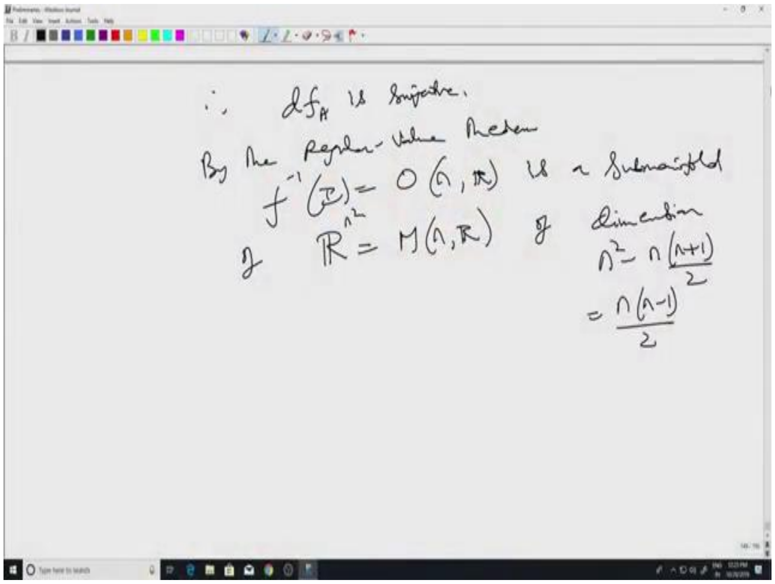
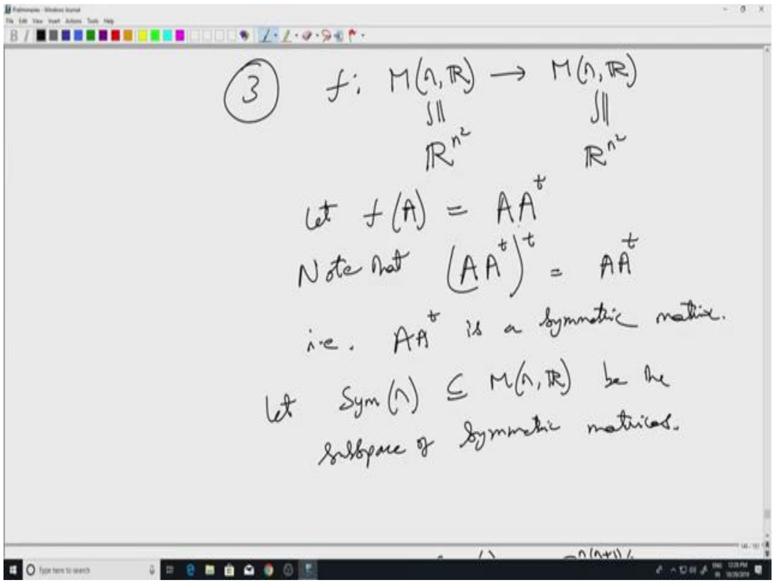
An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 25
Regular Value Theorem

(Refer Slide Time: 0:46)



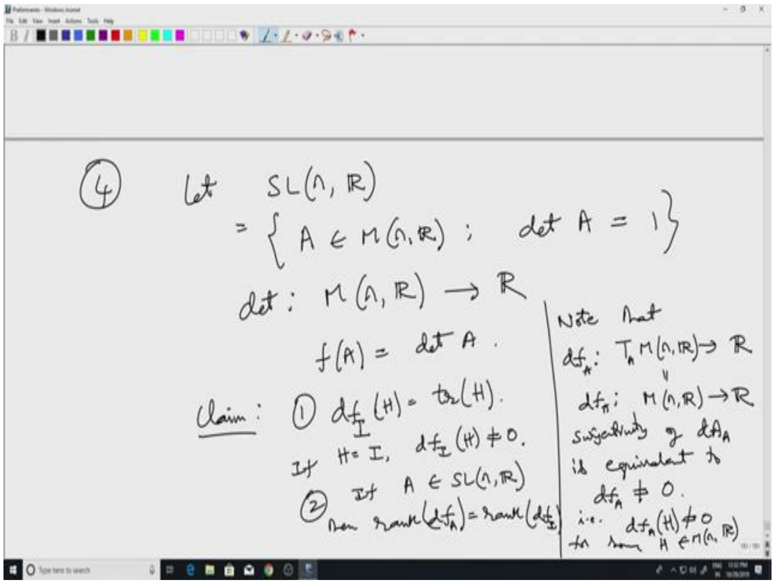
Hello and welcome to the 25th lecture. Last time we had started listing some examples arising from the regular value theorem. So, I will continue with that and then we will talk a bit about tangent spaces to these, the sub manifolds that we obtain. So, I had ended with the example of the set of orthogonal matrices, which is called the Orthogonal Group ONR. And this turned out to be a sub manifold of MNR which is essentially Rn square and it is dimensionless n into n minus 1 by 2. This was where we had stopped last time.

(Refer Slide Time: 1:30)



Now, there is one more interesting example that I would like to discuss in this which arises from matrices. So, let us see. So, this was example number 3.

(Refer Slide Time: 1:36)



Let SL N, R be the set of n cross n matrices with determinant of A equal 1. So, let us prove that this is also a manifold. In this case the function, so we would like to apply the regular value theorem and function for which this is the level set is clear from their definition, namely it is a determinant.

So, let us consider the determinant function from MNR to R. So, let us call, let us give it a name F of A equals det A. Claim, I will make two claims. First thing is that in order to, so one would like to check the rank of the derivative of F at any point A in SL N, R. Now, I claim that, so I one would like to check that, so the derivative has full rank.

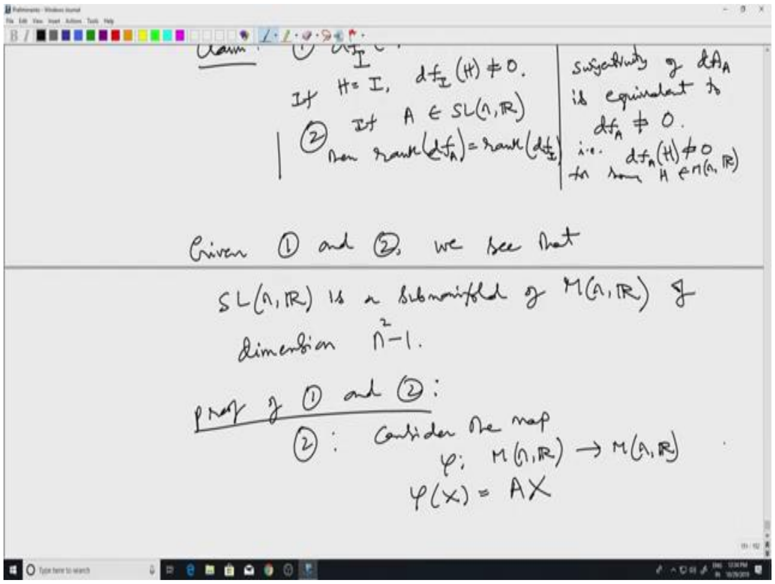
Well, in this case, the target is R, so we just want to ensure that the derivative, so here I will remark that note that DF at any point A would be a map from TAMNR to the tangent space of R at F of A but we know that this is just R and likewise, since MNR is a vector space, we know that this is MNR itself. So, DFA is just the map from MNR to R and to say that this, so we would like to know whether this is surjective or not as a linear transformation from this vector space to this vector space.

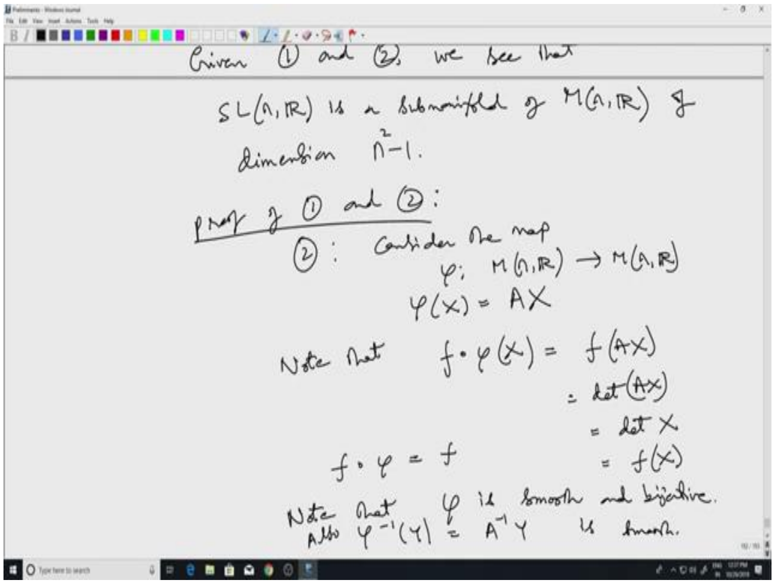
Well, since the target is R, surjectivity amounts to saying that this is the non, this is a non-zero linear transformation, I mean linear transformation, which is not identically 0. So, surjectivity of DFA equivalent to DFA not equal to 0, so in the, here not equal to in the sense of linear as a linear map. Well, yeah right. So, that is all that one has to check. Now, let us see, so i.e. and this amounts to i.e DFA of H is not equal to 0. Well now 0 is a real number for some H in M n R.

So, I just have to find but I need a formula for DFA. Now, first claim is that DFA, DFI I will take A equals I, notice that the identity matrix is an element of SL N, R trivially and DFI is a map from DFI of H turns out to be just trace of H. Now, in order to show that the derivative identity has full rank, as I have written in the right column here, it amounts to saying that I can find a single H for which this is not 0.

And there are lots of choices for which, so you just start to find some matrix whose trace is not 0 and you will be done. So, if, for example, you can take H to be identity itself, H equal to I DFI of H is not 0. So, therefore, the derivative identity has full rank but what about other points? The second claim is that, if I take any other point in SL N, R then rank of the derivative of F at that point is the same as the rank of DFI.

(Refer Slide Time: 7:29)





If we have these two claims, then we will be done because given 1 and 2, we see that SL N, R is a sub manifold of MNR of dimension. Remember that the dimension given by the regular value theorem is just the dimension of their domain minus the target, so dimension of domain is dimension of MNR which is n square and target is 1 dimension. So, we mediate, this immediately follows.

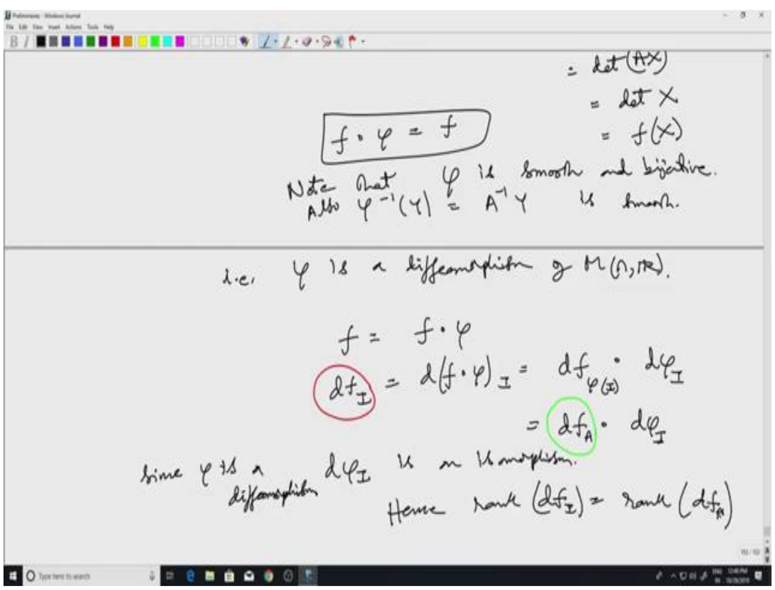
Now, let us see why 1 and 2 are true. So, proof of 1 and 2. So, let us start with 2, the easier one. Well, so I have to start with some A. Consider the map phi from MNR to MNR given by phi of X equals AX. So, this is the same A that I am starting with here. And I just multiply X with A.

Ok, so now if we note that F of phi of X is F of AX, determinant of AX. Just determinant of X, since determinant of A is 1 which is the same as well, it is F of X. So, this tells us that F composed with phi is just F itself.

If I take derivatives, before I take derivatives, let us observe that, note that, phi is smooth and bijective. Well, it is smooth simply because all it is that is happening is that I am just taking multiplying X with a fixed matrix A. So, if I look at the entries of phi, the n square entry is component functions of phi, they are just linear combinations of entries of X. So, phi is trivially smooth and its bijective because I can explicitly write down the inverse.

In fact, the inverse is phi inverse of Y is just A inverse of Y, maybe I should write this more clearer. So, phi inverse of Y is A inverse Y. The inverse of the matrix multiply by this. So, we have an inverse and again, for the same reason that phi was smooth, phi inverse is smooth as well. It is just multiplication by a fixed matrix and the component functions of phi inverse are just linear functions of the input. Also, phi inverse is smooth.

(Refer Slide Time: 12:16)

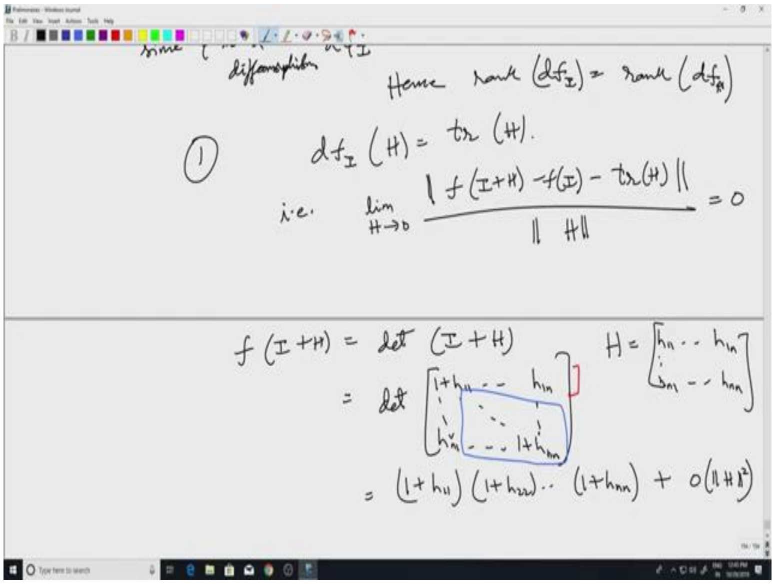


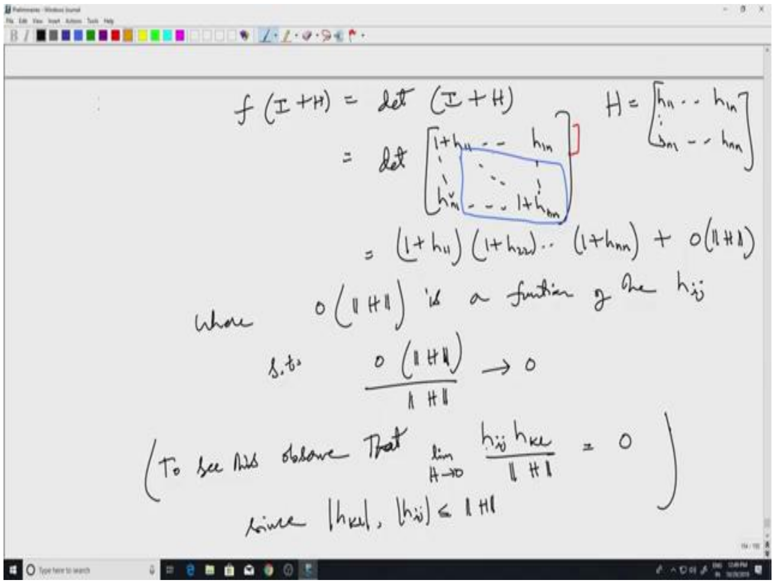
So, in short, phi is a diffeomorphism of MNR. So, now let us go back to this equation here and take derivatives on both sides. Now, that I know phi is smooth and so on, I take the derivative on both sides and use the chain rule. So, I will get so I will take the derivative of the right side at the point A, so rather at identity at I. So, F equals this so DF at identity is D of this at identity by chain rule this is DF at phi of identity composed with d phi at identity.

Well, phi of identity by definition of this map phi, phi of identity is just A, so this is DFA composed with d phi of identity. Now, I do not need to compute d phi at identity. So, all I need to know is that, well, I already know that phi is a diffeomorphism so that its derivative is an isomorphism at every point in particular DF at DFI is an isomorphism, since phi is a diffeomorphism, DFI is a isomorphism and then I am done because my object of interest is the rank how to relate the rank of derivative of F at identity, that derivative of F at some other point A.

And these two are related by just this composing DFA with an isomorphism. When I do that, I do not change the rank. So, the rank of this side DFA composed with this is the same as rank of DFA. And this equation tells us that, that is the same as the rank of DFI. So, hence, rank of DFI equal to rank of DFA.

(Refer Slide Time: 15:29)





And as for the other thing, which is DFI, derivative of F at I of the determinant function is just given by trace, trace of H. This is the other claim. Now, there are a couple of ways of doing this. One is to use the exponential of matrices. Here, I will take a more straightforward approach. So, we would like to claim so i.e. limit H going to 0.

Let us go back to the definition of derivative of H going to 0 of F of I plus H minus F of I minus DFI of H, which is trace H divided by nom H equal to 0. So, this is what the definition tells us. Now, let us calculate, let us estimate F of I plus H, the second term here is F of I, which is just 1 F of I plus H is determinant and of I plus H.

If I write H as, as usual as h11 h1n hn1 hnn, then this would be determinant of I just had 1 on the diagonal, 1 plus hnn, the other entries are the same h1n hn1. Now, the point is I would like to, so when I expand the determinant, I the plan is to expand the determinant along the first row. So, let us take this first row and expand it along the, calculate the determinant by expansion along this. So, I will get, for instance, the first term will be 1 plus h11 times the minor corresponding to that would be this matrix.

I want to isolate those terms which are linear in the Hij. So, I want to club all the terms in the expansion, which have, which are a product of more than one, product of at least two of the H edges in one place and those which have just which are linear in its Hij in one place. So, it can, it is easy to check that the linearity part will come it has to come from this term.

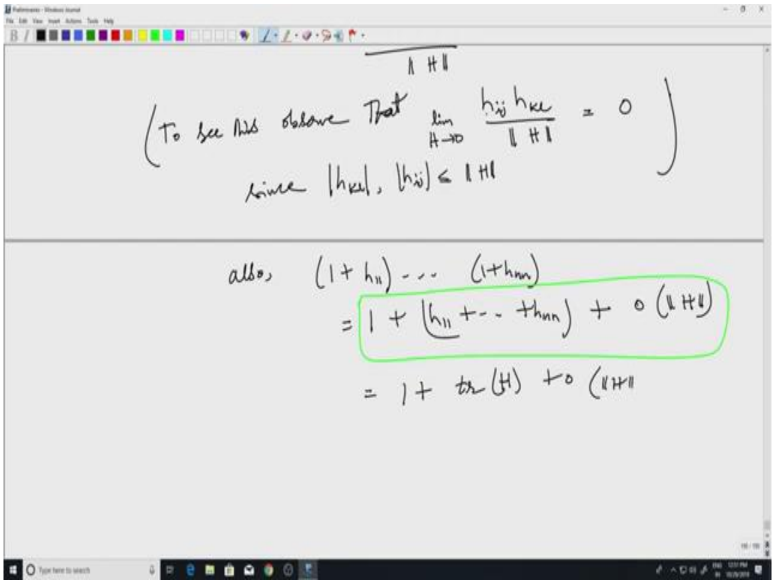
All the other terms in the first minor, as well as all the other terms in the first minor as well as all the other things when in the first row, everything else will contribute, will have a, will be a product of two of the Hij. I will just write that as small o of non H square where small o of non H square is a term, not a term is a function of the function of the Hij such that this function when I divide by nom H square actually not nom H square, nom H since I have used the small o notation.

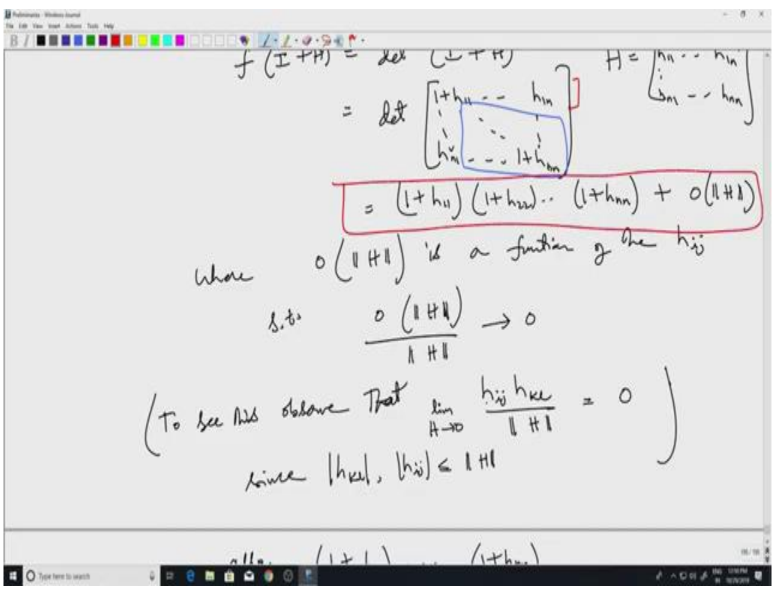
And likewise here the small o notation means that this divided by nom H goes to 0. So, essentially I have clubbed, so these are things which involve as I said, product of more than note that, so let me note that if I have two terms, Hij Hkl then any such thing will go to 0 as H goes to 0, to see this observe that this equal to 0.

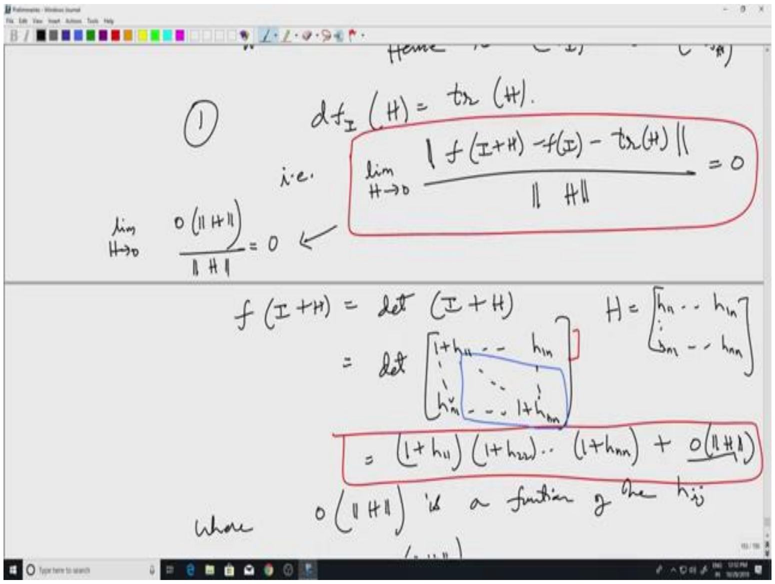
So, the moment you have a product of two or more of the Hij I have this property and well, this is 0 is quite clear because Hij will be less than or equal to nom H which are both Hij and Hkl are less than or equal to nom H. So, the product when I take absolute value in fact I can put absolute values here. So, product hkl, comma this both of them are less than or equal to nom H, so the product will be less than or equal to nom H square.

So, this entire thing less than or equal to nom H square divided by nom H and that will go to 0. Observe that since this happens, Ok so the every the only term in the expansion of that of I plus H, which will not be a product of more than one Hij, arises from this the first time here. But even that there is only one term which has which is linear in the Hij.

(Refer Slide Time: 23:42)





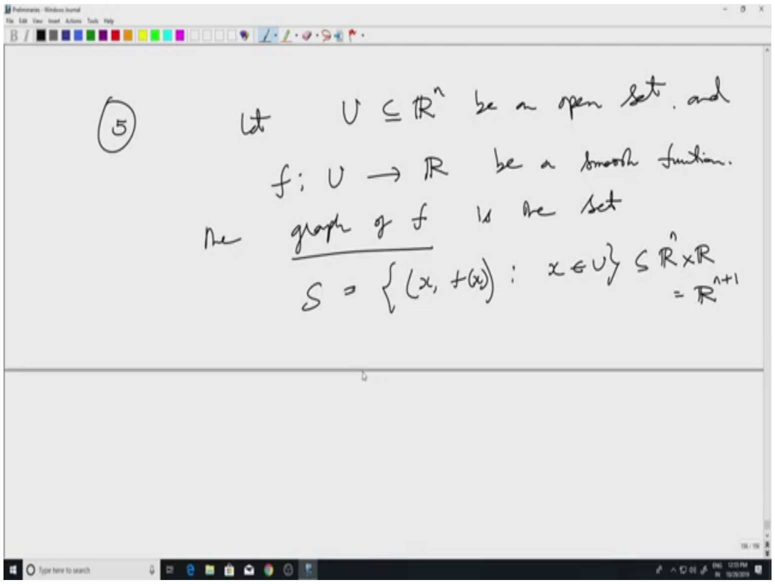


Also, there is a term 1 coming from the product of all the ones. And then there is this term Hnn. And the remaining terms again involve product of more one Hij. So, I will again put small o of nom H. So, this when I plug in these two things, one is this, this expression here, the second one is this and let us plug both of these back into this main expression here. The main expression is this, so I plug this into that. So, F of I plus H first I write like this and put it here.

The one note is that and then I use this one. So, I get a one and then this is exactly trace H. So, what I have here is one plus trace H plus small o of nom H. And so it essentially an F of I is also 1, so that one cancels with the one coming from F of I plus H and the trace H cancels with this trace H here. So, I am left with order of those who have one small o of nom H is coming from here and another one coming from this.

So, I can combine the two and get another small o of nom H. Another function, so the limit of this will boil down to so this will boil down to limit H going to 0 of small o of nom H. By the way, here, I need not have used the nom sign since the target is real numbers, just absolute value would have been fine but that is anyway, that is not so important. So, limit of this divided by nom H means is what I am left with, and by definition of small o of nom H this is 0. So, that proves everything that the derivative identity precisely trace.

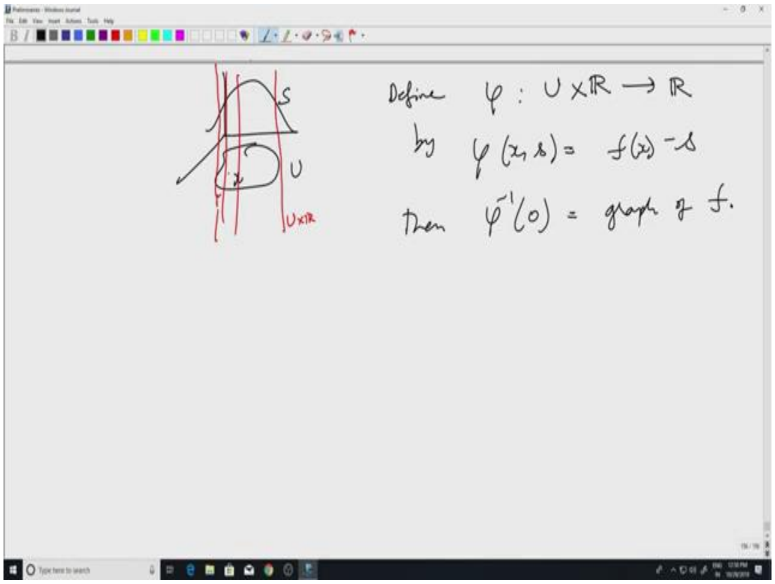
(Refer Slide Time: 27:14)



So, now, the next example I would like to discuss is something quite familiar, which is the graph of a smooth function. Let u contain Rn be an open set and F from u to R be a smooth function. So, the graph of F is the set S equal to set of all X, comma F of X.

And this X and u, so this I am adding one extra component X was already in Rn, X was, X is in u which is in Rn and I am putting one F of X. F of x is this the real number. So, this is a subset of Rn cross R which is Rn plus 1. So, the graph is a subset of Euclidean space of one higher dimension and this is precisely what we deal with in one variable calculus of for instance, where n equals 1.

(Refer Slide Time: 29:06)



Now, so pictorially u is some open set and the graph this is S, so at each point X essentially one can think of it as the second coordinate as height over X. So, at the point X you go height F of X it can go above if F of X is positive or below if F of X is negative. Now, I would like to say that the graph of a smooth function is always a sub manifold. You do not need any condition on the derivative of F. That is F can be a constant, for instance.

So, and in fact, one can directly show that this is a, one can explicitly construct a chart, a single chart on S. But let us use the regular value theorem and see what that gives us, phi of X, S is F of X minus S. And this is a function defined on u cross R, which is an open subset of Rn plus 1. So, this is in this picture it is a sort of cylinder u cross R will be a cylinder, solid cylinder over u, u cross R, so this.

And o on that open set, I am defining this function. This is smooth then before smoothness note that phi inverse 0 will be set of all points X, S such that F of X equals to S. So, in other words the second coordinate is F of X. But that is precisely graph of, the graph of f. So, let me stop at this stage. And then in the next class, that is compute the derivative of this phi and show that the conditions for the regular value theorem are satisfied. So, we will stop here. Thank you.