An Introduction to Smooth Manifolds
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Lecture 28
Vector Fields 1

Hello and welcome to the 28th lecture in the series. Today I will talk begin by talking a bit about smooth maps with reference to sub manifolds, and then we will move on to vector fields on manifolds.

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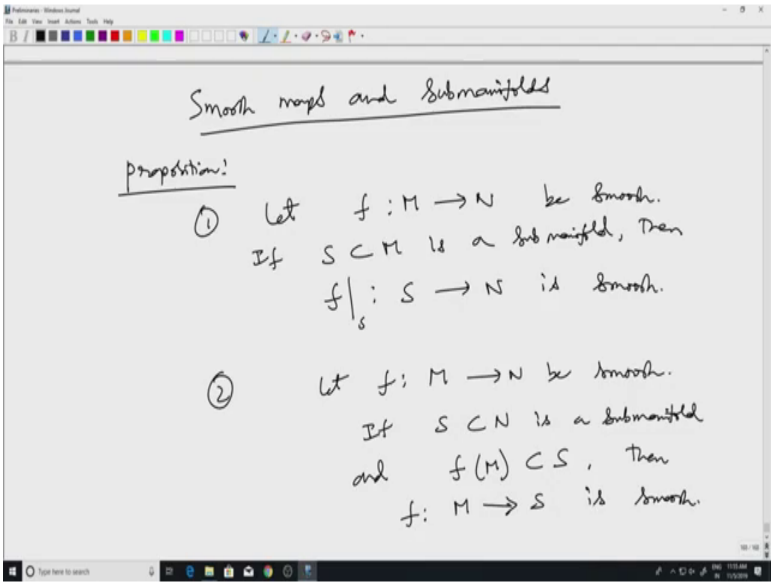

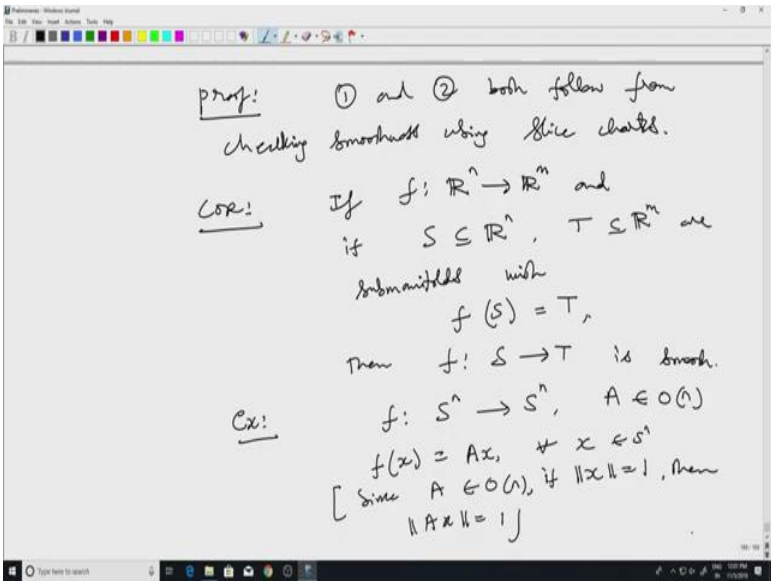

So, let us start with smooth maps. So, here is the proposition, so I have the first one is that let f from M to N be smooth as usual M and N are manifolds smooth manifolds and I have a smooth map, if S contained in M as a sub manifold, then the restriction of f to S from S to N is smooth. So, if I have a map smooth map on the big manifold and if I have sub manifold of the domain and if I restrict the smooth map to the sub manifold it will continue to be smooth.

Second thing is let f from again from M to N be smooth. This time if S I take as sub manifold in the image is a sub manifold and suppose I know that the image of f is actually contained in S, then I can regard f as a map from M to S, the claim is that this map is smooth and the proof of I would not go into the details the proof immediately, this follows from 1 and 2 both follow from checking the definition checking smoothness using slice charts.

The way we define smooth maps we have to choose some charts, and then see what the map looks like. So in both these cases use slice chart for the sub manifold and pretty much any chat for the other things and then you are done. Now, the use of this proposition is that in at least in one case it in one case it is easy it will be easy to check that the big map, map f on the whole manifold M and N will be smooth, then one does not have to bother using charts for S and so on, one can directly say that the restriction is smooth in the first case and in the second case as a map into S is smooth.

So, as a corollary, so the corollary is the following, if f from this is the case I had in mind we know if f is a map from Rn to Rm, we know that smoothness in our smoothness is equivalent to the usual definition of smoothness that we had and so one can basically check that f is smooth relatively easier in a easy way. And if S contained in Rn, T contained in Rm are sub manifolds with f of S going to T.

So, let me just change this write S in a slightly better way, S so suppose I have this condition that f this sub manifold is taken to this sub manifold, then I can regard f as a map from S to T and this map is smooth. So, in short if I have a map on the one can proceed to use this corollary, one can proceed in the sort of reverse direction, so suppose I have a map between two sub manifolds of Rn and Rm f from S to T, if I know that f extends to a smooth map between the corresponding Euclidean spaces, then I know that the original map I had is smooth.

So, the point is that I no longer have to vary about charts and so on just if it extends to whole of Euclidean space just work with the usual definition of smoothness in terms of partial derivatives and one is done. So, as an example let us take something that I had briefly touched upon in the beginning of this course. So, let us take a map from a let us take f I will start with the map from Sn to Sn the n dimensional sphere and this f of x equal to Ax.

So, f from S let us start A in the orthogonal group o n and so I will define f of x equals Ax, for all x in Sn. So, it is just the rotation matrix acting on elements of the sphere rotation matrix acting on elements of the sphere and we know that if norm x since A is in orthogonal matrix, if x norm x equal to 1, in other words if x belongs to the sphere, then if x if norm x equal to 1, then norm of Ax is also equal to 1. So, this map is a well define map from Sn to Sn.

And now suppose I want to claim that this map is smooth, the straight forward approach would toward start with some charts on Sn and then express the map as a map between two open sets in Rn and then check the smoothness, but that is bit inconvenient, so since we have to use charts and the formula for the charts is well in this case it is not that complicated but still one can avoid all that just by using the proposition the corollary here.

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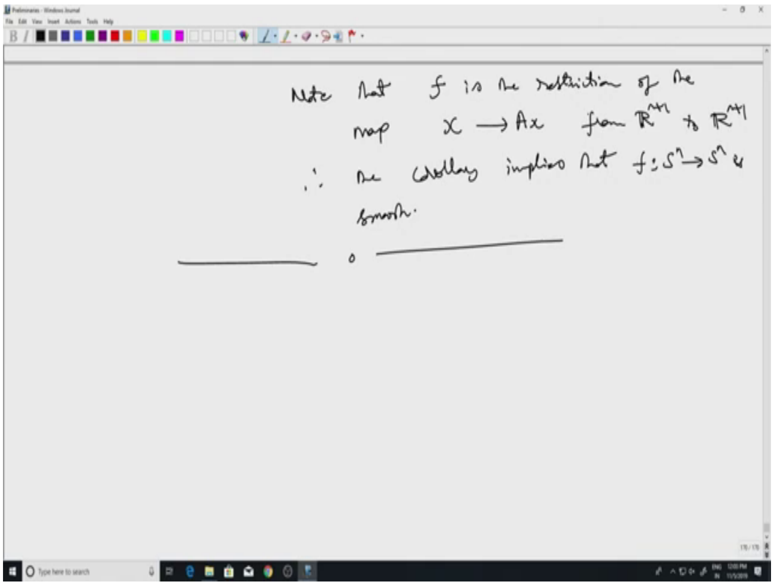

By the corollary no before the apply the corollary, let us not that note that f is the restriction of the map x going to Ax from Rn plus 1 to Rn plus 1. So, since well this map from Euclidean spaces is just a linear map and we know it smooth. Therefore, immediately the corollary tells us the corollary implies that f from Sn to Sn is smooth. So, we can avoid the use of charts all together. So, that conclude our brief discussion of smooth maps and the regular value theorem.

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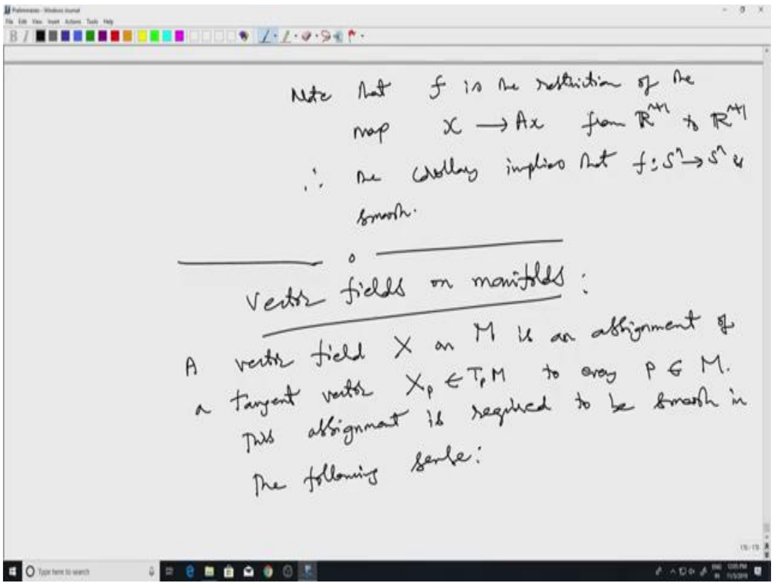

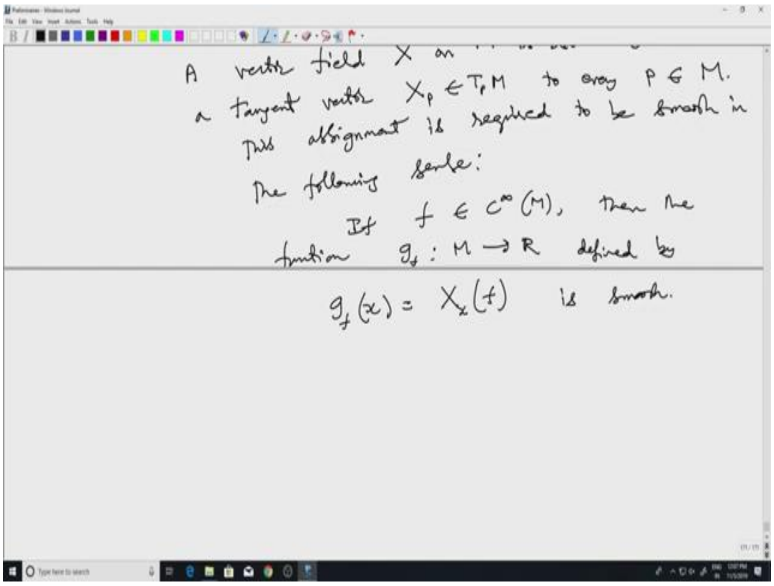

Now, let us move on to the next topic which is the motion of a vector fields, vector field on vector fields on manifolds. So, a vector field X on M is an assignment of a tangent vector Xp in TpM to every p in M. So, to every point in the manifold we are picking out a specific tangent vector X which I denote by X subscript p. And this assignment is required to be smooth in the following sense. First, I will give an abstract definition of a smooth vector field, then we will see what this means and local coordinates in coordinate chart.

Well following sense, so if f is a C infinity function, then the function g subscript f from M to R define by gf at a point x is I just I have a vector field, so at each point X I get a tangent vector at the point X which is a derivation. So, I can act this derivation on the function and I get a real number. So, the point is that if I start with vector field and if I am given any function I will get a new function from M to R. And we require that this function gf is smooth. So, this is this what we mean by smooth vector field. Now, so this has well every notion that we define can be we can always use as coordinate chart to see what exactly it means.

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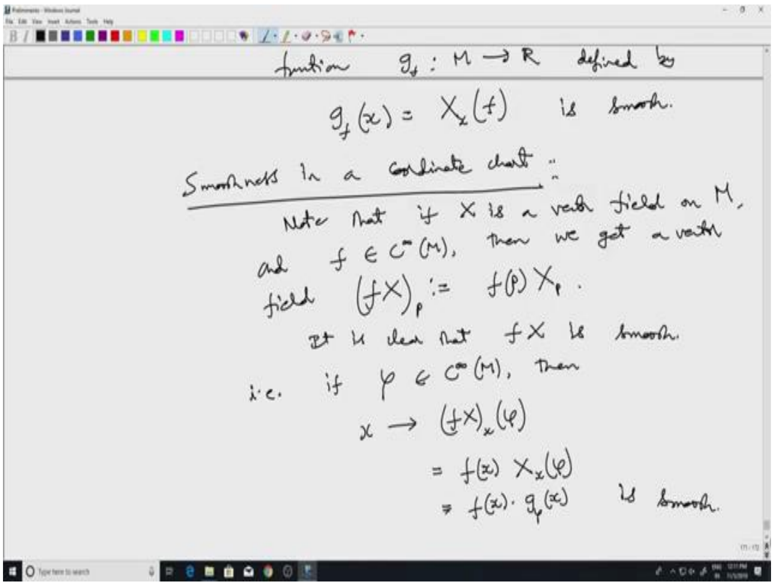

So, here for example let us see what this motion of smoothness means if I take a coordinate chart. So, I will give an equivalent definition of smoothness, smoothness in a coordinate chart. For to do this let us observe a couple of things, note that if X is a vector field on M and f is a C infinity function on M, then I get a then we get a vector field fX f times X and this is defined, so in order to say what a vector field is I have to specify what it is what tangent vector we get at every point.

So, let us take a point p and this is defined to be f of p which is just a real number times Xp which is the tangent vector given by the vector field X and the point p, then we get a vector of it. And it is clear that this new fX is also smooth, fX is smooth in the sense that we are defined earlier i.e if I take another C infinity function if fi belongs to C infinity M then this what I called this gf then the function gf just consist of taking the vector field at the point x and the acting it on the function f.

So, here let us take a point x this is send to the now the vector field is fX at the point x acting on fi which is f x which by definition is fx times X this. And this with our earlier notation this is fx times g fi of x. So, in other words this if I start with any fi the new function whose smoothness I am concerned about is just the product of this fixed function f and this function g fi which I know is smooth because x itself is smooth, f times g fi will be smooth this is smooth. So, in short I can just given a smooth function given a vector field and a smooth function I can multiply the vector field by the smooth function and get another vector field.

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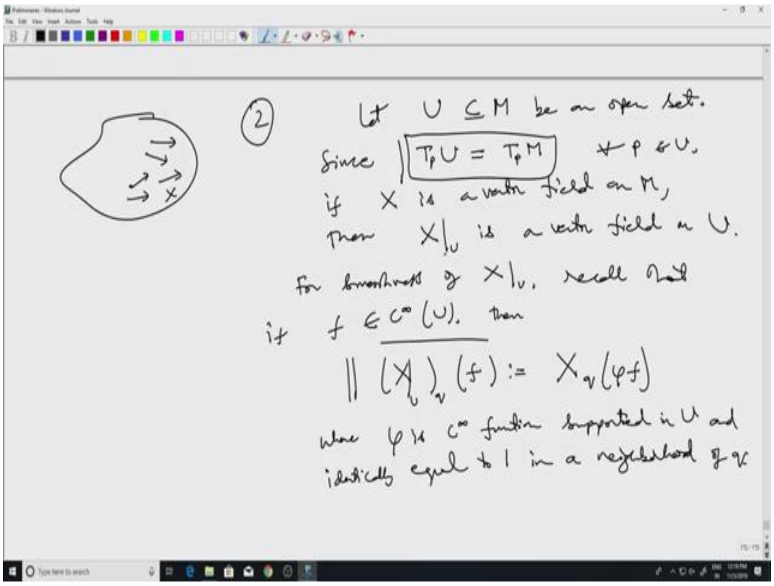

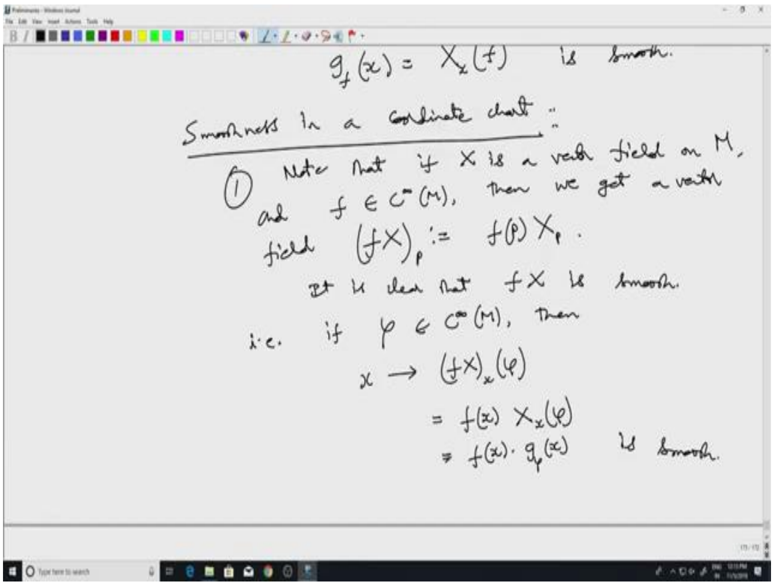

So, intuitively one thinks of it like this, so after all what is a vector field schematically at every point I have tangent vector and if this is my vector field X and if I have a function all I am doing is I am just taking these tangent vectors and multiplying by the constant at the point X I will multiple by the constant f of x. So, in this vector it will shrink or expand and the it might get reversed also depending on whether f of x is negative or positive, but essentially we stay along the same line at each point and but the direction or the magnitude of the vector can change depending on f.

So, that is the one observation which is useful second thing is that, so this is one thing, let U belong U subset of m be an open set, since we already seen that the tangent space at U a tangent space at a point p in U is actually we have identified with tangent space to M itself for all P in u. If X is a vector field on M, then X restricted to U is a vector field on the manifold U, we know that every open set in a manifold is itself a manifold, so all I am just I can just restrict it to look at the values of X for points in U and I get another vector field I get a vector field on U.

And as for smoothness I know that the original vector field on the all the M is smooth and if I want to say that this is smooth, it just depends on this identification that we have here. So for smoothness of X restricted to U recall that after all how did we identify the tangent to U with the tangent space to M, well suppose recall that X restricted U if have a C infinity function we called f as a C infinity function on U, then the action of X restricted to the action of Xq, X restricted to U at the point q maybe I should put the q outside acting on this function f the point is that a tangent vector to U will act on C infinity functions which are defined on U not necessarily on all of M.

But we saw that we can always extend any such function on U close to a point extend to a C infinity function on M, such that the extension agrees with the original one in a neighborhood of any given point q, that was our process, so this was define to be Xq of fi f where fi is a C infinity function supported in U and identically equal to 1 in a neighborhood of we are being using this repeatedly neighborhood of the point q.

This will immediately imply that a well, what I written here is just the definition the identification that we have here, so I just spelt out this identification explicitly in this and if one wants to actually check smoothness, one goes by the identification so I would not go into that it follows immediately from this that if f is so in fact if f is a C infinity this identification also shows immediately that if f is a C infinity function on U then this new map q going to this thing here what I have here is actually a smooth function U because what I this is a the right hand side in fact will be a smooth function on all of M, so when I restricted to U I get a smooth function on U. So, it is clear that I get something smooth.

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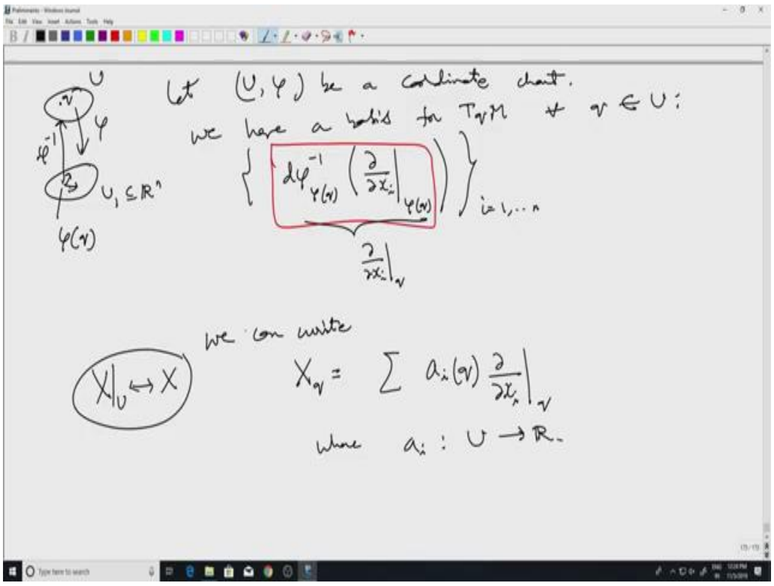

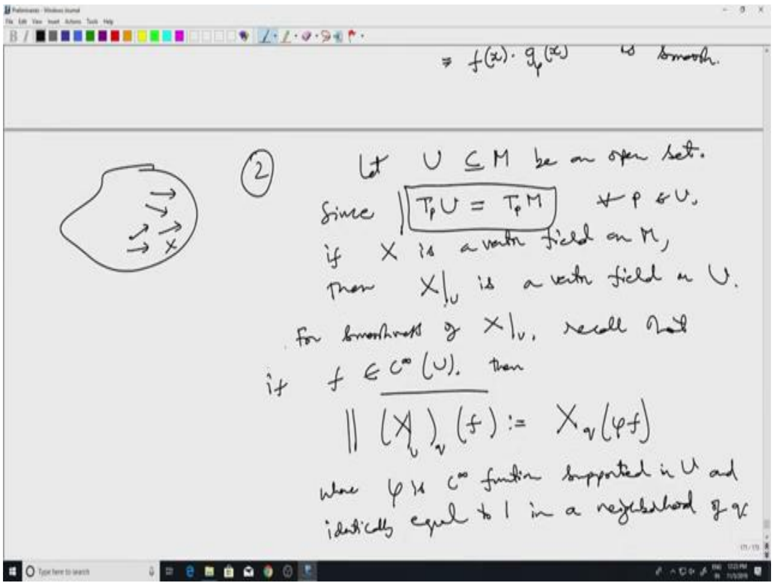

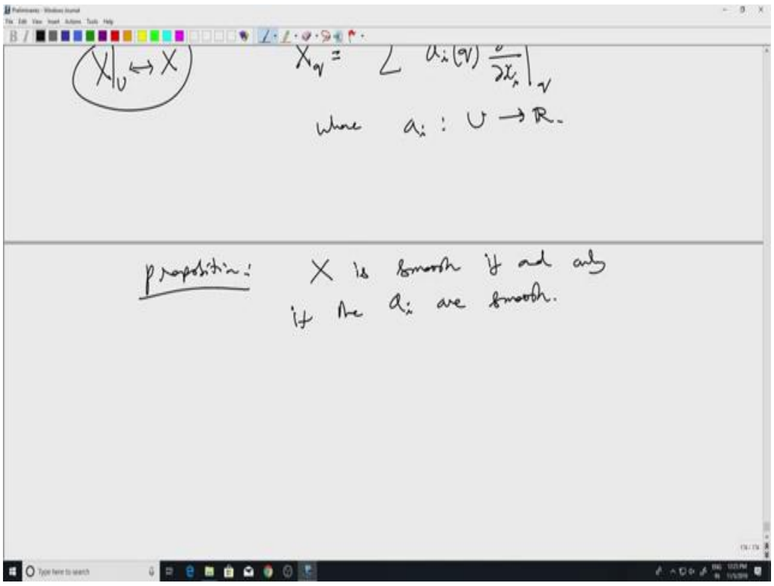

Now, given this let us fix a chord now next thing is let us, let U fi be a coordinate chart. We have a basis we have seen this before we have a basis for TqM for all q in U. So ,namely we know that by definition this is fi is a diffeomorphism on to U1 and Rn and on Rn I have natural basis for the tangent space at every point namely the partial derivatives along the n directions evaluated at that point.

So, and since fi is a diffeomorphism the derivative of fi is a linear isomorphism. So, I can just use the derivative to transfer this basis to this, actually I will have to go in the opposite direction fi inverse will be from here to here, so at each point I have a basis use the derivative of fi to push this basis to basis of the corresponding point to the tangent space.

So, what I do is d fi inverse so if I start with a point q suppose I want to basis for the tangent space at q I go to fi of q here this is fi of q use the basis here and use the derivative, so d fi inverse at the point fi of q of the basis in Euclidean space which is del by del xi evaluated at fi of q, so this derivation I just push it back to this, so look at this i equals 1 to n. And this is a somewhat combustion notation the way I have written it normally this is denoted by this del by del xi at q.

There is a chance of confusion with this notation, but it is once one has to keep in mind that when you see this del by del xi evaluated at q what one literally means is what I have here this thing. So, as long as that is kept in mind it can freely use del by del xi at q. So, we have this basis and now we also had a vector field I started with a vector field on the all of M, the second observation was that I can restrict it to U and get a vector field on U.

Well, so let us express we can write X at q so this is supposed to be so this is the restricted vector field I no longer I will drop the notation X restricted to U instead of this I will just use X itself. So, with that change so X at q will be I can it is a tangent vector by definition it is a tangent vector at q, so it is an element of Tq U or TqM. So and I have already have a basis so I can write it as sigma ai q and using this convention so I will write it as del by del xi at q.

Here, where ai are now functions from U to R. So, the claim is that which I prove next time. So, what I want to claim is the smoothness of the vector field proposition X is smooth if and only if the ai are smooth . So, we will stop here I will resume with the quick proof of this proposition next time. Thank you.