An Introduction to Smooth Manifolds
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Lecture 30
Vector Fields 3

Welcome to the 30th lecture in this series. So last time I introduced the notion of a vector field on a manifold. Now, let us discuss some, explore some examples.

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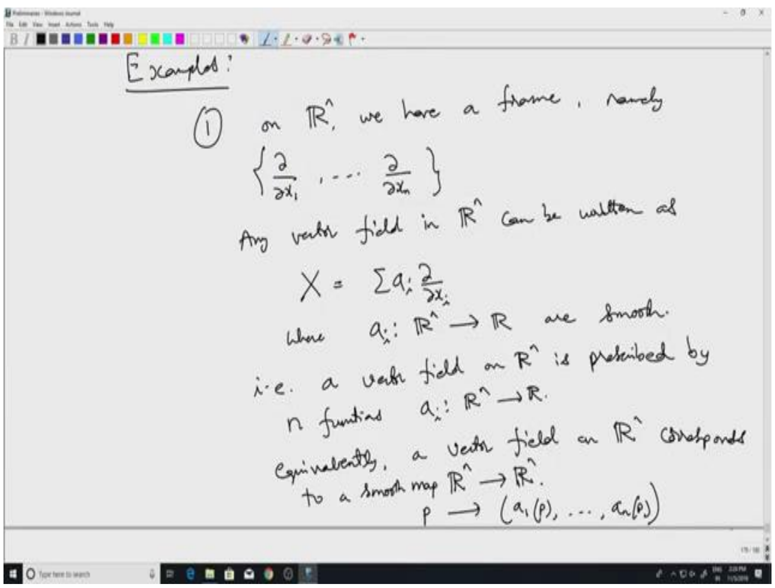

So Rn is regarded as a manifold, we know that we have a frame, namely del by del x1 at a point P, del by del xn at P. So, the partial derivative operators, actually let me not mention the P this, the partial derivative operators give a frame.

So, in other words, a basis for the tangent space at each point and every vector field, any vector field in Rn can be written as x equals sigma ai del by del xi, where the ai functions are real valued functions on Rn are smooth. So, in other words i.e. a vector field on Rn is described by n functions ai. So, this is the same as saying equivalently, a vector field on Rn corresponds to a map from Rn to Rn to a smooth map from Rn to Rn because the smooth map is a P going to a1 at P comma an at P.

Now, this way of thinking about a vector field as a map of from Rn to Rn is useful when one talks about the integral curse which I will discuss next when we come to ordinary differential equations in integral curves in the context of Rn, then this viewpoint is helpful.

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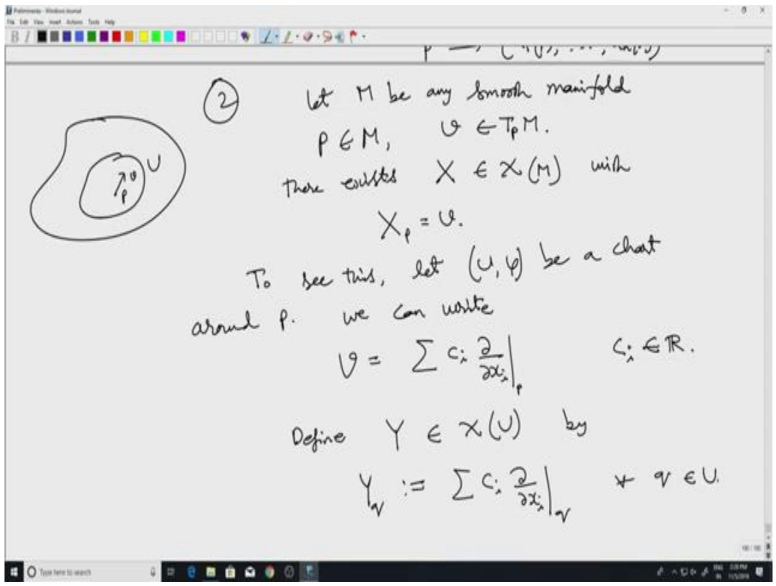

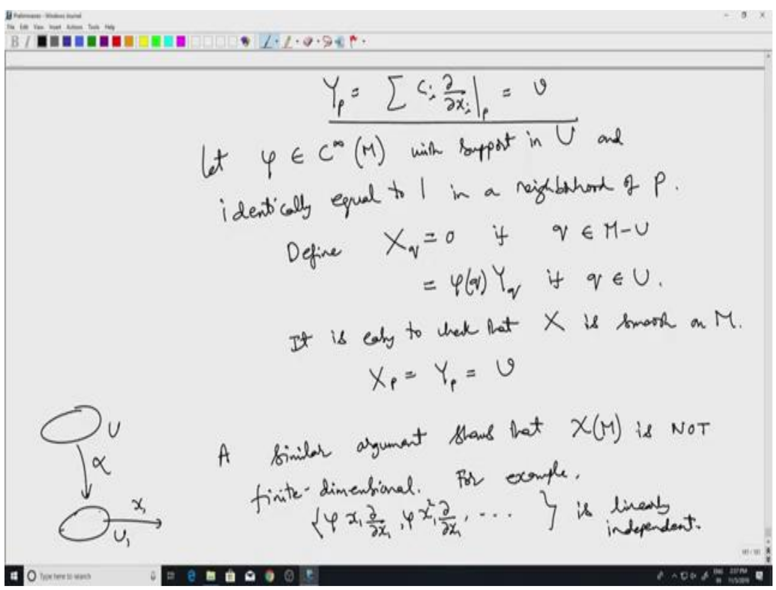

So this is an Rn now on the, let us move to the most general case namely any smooth manifold on let M be any smooth manifold, the point in P be it a point in M, v vector tangent vector at P.

So, then I claimed that there exists a vector field X on M with XP equals V. So, another way of saying it as any tangent vector can be extended to a vector field on the whole of M and this is immediately seen by looking at a chart around P, to see this let u, v be a chart around P. We can write as usual, we get a frame namely the coordinate vector fields give rise to a frame on u. So therefore, we can write V equals the some constant ci del by del xi at the point P where the ci Rn are just real numbers.

Define, so at least locally, so this is my u, this is the point and I have a vector v, at least u I can define a vector field which extends V in a very trivial way. Define, Y to be a vector field in u by well, I just take Y at any point Q. I define it to be I use the same formula as I have here except that of course, instead of del by del xi at P I will be putting del by del xi at Q but the Ci will be the same, so ci del by del xi at Q for all Q in U.

So clearly, I have already written Y is a vector field on u which according to our definition already mean, I am assuming that Y is smooth, but in fact it is quite clear that Y is smooth simply because these functions are constant functions. We have seen that smoothness is equivalent to this, the co-efficient functions being smooth, but here they are actually constant. Now Y is still defined only on u and but the good the thing about Y is that, Y at P is just ci del by del xi at P which is by definition v.

So, at least we have been able to extend the vector tangent vector v to a vector field Y only on u though. So, if I wanted on the whole manifold, I just do the usual trick of multiplying by it with a cutoff function. Let fi be a C infinity function on M with support in u and identically equal to one in a neighborhood of P defined as usual. Thus, whenever we want to extend something from a coordinate from an open set to the whole manifold, we multiply the locally defined quantity.

In this case, Y be the cutoff function and define the vector field to be 0 outside u define X, Q equal to 0 F, Q is belongs to M minus u. And define X, Q to be fi Q times Y, Q, F. Q is in u and define that fi has support in u will ensure that this X is smooth in where it is easy to check X is smooth on M with this definition. It is certainly smooth on u and it is smooth on M minus u closure.

The only issue as usual is in the boundary of u and there it continues to be 0 and in fact it will be 0 in the neighborhood of the boundary. And one can argue that it is smooth and check, it is easy to check that X is smooth on M. So this standard way of extension will work here all right. And since fi is identically 1 in a neighborhood of P, well all I need is, in fact, that fi at the point P is 1 the neighborhood is also not necessary.

So fi of P is one will guarantee that X at the point P is the same as Y at the point P, which is same as v since that is what we had here. So, any vector can be extended to be a smooth vector field on M and in fact the same sort of idea will show that similar argument shows that this similar argument shows that the set of vector fields on M is not finite dimensional and so here what one does is again, one can start with for instance, it is quite straightforward to see this.

You start with a chart and take 1 say del by del x1, which is a vector field on the chart. One can multiply del by del x1 by various functions defined on u, for example one can multiply by powers of x1 and then extend it to all of M, for example it is enough to show that it is not finite dimensional, we can exhibit that linearly independent infinite number of linearly independent vectors that will do the job.

So here, I can just take x1 times del by del x1, x1 squared times del by del x1. Everywhere I have to multiply by this fi etc. This multiplying by fi will give a vector field on all of M. And since v is identically 1 in the neighborhood of P in a small neighborhood element in that neighborhood, I just get these vector fields, x1 del by del x1, x1 squared del by del x1 etc. Of course here, as usual, when I x1, what I mean is x1 is a coordinate vector on u, not on u1.

X1 is defined here, just the first projection. And if I want something on u, I will as usual I will have to compose with fi. In other words, look at the first coordinate function of, so this fi should not be confused with the chart map. So, let me call the chart as something else, I already used fi for the cutoff function. So let us say, let us call it alpha. So when I say x1, what I mean is alpha 1, the first component of alpha, etc. and this del by del x1, we know what it means.

It is del by del x1 here, which has been transported back to the manifold by the derivative of alpha, alpha inverse. So for example, this set is can be is linearly independent.

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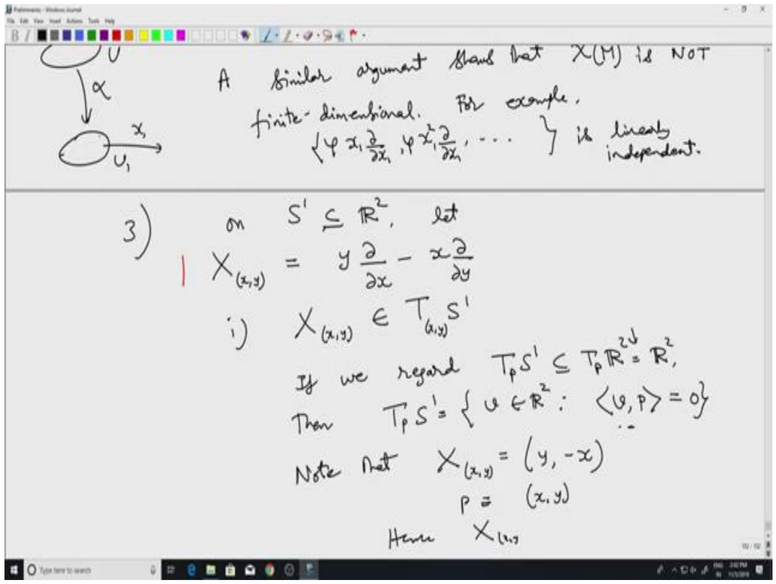

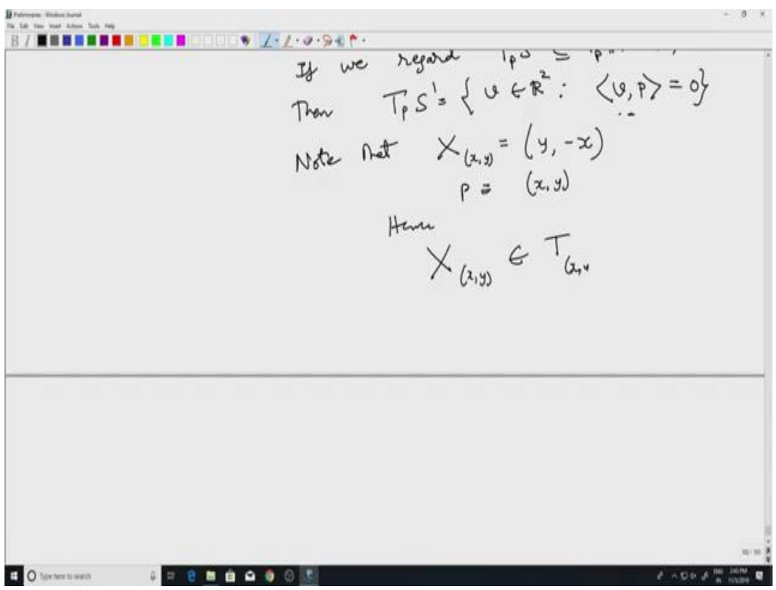

Now, let me give another an interesting vector field on R2, not on R2 in fact on S1 on the circle, on S1 the one dimensional manifold s1 which is which we regard as a subset of R2 as usual. Let X the vector field X at the point X comma Y.

I define it to be Y times del by del x minus x times del by del y. So now, there are two, so I want to claim that this X is a vector field on S1. So there are two things to check. First of all, this what I have defined here is certainly a tangent vector to R2, but is it actually a tangent vector to S1 that is one thing. Second thing is, as usual is it smooth as a vector field on S1? It is again, it is smooth as a vector field on R2, this right hand side is well-defined vector field on R2 with smooth, that is so is it smooth on S1?

Now first, let us check that this is indeed, so first point is X that x y should belong to the tangent space at the point x y to S1. Now we know that the tangent the, we have seen that if we regard TPS1, the subspace of TP R2 and which is which we identified with R2. This, what I have written here will give rise to a subspace of R2 for every P in S1, so as the point P changes I get different subspaces of R2.

So if we think of TPS1 as a subspace of R2, then TPS1 is this subspace, we have already seen this when we talked about regular values, TPS1 is all v in R2 such that in a product of V and P is 0. So in particular, well in this case, so now if I coming back to this, here I have written X coming back to this here I have written this X at X Y as a derivation it is the partial derivation derivative of vectors.

But if I think of it as a element of R2, then note that X at the point x y would just correspond to after all this identification here, this equality here just involves whenever we have a derivation at P can be written in terms of the partial derivative operators with some coefficients and this equality just amounts to putting the coefficients together as a coordinate vector that we have seen that when we first talked about tangent vectors.

Well, if we do that for this X, then this is the same thing as Y comma minus X. And the point P is just X comma Y, this P is X comma Y. So, to see that this X belongs to the tangent space in P all I have to do is check the product Y comma minus X and X comma Y is 0 like here, which is of course the case. Hence, X at X comma Y belongs to T X Y at S1.

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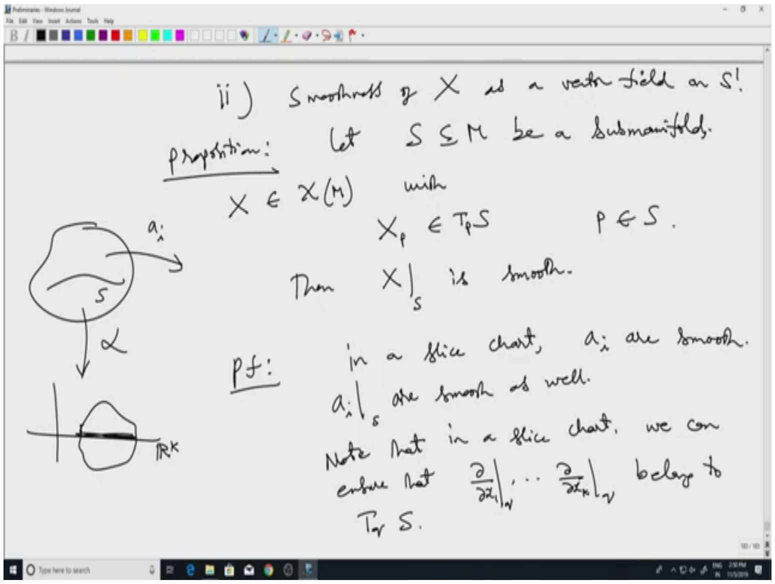

Now coming to the next issue is smoothness as usual, smoothness of X as a vector field on S1. Here again, it is a rather than worry about the special case, let us prove a more general thing, which is true for any sub manifold. So here is a small proposition, let S contained in M be a sub manifold, sub manifold X a field on M with, suppose X has this additional property that X at a point P if I start with a point P in the sub manifold S, then X at that point should actually lie inside the tangent to the sub manifold.

So TPS which is as usual regarded as a subspace of TPM, so to begin with except P is for arbitrary vector field X at a point P, wherever the point is, it would not care if the vector field is arbitrary then there is no requirement that X at P should belong to TPS, but we demand this condition, then if I restrict X to S, I get this is smooth. The proof is quite simple, it is pretty much anything involving sub manifolds, it is the easiest.

It is easy to see any statement in terms of a slice chart. So that is what one would do here. You would just take a slice chart on P, in a slice chart, so ai a smooth, ai the usual coordinate functions. The thing is that if ai are smooth so ai are functions on this, so here is a sub manifold S the slice chart is taking me to essentially putting the sub manifold S in RK, this is RK and s is going to this part under a slice chart.

So this ai is a functions here as I defined. Well, if ai is a smooth on this, then using this slice chart ai are smooth, then it is easy to see that ai restricted to S, the smooth as well. Actually, this does not involve anything, I mean, it is just a statement about smoothness, the earlier remark that I made that if you have a smooth function on a smooth map on a manifold, then it is restriction plus sub manifold is also smooth, that is all that I am saying.

But the slice chart condition is used because these coordinate vector fields that I get here note that, in a slice chart we can ensure that del by del x1 at Q etc. del by del xn, del by del xk Q are a tangent to belong to the tangent space to the sub manifold. So the very definition of a slice chart is that it is a chart on the bigger on the full open set M which when restricted to S intersection that open set will give rise to a chart on S itself.

So therefore, we have that these things del by del x1 etc. all the way up to xk, they are all tangent to S and since these functions ai are smooth. So actually, all I care about is a1 up to ak, they are smooth. So X restricted to S we will be by this condition, the fact that I have this condition X restricted to S will be a linear combination of this del by x1 up to del by del xk, with the coefficient functions a1 up to ak. So therefore, that will also be smooth. So we will stop here, and next in my next lecture, I will continue with more examples of vector field and I will talk about lie groups as well, thank you.