**An Introduction to Smooth Manifolds**
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**Lecture 21**
**Inverse Function Theorem for Manifolds**

So, hello and welcome to the 21st lecture in the series. Today I am going to talk, well, we will continue where we had stopped last time. And now that we have, the notion of a smooth map, the derivative of a smooth map and we know that the dimension of the tangent spaces what we expected to be.

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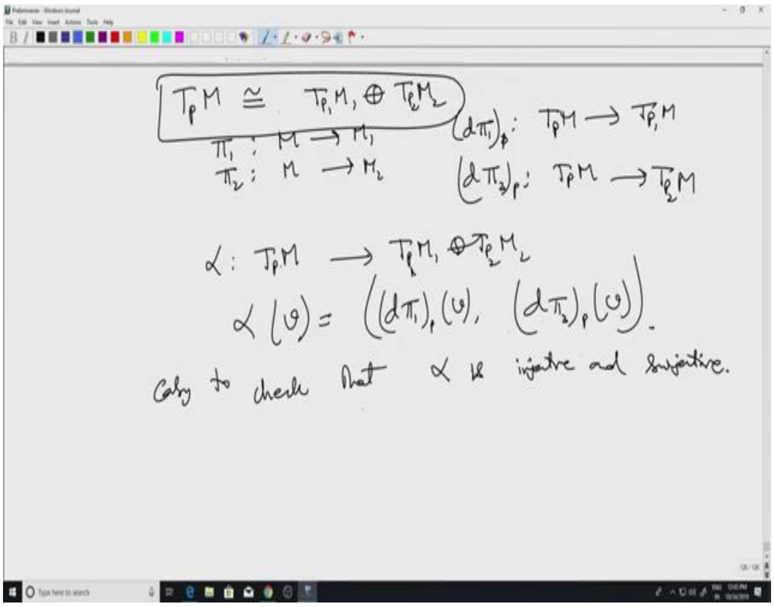


We can carry over several of the most important results in the Euclidean case, to the case of the manifolds.

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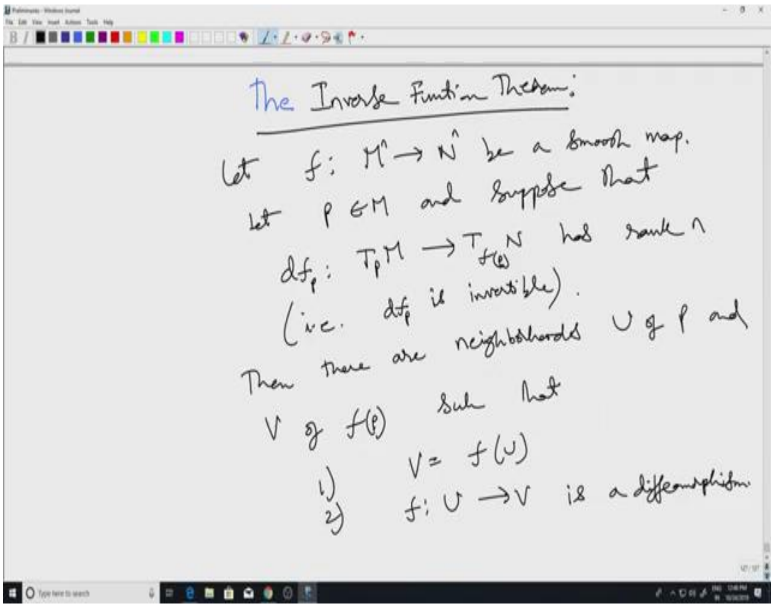


So, in particular case let us state the, the inverse function theorem for manifolds. So, here the setting is, let f from Mn to Nn. So, the point is that both M the domain manifold and the target manifold should have the same dimension, be a smooth map. If, let P belong to M and suppose that dfp from TpM, Tf of p N has rank n. In other words, dfp is invertible, it is a bijective linear map that is the, as rank n.

Then, there are neighbourhoods U of P and V of f of p. Such that, V is the image of u, and f from u to v is a diffeomorphism. In other words, f from u to v is a bijective and the inverse of f is smooth as well. This is the, exactly the direct generalization of the classical inverse function theorem.

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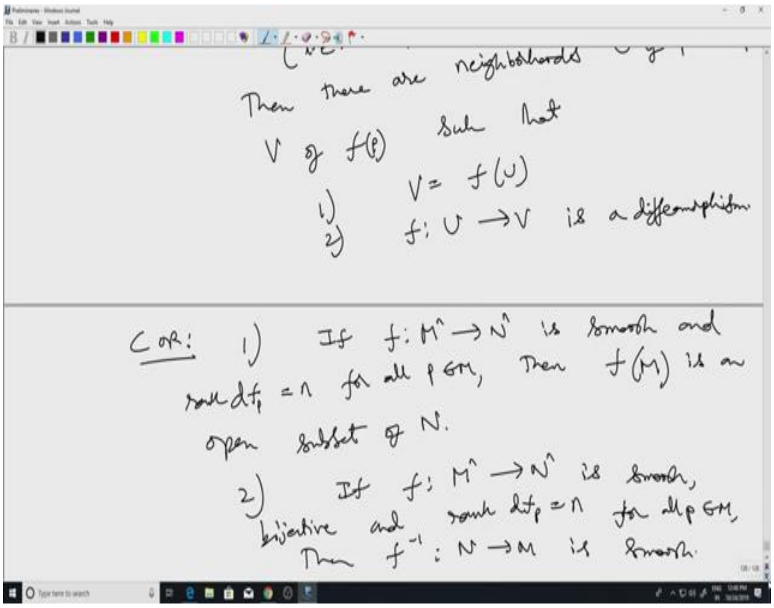


And similar, whatever we had corollaries we had, in that setting they continue to hold here. So, corollary, if and dfP, rank of dfp equal to n for all p and m, then f of M is an open subset of N. Argument for this is exactly the same as the, what we had for maps between open sets and Euclidean spaces.

And the second thing we had in that setting was that if f from Mn to Nn is smooth, bijective and rank dfp equal to n for all p and m, then f inverse from N to M is smooth. So, again the proof is exactly the same as what we have for the classical this inverse function theorem.

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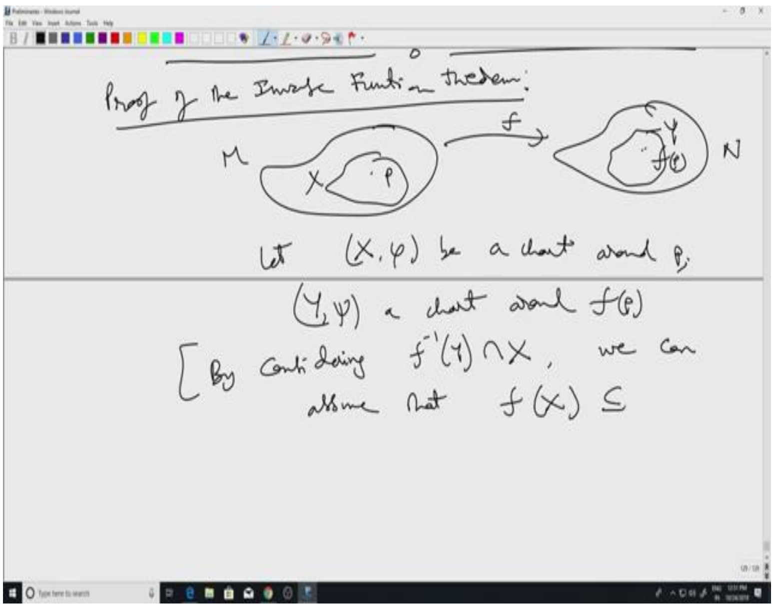

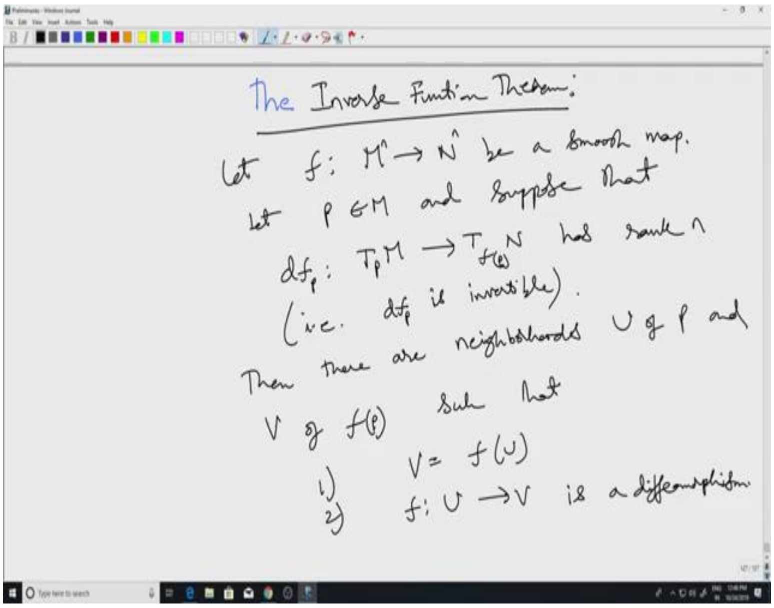


Now, as for the proof of inverse function theorem manifolds. So, let me just make a few remarks. Proof of inverse function theorem. So, this is M, this is N and have a map f, p is a point here, f of p is point here. The conclusion of inverse function theorem, is a local one. In other words, we are asserting the existence of a neighbourhood of p and a neighbourhood of f of p. So, we are not saying anything behaviour of f globally, which is not surprising since the assumption of the invertibility of the derivative is only at the point p.

So, because of this local nature, what I can do is, I start with a chart here. Let us, start with some chart here. Let, I will call it some X, P, be a chart. So, this is my X around P, and I would like to choose a chart here, Y, Y psi a chart around f of p. So, the essentially, I would like to reduce to the Euclidean case, the only thing is that as usual.

So, I would want to have f of X the image of this open set X, I would like it to be contained in Y. And we know that we can always achieve that just by taking, we have seen this before even in the definition of smoothness this issue arises. By considering the open set f inverse Y, intersection X, we can assume that f of X is actually contained in y, this is not an issue.

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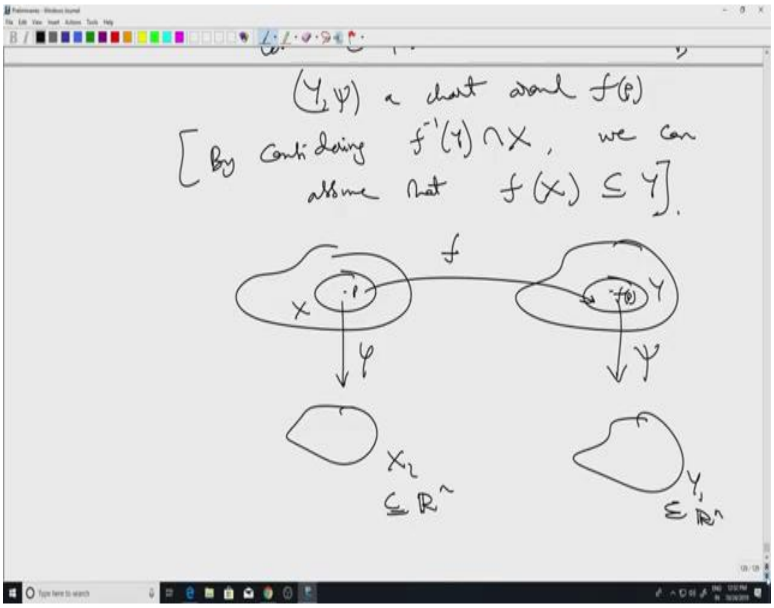


So, once we have that, perhaps I should have left some space there. So, this was my XP, f of p and this is the chart y and I know that f is taking this open set to this open set. So, this is a chart, I have this map p here, psi here, let us call it X1 and Y1 these are, X1 and Y1 are open subsets of Rn. Well, so far I have not used the hypothesis on that derivative this is just a general thing.

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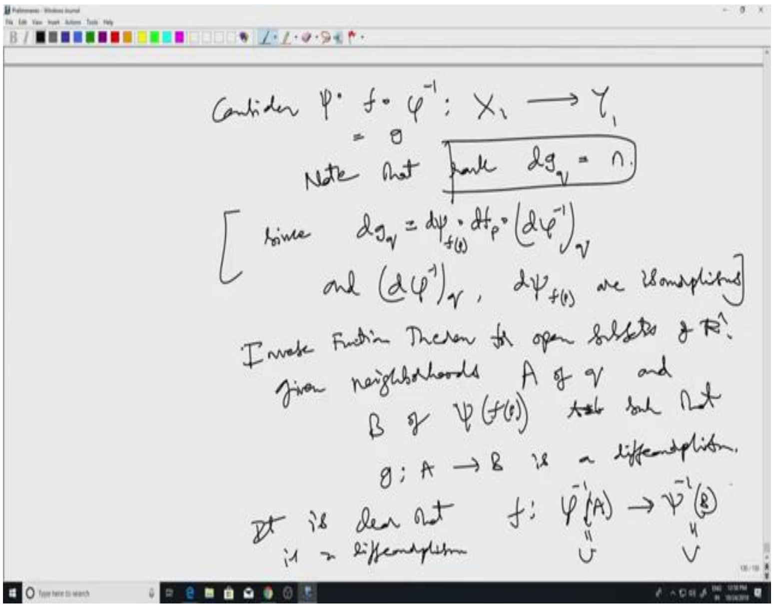

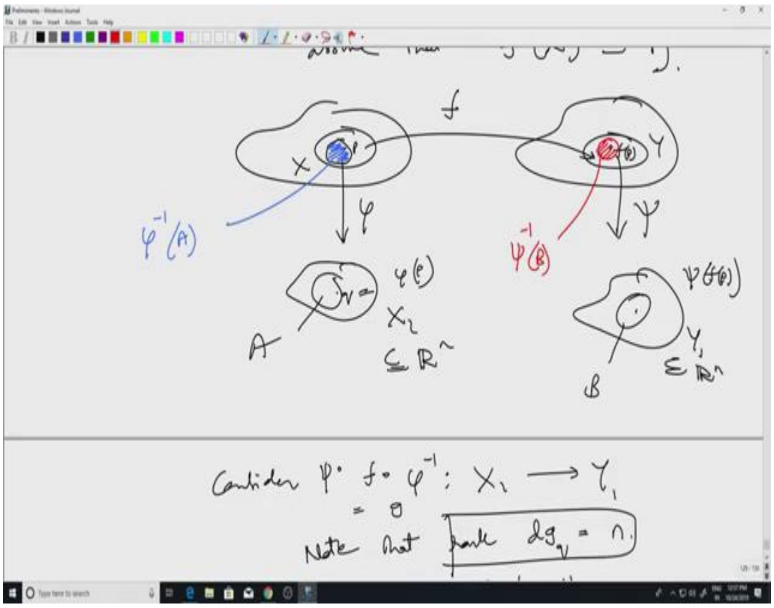

And, then now, what I do is consider, the map f in these coordinates. In other words, I look at P inverse compose with f, compose with c, that is a map from X1 to Y1. The point P, let us say this goes to q, q equal to phi of P. Now, the fact that we would like to say, so let us call this map as some map g. So, note that rank dgq, is also is, equal to n.

This is because well, since, dgq it is matter of g is this composition of these three maps. So, I just apply the chain rule. So, first I have d phi inverse at the point q and then phi inverse is going to take me to dfp, and then dc at f of P.

And, well we know that by definition almost that this phi and psi the chart maps are actually diffeomorphism. So, and we also know that when we have diffeomorphisms the rank of derivative is the full rank. So, the in other words, the derivatives are linearized morphism between corresponding tangent spaces.

So, and d phi inverse at q, d phi at f of p are isomorphism. So, these I am basically, this dg at q and df at p are related by pre composing and post composing by isomorphism. Therefore, they have the same rank. That is what I am claiming here and then the classical inverse function theorem, inverse function theorem for open subsets of Rn, gives neighbourhoods. Let us call it A of q and B of, well it is psi of f of P.

So, this point is psi of f of p. Such that, g from A to B is a diffeomorphism. So, once we have this A and B, then we just pull them back to the manifolds and we get the corresponding statements. So, A would be this and so, this would be A, this would be B. So, what I do is I just take phi inverse of A, it will give me a smaller subset here, this is phi inverse of A, and likewise here I will take psi inverse of B. That will give me some small open set around f of P.

It is clear that, f from phi inverse A to psi inverse B is a diffeomorphism and this is what we are going to call. So, recall that in the statement of the inverse function theorem, what we asserted was that there are open sets neighbourhoods U of P and V of f of p. So, we just call these, these two things as U and V and we are done.

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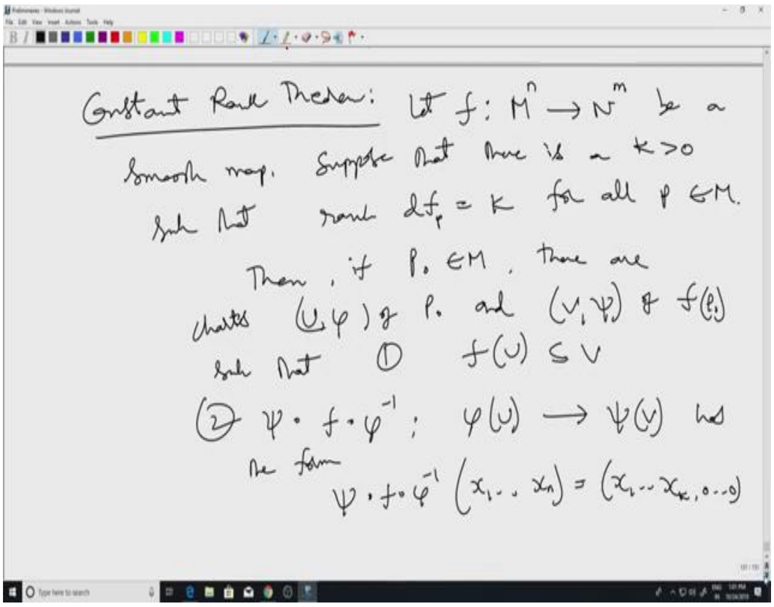

So, likewise, we also have the constant rank theorem which again the proof is exactly similar. Given the Euclidean statement, you just reduce it to the Euclidean, even the Euclidean constant rank theorem one can get this. So, let me just state it, constant rank theorem. Now, we do not assume that they have the same dimension be a smooth map.

Suppose that there is a K greater than 0, such that rank dfp equal to K, for all P in M. Then at any, if P naught belongs to M, there are charts U phi of P naught and V psi of f of p naught.

Such that phi inverse composed with f composed with psi this would be a map from phi of u to psi of v.

Here, before I make the statement as usual I want to say that the first thing is such that the small condition that f of u contained in v and this map from here to here has the form psi composed with f composed with phi inverse of x1 x2. So, let us assume the dimension here is n, so x1, x2, xn is just x1, xk, 0, 0, 0. As usual I mean as in the Euclidean case, if these zeros may not occur depending on the rather m is greater than n and what k is. So, these zeros may or may not occur and in fact so.

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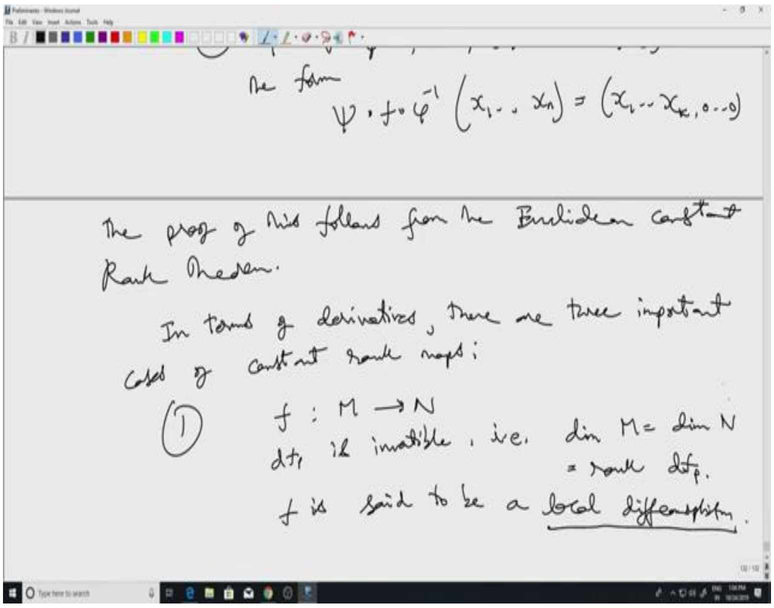


The proof of this follows from the Euclidean constant rank theorem. So, as in the Euclidean case there are three important classes of maps, one we have already seen the notion of a, we have a diffeomorphism between two manifolds. The other two classes of maps. Well, actually apart diffeomorphism there is in terms of diffeomorphism of a map involves not just condition of the derivative, but rather the map itself by bijective surely in terms of the derivative, there are three classes of maps.

In terms of derivatives, there are three important cases of constant rank maps, here I do not need capital letters of constant maps. The first one is, I have f from M to N, here the first condition is that dfp is invertible. So, i.e. dimension of M equal to dimension N equal to rank of dfp.

In this case f is said to be a local diffeomorphism and the terminology is clear enough. Because, this is the setting for the inverse function theorem, the inverse function theorem tells us that the given any point here can find and open set or open neighbourhood which is diffeomorphically map open, open subset of the image point, open neighbourhood of the image point. So, therefore, we have this name here local diffeomorphism

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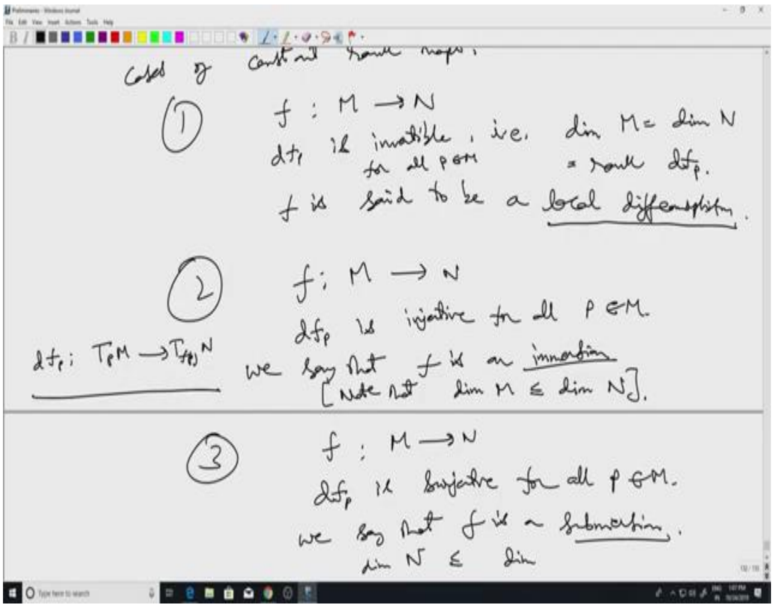


The other one is f from M to N dfp is, here I should mention the dfp is invertible for all p and m, dfp is injective for all p in m in this case again, this will force we say that f is an immersion so, in this case this will force note that dimension m should be less than or equal to dimension of n.

Because, after all the dfp is a map between tangent spaces. So, dfp is a map from TpM to t f of p n and if this map is to be injective then the dimension of the image vector space is at least the dimension of the domain vector space. These vector spaces are the dimensions equal to the corresponding manifolds. The last case is when I have dfp is surjective for all p in m here, we say that f is a submersion in this case the dimension of N.

So, again the tangent map, the differential would be a map between two vector spaces and we are claiming that the assumption is that this is surjective. Therefore, the dimension of the image vector space should be less than or equal to dimension of the domain and it translates to a statement about the manifolds so, dimension of N less than or equal to dimension of M.

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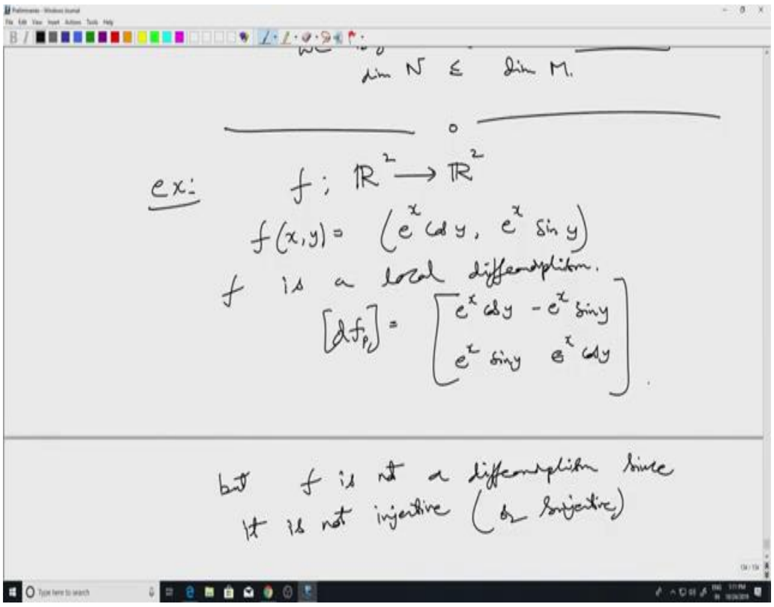

Now, let me quickly give an example of a smooth map between from R2 to R2 to illustrate this sum the difference between the local diffeomorphism and then actually diffeomorphism. So, let f from R2 to R2 be f of x, y equals e to the x cos y e to the x sine y. So, this you can recognize that this map is nothing but the map f of z is e to the z in complex coordinates. Here, I have just written it in Euclidean coordinates. Well, this map is a, f is a local diffeomorphism.

So, in other words the derivative of f is invertible at all points in R2. So, to see that you just write down the derivative we know that the derivative it (())(26:02) the notion of derivative is the same. The two notions of the abstract notion of derivative and the classical notion of derivative are related by isomorphisms. Therefore, when we talk about rank we can work with either one in the Euclidean setting.

So, I hear I will take the classical derivative and the classical derivative is the matrix of that is just the Jacobian matrix e to the x cos y, e to the x sine y, e to the x sine y then minus, well. So, here I should get. So, here I am differentiating. So, e to the x cos y. I should get a minus sign somewhere. So, when I differentiate this here I think I want to know and the point is that this matrix is invertible. Whatever, x and y are its determinant is just e to the power 2x and f is a local diffeomorphism.

But, f is not a diffeomorphism. Since, it is not for instance it is neither surjective, since it is not injective or for that matter or surjective the value 0, the 0, comma 0 is not assumed by this map and it is not injective. Because, of the periodicity in the y variable. So, one can have local diffeomorphisms, which are not diffeomorphism.

So, let us stop here in the next lecture we will get an important way of using the constant rank theorem, we will get an important sort of procedure of generating a large class of sub manifolds. So, we will stop here. Thank you.