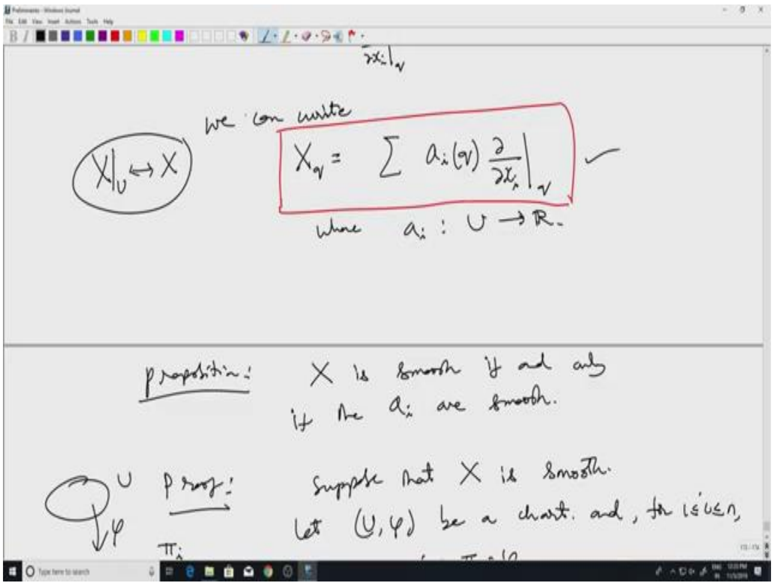
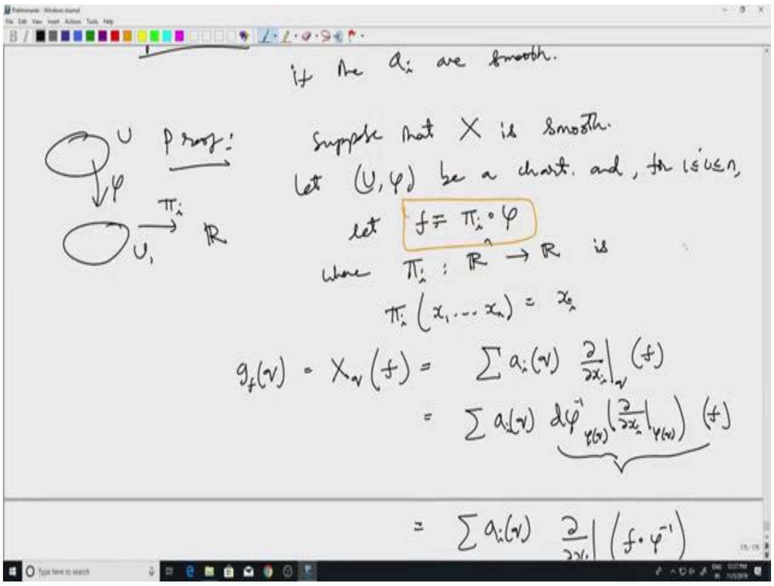
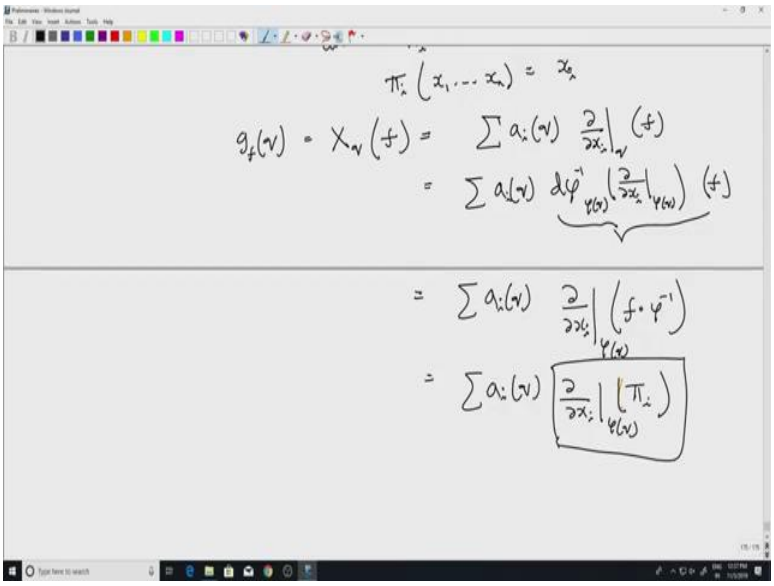
An Introduction to Smooth Manifolds
Professor. Harish Seshadri
Department of Mathematics,
Indian Institute of Science, Bengaluru
Lecture 29
Vector Fields 2

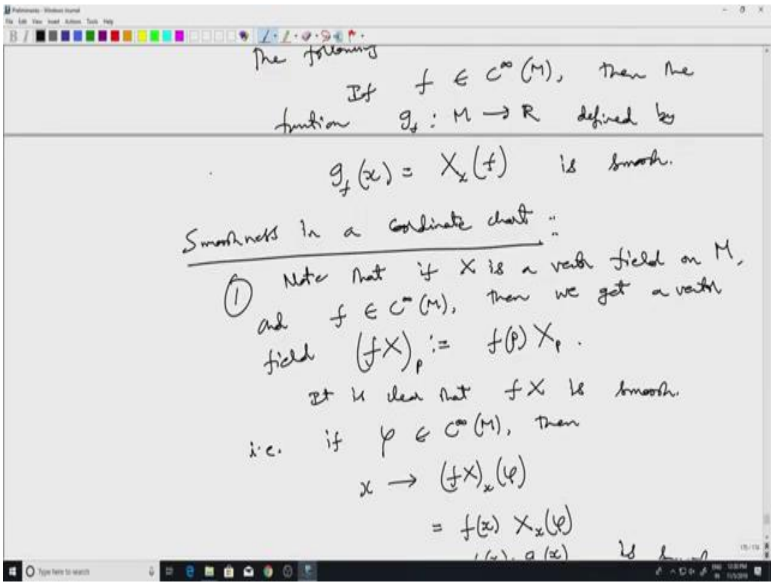
Hello and welcome to the 29th lecture in our series. So, last time I just started discussion of vector fields and I wanted to state the smoothness of a vector field in terms of what in terms of coordinate charts.

(Refer Slide Time: 00:48)









So, the proposition was this let us quickly see what we had the proposition what that if I take a local chart then in that local chart I have a natural basis for the tangent space at each point and therefore, the vector field for that matter any individual vector can be expressed as a linear combination of this basis. So in particular express the vector field like this at each q I have this. So, I get a n functions even up to a n on the chart U and the claim is that the smoothness of the vector field is equivalent to smoothness of these functions, that is it.

So, let us see why that is the case. And this is a somewhat this way of thinking about of a vector field less more concrete, because after all what we are saying is just that we are starting with some standard vector fields del by del xi and multiplying them by smooth functions and adding them up. So, on the other hand it is always preferable to have a coordinate free definition of smoothness, so we started with a coordinate free definition then we would like to say that is equivalent to this.

Now, let us see, so I with an if and only if statement suppose let us start with the, suppose that X is smooth in a in the original sense that I define which is a coordinate free sense. So, we know that, now let U fi be a chart, so I know that I can restrict X to U and get a vector field smooth vector field on U. So, now I want to conclude that these functions are smooth. Well, what I do is it is let this be a chart and I will define some functions let f equals pi i composed with fi.

So, here as usual U and there is a fi plus U1. Now, this pi i are just the projection maps on Rn. Let f equal to f pi i composed with fi, so here i is between any number between 1 and n, let f equals this where pi i is the usual projection map on Rn to the ith coordinate, pi i of x1 xn is just xi. So, I just look at in effect all I am doing is, so this is to R this map is I am looking at the ith this fi has n components fi 1, fi 2, fi n, I am just looking at the ith component of fi, that is what this composition means.

So, now let us see what let us act X on this function f and see what we get. X composed with rather X at the point q acting on the function f, let us see what this is? This is our what we called gf q earlier when I wanted to talk about smoothness of a vector field, so this by definition in local in this basis that I have, so I will use this ai q del by del xi at q acting on f.

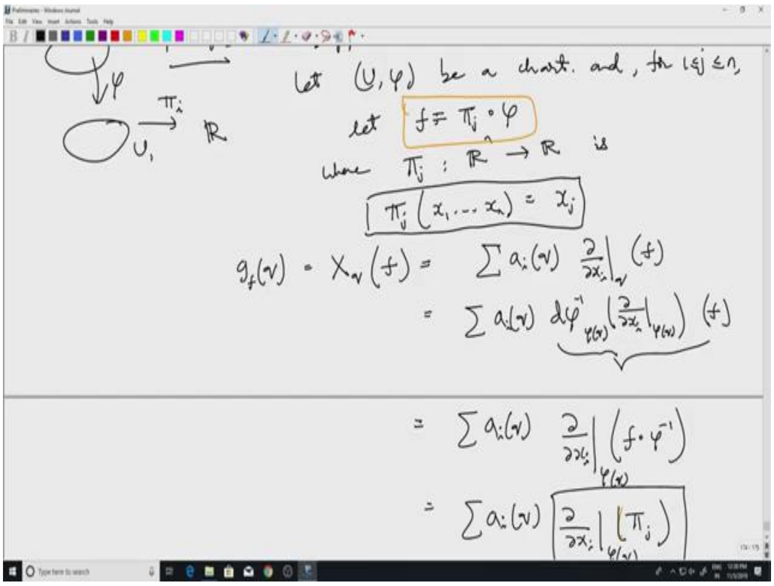
Now, this thing here del by del xi q remember that del by del xi at q was actually given by this expression here, so this is actually d fi inverse at fi of q and then del by del xi the usual del by del xi at fi of q again acting on f. This looks rather complicated but actually it is just going by definitions it works out to be something simple.

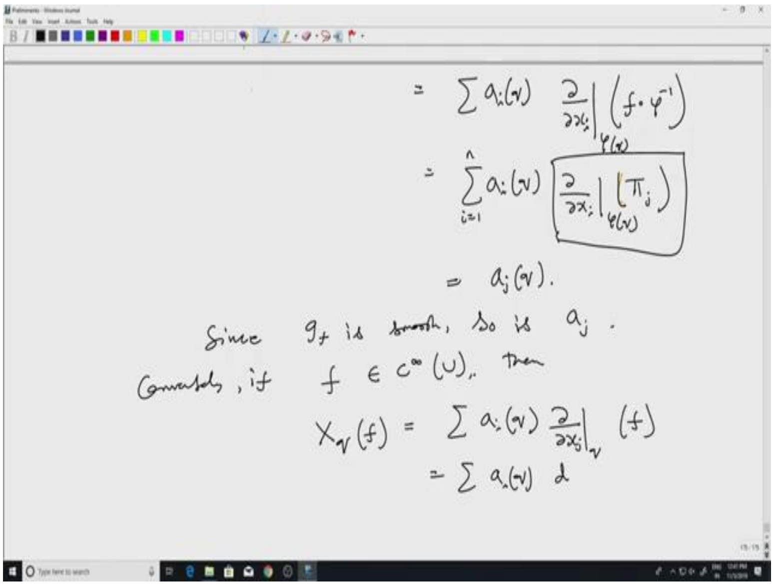
So, well this is a tangent vector and this is a the differential of a smooth map and we know the way we define the differential of a smooth map, this is this expression here is by definition del by del xi of at fi of q acting on essentially all I have to do is, I have to compose f composed with fi inverse, that I just by that definition of the differential d fi inverse.

Now, I will use the fact that I have taken a specific f, f is equal to pi i of fi, so this is del by del xi of fi of q, f composed with so the point is that f is this it is pi i composed with fi, but there is a fi inverse here, so that gives me fi composed with fi inverses identity, so I am just left with pi i. So, this would be pi i let me write it a bit clear, so this is pi i.

And we know that now this is just the this expression here, we can forget about the manifold actually, we are just in Euclidean space and all we are doing is we are taking the ith partial derivative.

(Refer Slide Time: 10:05)



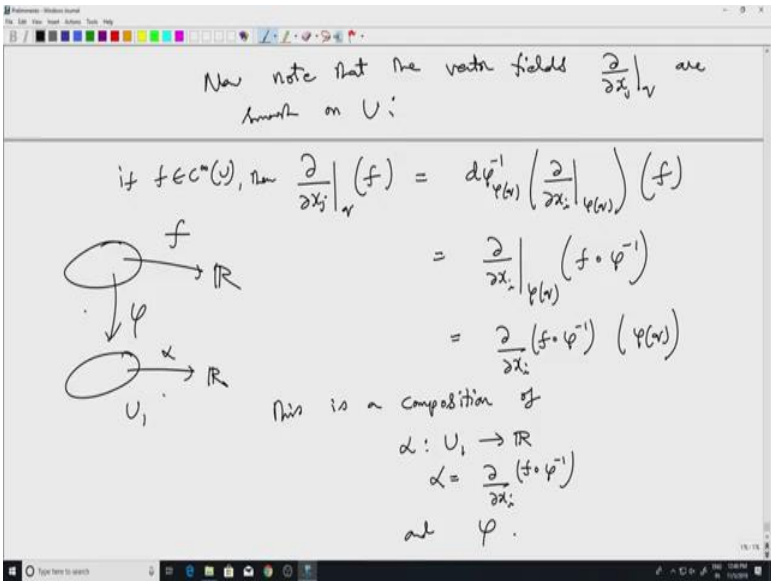


I should be bit careful, let just then index to a bit careful about the index, so let me use the index j here rather than i, so pi j and this is xj and here as well I would need to be because I am using i for the summation index. So, let us take a j and then here to I would then have pi j. So, what we are doing is we are taking the ith derivative of the jth projection map.

So, and we know that and jth projection map, so here also I should put the j, pi j, so if I take the ith derivative of this map here the only time it will be 0 unless i equals j essentially. So, that will be, so the only term which will survive those were here the summation is i from 1 to n, only when i equals j, I will get 1 otherwise it is 0. So, I am left with aj q. So, in short what this calculation shows is that the function aj q is exactly qf q where f is if we take f to be of that special thing or special form.

So and we know that gf q is smooth, since gf is smooth, so is aj for any j between 1 at n. So, that proves that smoothness of the vector field implies smoothness of this functions aj. And the converse is also quite clear, so conversely, if f is a C infinity function on U ,then Xq f is equal to ai q del by del xj at q of f again this is what this literally means is so del by del xj at q it is this thing here, so this expression here that I have here this one and so therefore this is d fi inverse, so let me, so in fact rather than stating it this was let us just state it in a cleaner way. So, at least a notational is simpler way a converse then we have this.

(Refer Slide Time: 14:16)



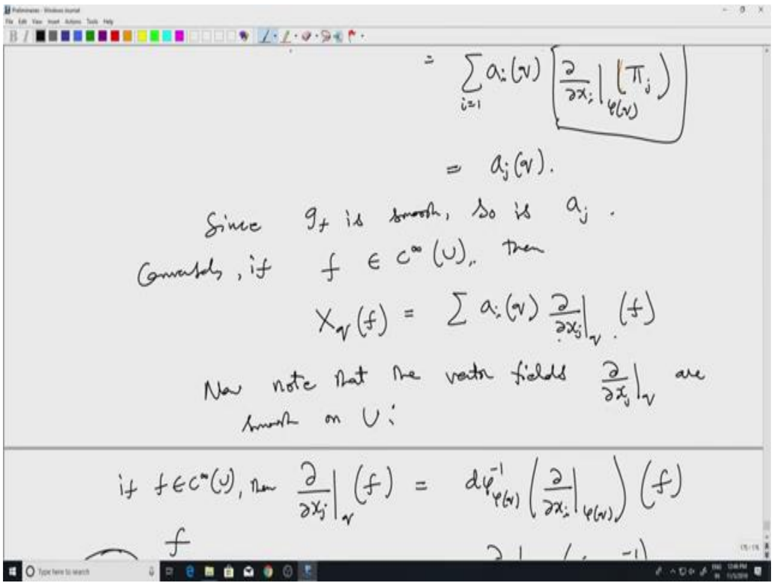
Now, note that the vector fields del by del xj at q are smooth on U. Vector fields smooth on U and to see this, this is quite clear but let us see this quickly, so after all what is this del by del j del by del xj, so I have to start with if f is C infinity function on U, then del by del xj at q acting on f is by definition equal to d fi inverse at fi of q of del by del xi at fi of q acting on f.

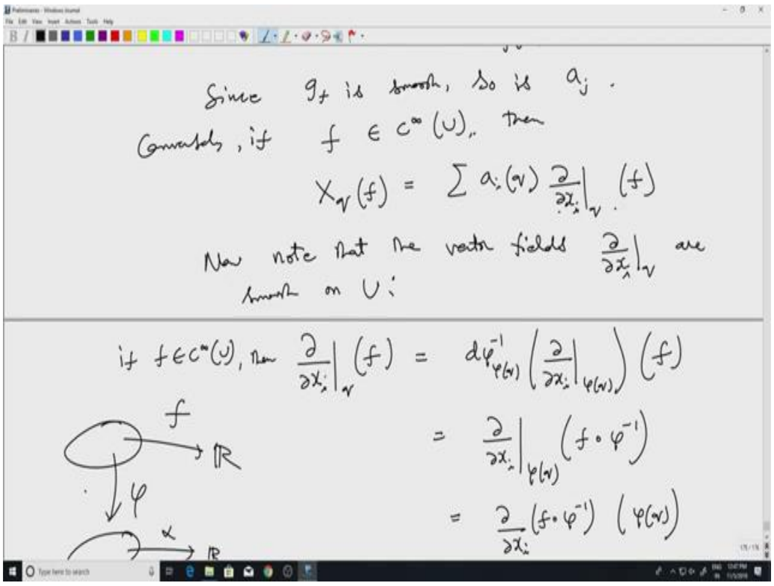
And by the same logic the as before there is a differential of a smooth map acting on a vector acting on a function and that is equal to del by del xi at fi of q, so you just do f composed with fi inverse. And this is nothing but del by del xi of f composed with fi inverse evaluated at fi of q. So, what this short calculation shows is that, if I take a C infinity function on U and act the coordinate vector field on that I get this expression final expression here.

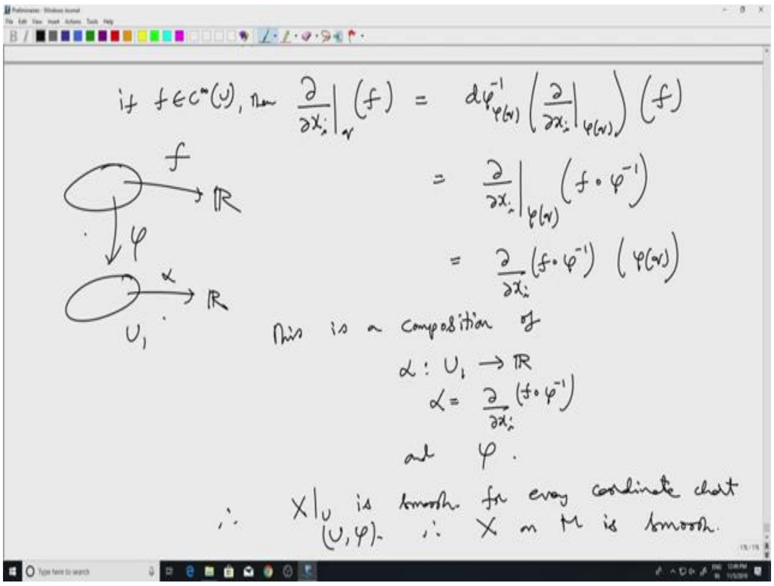
Now, I want to say that this is a smooth function of q, that is clear because well it is a composition of two smooth functions, this is a composition of after all fi is the function here to U1 and on U1 I have this function the partial derivative function, so let us call it alpha composition of alpha from U1 to R given by alpha equals del by del xi of fi inverse that should come second f composed with fi inverse f remember that f was here to R and this alpha is also to R.

This is a smooth function because f is a smooth map and fi is a different morphism, so f all partial derivatives are smooth, so alpha is smooth. And then of course the other map is fi itself and fi. So, first I come via fi here and then do alpha, so composition of this. So, therefore this brief calculation shows that this coordinate vector fields del by del xj at q which are actually this are smooth.

(Refer Slide Time: 19:08)







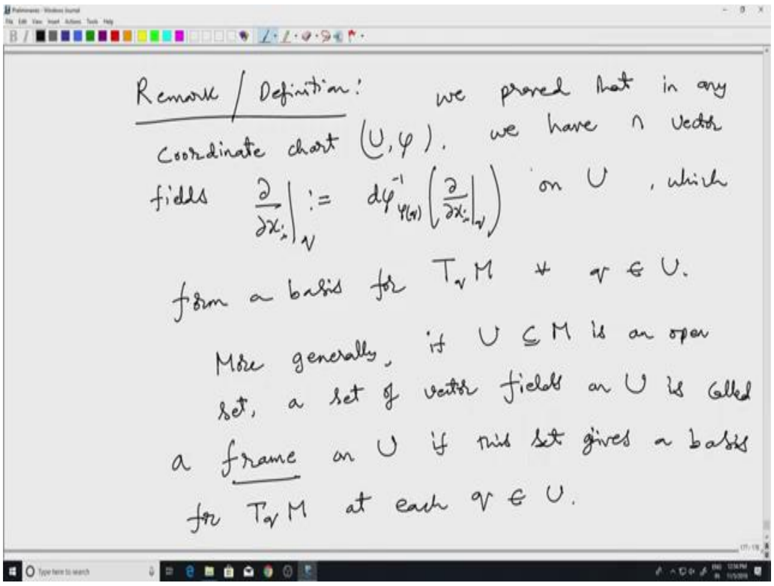
And now going back to the original vector field X, we have seen that this coordinate vector fields are smooth and I am multiplying by smooth function ai q, that is not j there it should be i and likewise here not that the vector fields del by del xj as for some reason the i's and j's got mixed up here, so the let us just stick to i so this is an i here, this is an i.

So, the coordinate vector fields are smooth and I am multiplying each coordinate vector field by a smooth function and therefore, the product is smooth and therefore the sum will be smooth as well. So, this what this we can conclude is that xq, so this where restriction of the vector field X, therefore X restricted to U is smooth.

For every coordinate chart we are assured that this functions ai that we get are smooth therefore, X restricted to every coordinate chart is smooth for every coordinate chart U fi well we have already seen that the smoothness of f smoothness of the vector field X on the whole manifold is equivalent to smoothness of X on any open set, here of course we are not I mean it is not necessary that we use every open set any covering by open sets is good enough and we have used coordinate charts.

So therefore, X on M is smooth. The point is smoothness is always whether it is a local issue whether we are talking about smoothness of maps or smoothness of vector fields and so on. So, that is more some technical discussion about smoothness.

(Refer Slide Time: 21:48)

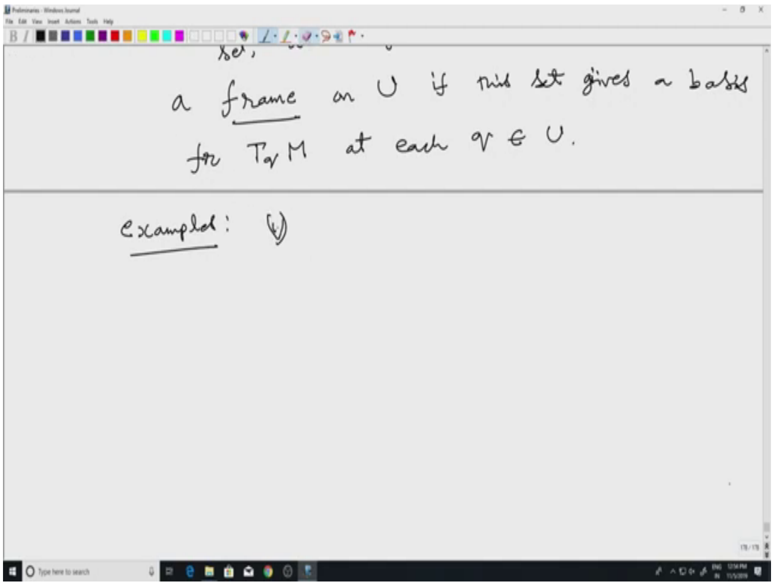


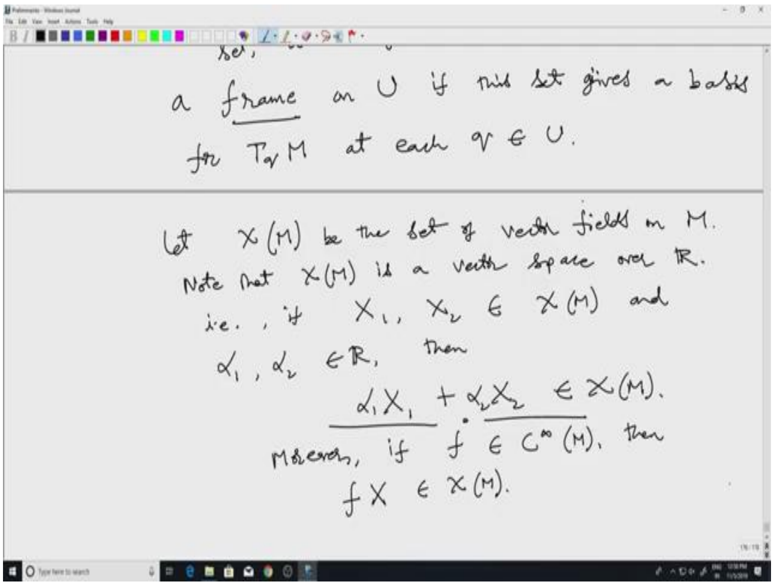
Now, let us I will make a remark slash definition in the course of the proof we saw that, we have proved that in any coordinate chart, U fi, we have n vector fields del by del xi at q which of course what it means I will put with in brackets, no may be not the bracket will make it to look a bit odd. So, let us just say as d fi inverse at fi of q del by del xi at q, so this is defined to be this.

We have n vector fields on U which form a basis for TqM for all q in U, such a set of vector fields as a name it is called a frame more generally frame, if U contained in M is an open set, a set of vector fields on U is called a frame on U if this set gives a basis for TqM at each q in U.

So, in general one cannot guaranty the existence of a frame if once start with an arbitrary open set for example if U equals M all of M we cannot say that there is set of vector fields on M which forms a basis for the tangent space at each point, what we can say and we have seen is that if you take U to be a coordinate chart, then there is frame on U. So, now let us move on to some examples for vector fields.

(Refer Slide Time: 26:15)





Well, we have already seen that, now it will be convenient to introduce a notation, so before I start writing down the examples let us, so let chi of M be the set of vector fields on M. Note that this is actually a vector space note that chi of M is a vector space over R. i.e, if X1 and X2 are two vector fields and alpha 1 alpha 2 are two real numbers, then alpha 1 X1 plus alpha 2 X2 is again vector field of in our definition the way we are defined a vector field is automatically smooth, smoothness is part of the definition, so now this statement here it is not saying much all it saying is that if I start with a vector field I can multiplied it by a real number and get another smooth vector field.

So, that is here alpha 1 X1 and alpha 2 X2 are smooth vector fields and if I have two smooth vector fields I can add them up and get another smooth vector field. So, it is an easy trivial exercise to check that some of two smooth vector fields is again smooth in constant times a smooth vector field is smooth in fact we have shown something stronger we have seen that if we have a not just a constant if we have a smooth function on M f times X is again a smooth vector field.

So, let us observe that in moreover if f is in C infinity M then f times X, which we define as which we define earlier is again a smooth vector field. Now, actually this, so the vector space part was already done before introducing this smooth this alpha 1 X1 plus alpha 2 X2 part, alpha 2 X2 being in chi of M ensures that chi of M is a vector space an additional comment is that if we have a C infinity function then f times X is also a vector field.

Well algebraically what this amounts to saying is that, the set of vector fields is not a just a vector space over R it is a module over the ring of C infinity functions, C infinity functions on M forms a it is a ring with usual addition and multiplication of functions and the fact that we can multiply a vector field by a function to get another vector fields is that a set of vector fields is a module over this ring.

But for us what is relevant is that first let us start by observing that this there always a trivial vector field and namely the 0 vector field, so at every point you just take the 0 vector, but in fact there is something much stronger what will I will briefly outline next time is the simple fact that if you have any tangent vector on a manifold there is a vector field on the manifold which gives the tangent vector at that point. So, let us stop here and we will resume next time. Thank you.