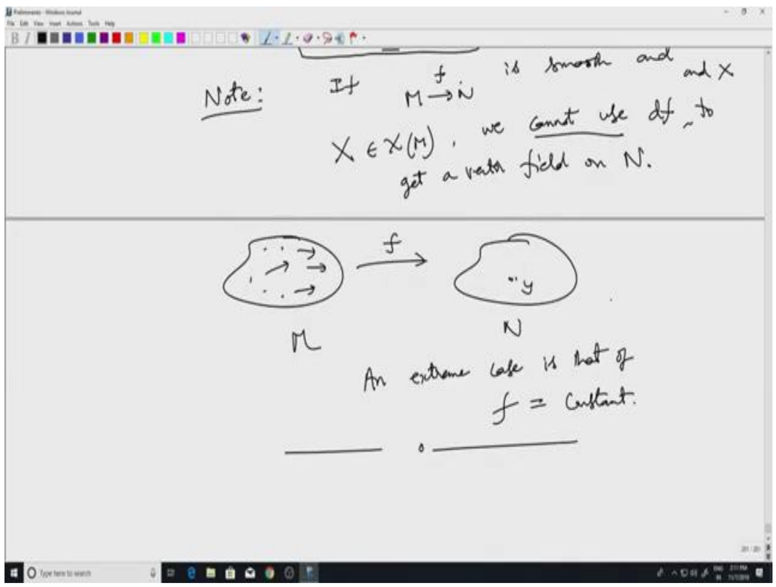

And the first thing is that we transfer the vector field on U to a vector field on U 1 by this equation here. And I had remarked that in general when you have a smooth map between manifolds and you have vector field on the domain. It is not possible to use the derivative of the map to transfer the vector field on the domain to that to a vector field on the target as an extreme case. One can consider an extreme case is that if equal to constant. So the whole domain is getting mapped to a single point.

Now the so the derivative of the map is 0. And so no matter what vector field you start with here you will get 0. And which will be so of course one is the 0 vector field on the target but that is not what one wants. Even if f is not the 0 map the problem. There are two problems essentially which I discussed last time the map not being subjective. And the map not being injective well if it is diffeomorphism then one can proceed like this.

What I have done here actually? So here I have to make a small correction. So this I am starting at I am starting at a point Y here so then I go Y of phi inverse and then well no it is I guess it is fine so I go by Y of phi inverse end up by phi inverse Y use the vector field at that point X at phi inverse Y then use the derivative of phi and get back to Y so right so that is okay. So the so as usual one has to be careful about smoothness however we have seen this before.

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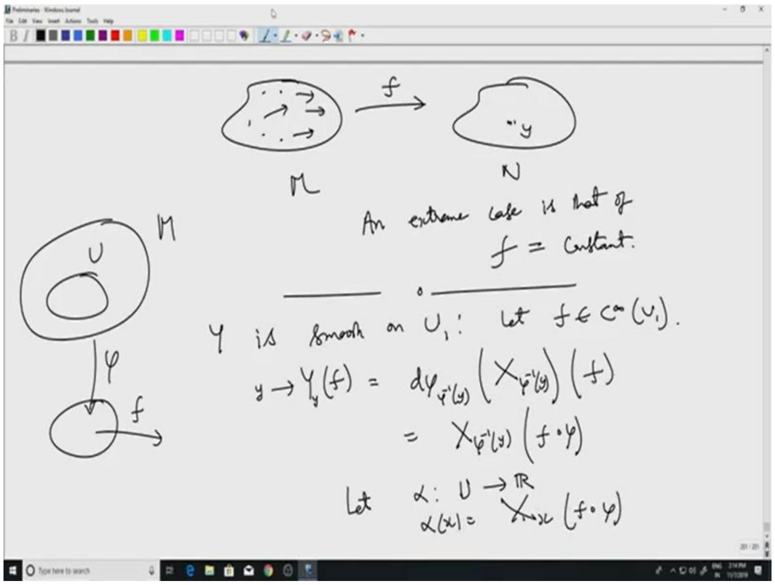

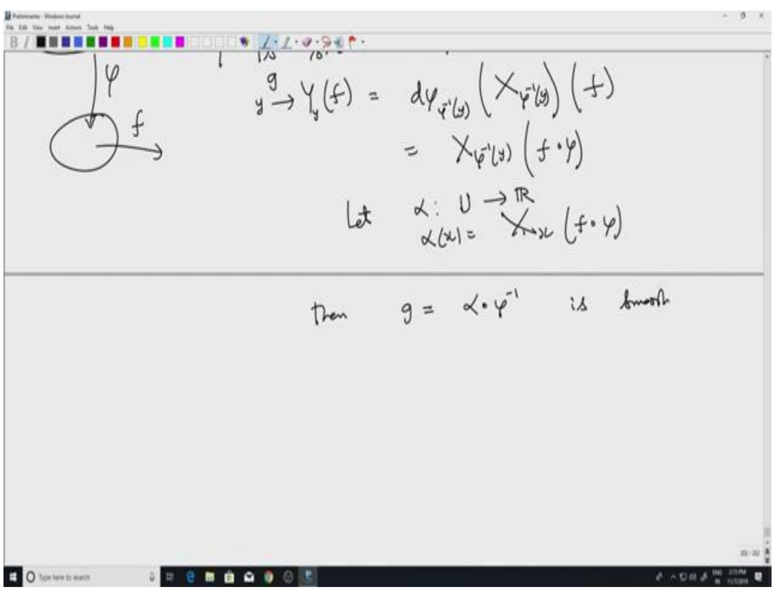

Y is smooth on U1 to see this what one would do is one would so let f be a C infinity function on U1 so we want to consider this function y by f as a function of so Y going to this so I want to see this is smooth well this is one way are the way is just to express this in coordinate functions and so on so in fact right okay let us do this one can do it both ways so I have a function f here this is V this is U sitting inside M so this now this by definition is so this now this by definition is whatever I wrote down here DV at phi inverse phi acting on X at phi inverse phi the whole thing should act on F and exactly like a couple of classes ago.

This using the definition of the differential of phi this is actually equal to X at phi inverse y then f composed with phi, the phi coming from here, X at phi inverse y of this. Well this can be written as a composition of two maps, one as, so let us write define two maps, so I want to define a map alpha from this is from U to R by alpha equals x; alpha at x equals x at X of F composed with phi, this is one map and the other map is, phi inverse, so then this map, Y going to capital Y subscript Y f, let us call this as, so let continue doing that.

Then g equals alpha composed with phi inverse. And we know that so the vector field being smooth by definition means that, so the very smoothness of the vector field by definition means that this alpha is a smooth map and since I am free in vs smooth because please add a few more films therefore G is smooth right.

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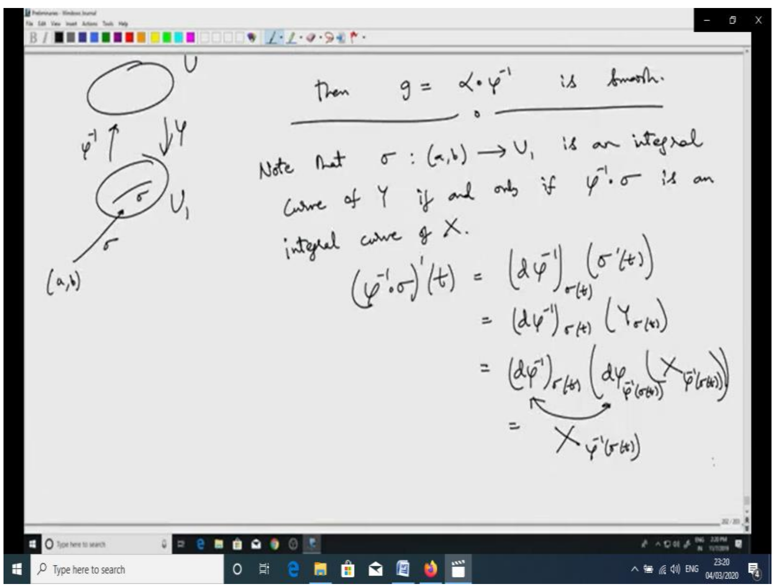

So it is immediately clear that this transferred vector field on u 1 is also smooth well right. Now the other thing to notice, note that sigma from an interval to U1 is an integral curve integral curve for or of Y if and only if so again the picture is like, this so this is phi U U1. So if I have an integral curve Sigma here, so literally sigma would be a map from here to here, so I can use phi inverse to get a curve, smooth curve in U and so the claim is that sigma is an integral curve of Y if and only if phi inverse with composed with sigma is an integral curve.

Actually let us say, instead of saying for, let us say of, integral curve of Y, integral curve of X. This again follows just from the definition of Y, so let us see one way, I mean let us do yeah, so suppose I start with an integral curve of Y, let us see why this is an integral curve of X. So, all I have to do is I have to differentiate, so I have to look at the phi inverse composed with sigma prime at t and I want to check whether this is actually equal to X at the corresponding point, X at phi inverse composed with sigma T. So, now it is just a matter of applying chain rule.

So the chain rule tells me that this equal to d phi inverse at the point sigma t, acting on sigma prime t. Well actually tells me that this is equal to d phi inverse sigma t composed with d sigma acting on del by del t but that is the same as this, the way we defined sigma prime t, it follows that this is, left hand side is the right hand side.

So, now we know that sigma is an integral curve of y therefore, I can plug that in here d phi inverse sigma t so and so this is Y at sigma t, so now but why it is sigma t by definition was, so let us recall why it is sigma t was just transferring, obtained by transferring the vector field X to this so d phi inverse d phi at the inverse of sigma t and then x at phi inverse sigma t. So, now we just, so this is should be in brackets actually. So now you just compose, so you just look at these two terms, this d phi inverse at sigma t and phi of this.

So the chain rule tells me that this composition is exactly identity, so I am left with x at phi inverse of sigma t, which is exactly what I wanted, so phi inverse sigma t, phi inverse composed with sigma prime is x at phi inverse composed with sigma. So this statement, so we had just checked that integral curves go to integral curves under this, just by taking phi inverse so with all that in hand now let us finally look at what the integral curve equation looks like.

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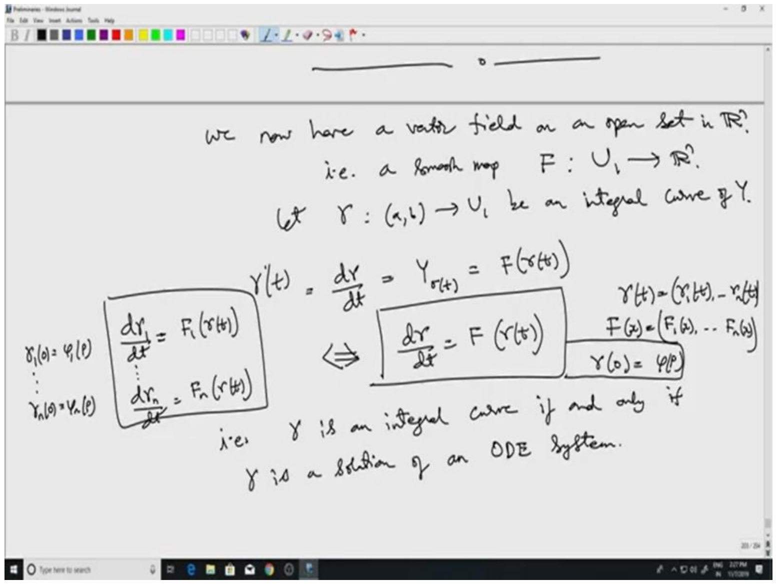

So we are now have a vector field on an open set in RN, as I pointed out last time the vector field on an open set in RN is the same thing as i.e. a smooth map on capital F, F from U1 to RN, vector field on an open set in RN does the same thing as a smooth map on this, so and now let us see what the equation, so the integral curve let gamma from a b to U1 be an integral curve of this Y so d sigma by definition, d gamma by dt, this is gamma prime t, gamma prime t equals d gamma by dt is supposed to be Y at gamma t, now if one thinks of the vector field as the smooth map then this is the same as F of gamma t.

So this is the usual going back and forth between derivations and elements of RN that is going on here, when I normally talk about vectors and tangent vectors and vector fields, at each point I, one thinks of them as derivations, at the moment you are in RN I can think of them as elements of RN that is this F of gamma t. So in effect what we have is d gamma by dt equals F of gamma t i.e. gamma is an integral curve of, if and only if gamma as this is a solution often. Now this thing here in the box is, of course I have written it as a single equation but recall that gamma is a map from, it is a function of one variable but the target is an open set in RN.

So actually gamma t is gamma 1 t gamma n t, so there are n functions of one variable and this likewise FS, f of X is F1 X1 Fn Xn. If one uses this coordinate expansion of gamma and f, then this thing here is equivalent to d gamma 1 by dt equals F 1 gamma t dot dot dot and then d gamma n by dt equals Fn gamma t, so in effect we have a system of n ordinary differential equations and, if and only if gamma is a solution of an ODE system and the fact that the right-hand side just depends on gamma t is and does not involve t as a separate variable is this is usually called an autonomous system of ODEs is so but let me just stick to the phrase ODE system to mean this.

So what we have done is we have reduced the problem of existence of integral curves to the existence of solutions to ODEs and I have omitted this starting point but we can retain that and here, for example, even though I have not mentioned that so gamma 0 equals so if we want a integral curve starting at P remember that phi inverse sets up a correspondence between integral curves in U1 and U, so if I want something starting at t, I should start here at phi of P so gamma 0 equals phi of P is the let me add this as a box here.

So here it would be gamma 1 equals phi 1 of P gamma n 0 equals phi n of P. So the idea whole point is to see that this the notion of an integral curve is the same thing as a solution to a certain ODE system, it is completely equivalent and our (())(21:18), of course the point is that one should be able to say something about this ODE system.

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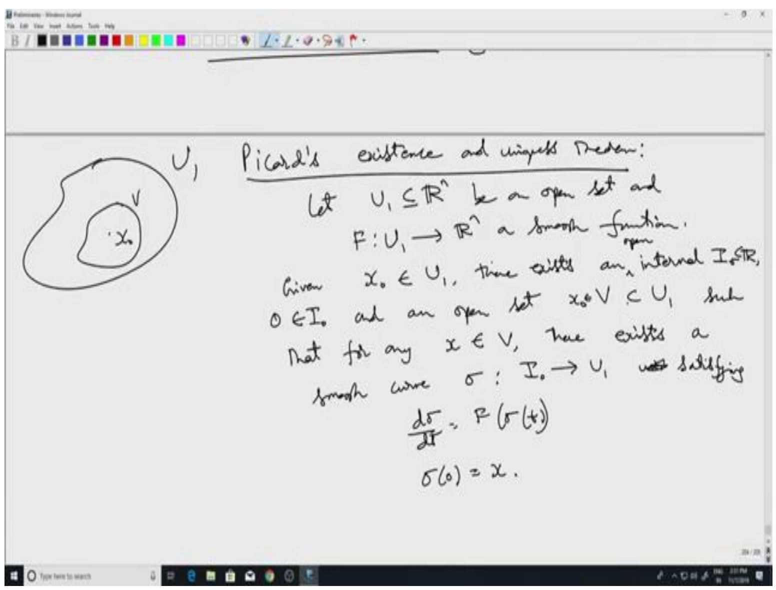

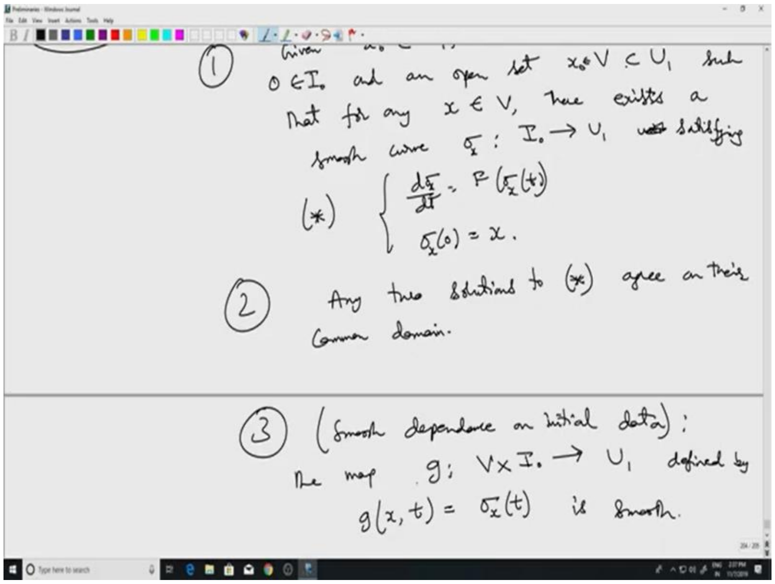

And for this we have the powerful existence and uniqueness theorem of Picard, so let us state this. Let one contain I will just keep the same notation as I had before be an open set F from U1 to RN a smooth function. Given X naught in U1.There exists an interval, an open interval, let us just call it I naught in R with 0 in I naught and an open set V which contains X naught and it is contained in U 1 such that so this is U 1 X naught at some point V is a smaller open set and I want to claim that for any point inside this V, such that for any X in V there exists a smooth solution to their exists smooth curve sigma from this interval I naught to U1 with satisfying d sigma by dt equals F of sigma t and then sigma 0 equals x.

So, what I am saying is slightly stronger than what I had stated earlier in the context of integral curve here what I am saying is that if I take any point X naught and there exists a neighborhood of X naught which I called V such that for any point in the neighborhood I can find an solution to this equation starting at that point X and that solution is can we can assume it is defined on the same interval I naught, so this interval I naught is the role of that is at this stage it may not be clear but it is very important. It typically very much depends on which point you are at,

At some points the solution may be defined for long time interval, at some point it may be very short interval so here we are saying that as long as we are close enough to a given point all the intervals on which the solutions are defined we can assume that the they are all defined on a common interval satisfying this. This is one statement. Second statement is that any two solutions, so let us call this star, any two solutions to star agree on their common domain, so in other words if I have a solution to this equation system with the same initial data sigma 0 equals x, I have two solutions both of the domains will have to contain 0 because I need to make sense of sigma 0.

So if I have two solutions then the domains may be different but if you take the intersection of the domains that will be an open interval containing 0 on that they both agree. The third thing is that this is called smooth dependence on initial data. So, let us define a map from the map phi, no I have already used phi, so let me use some other notation, the map let us say g, from this V cross that I naught to U 1, defined by, so here in the previous in the step one I said there exists a smooth curve sigma, now this sigma is solution to the differential equation and it starts at X.

So, let us keep track of that X by putting a subscript here sigma X, so and this products X here so now this map g is defined by g x t equals sigma x t so in other words what this map is doing is easy to explain, you start with a pair x comma t well you just look at the solution which starts at x, travel along that for time t that is this, so the claim is this map is smooth. Now this is stronger than saying we know that each solution to this, differential equation is smooth but what we are saying here is that when we regard the solution not just as a function of here.

Also I should put in x, when I regard the solution not just as a function of t but also of the initial starting point x so for this new g x is also a variable, even then the, there is smoothness so as you change the starting point, then the solution sort of you get different solutions, they all change smoothly that is what this is saying. So, this is the fundamental theorem which I am not going to prove and let us assume this, once we assume this then the integral curve set up becomes a simple corollary statement of this.

It is not even a corollary it is just a restatement. So for instance what we did was we started with a manifold, we had, we took a chart, transferred the vector field to an open set of, subsets of RN, then we saw that we get an ordinary ODE system namely this. Now we apply Picard's theorem to conclude that there is a solution to this, the way the given starting point and we transfer it back to the manifold y f phi inverse to get a integral curve for the original vector field.