An Introduction to Smooth Manifolds
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Lecture 18
Dimension of Tangent Space 2

Hi, welcome to the 18th lecture in the series. So in the last lecture I was in the process of completing the proof that the tangent space n dimensional and so I was almost done.

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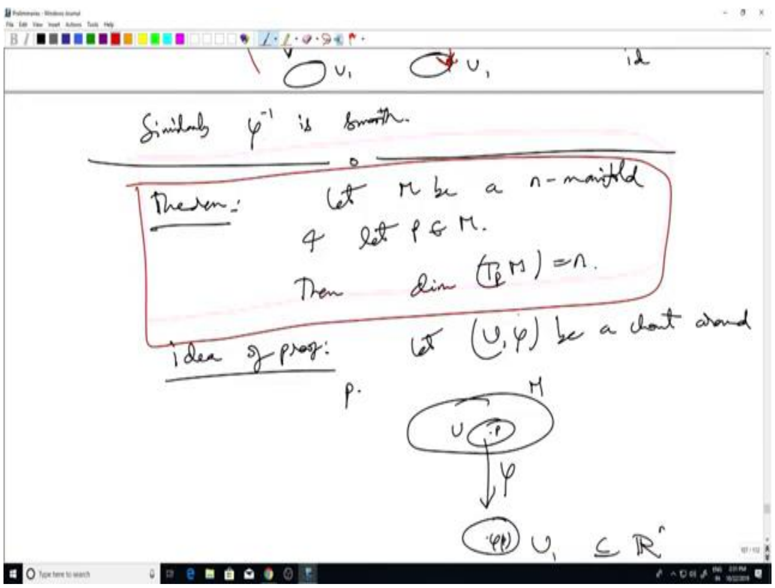

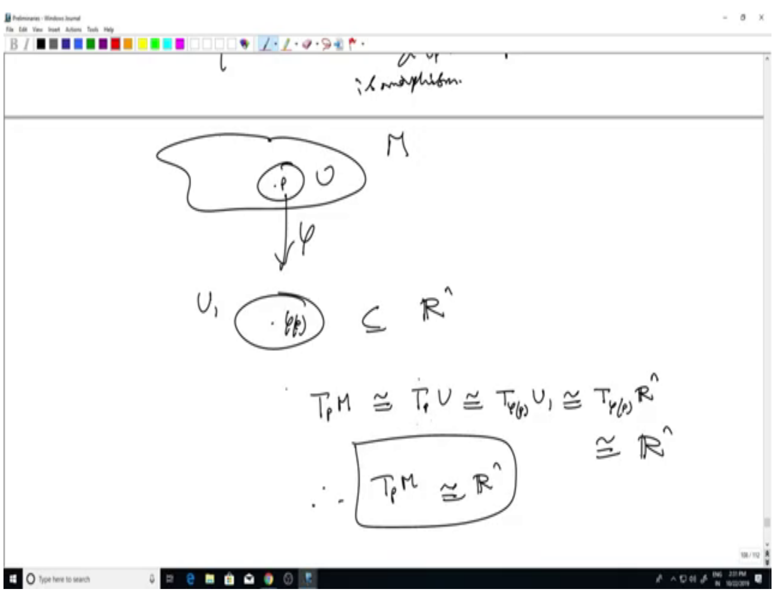

Now let me resume from the, this point so if we recall quickly what we have the strategy of the so here is the main statement the theorem is that this is what we have aiming for n manifold and TPM has dimension n. So the idea of the proof was as follows we have the, we take a chart around the point P, U comma phi and then the main idea is just this series of isomorphisms.

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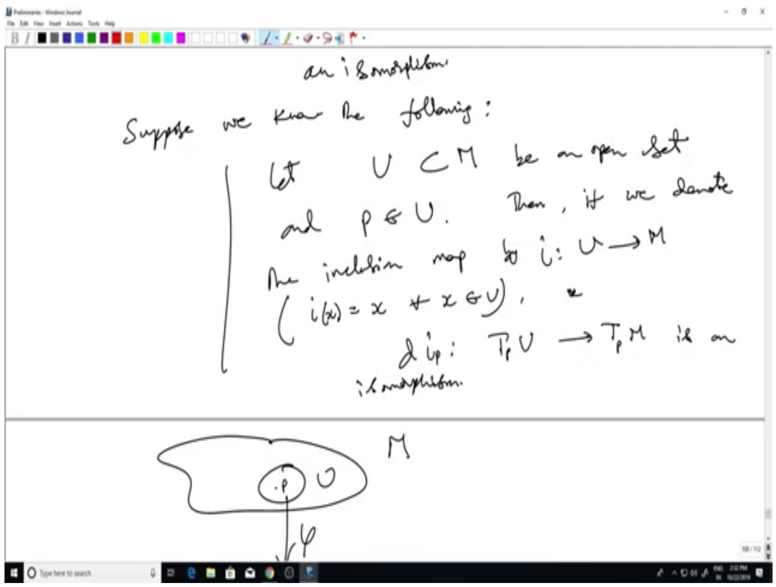

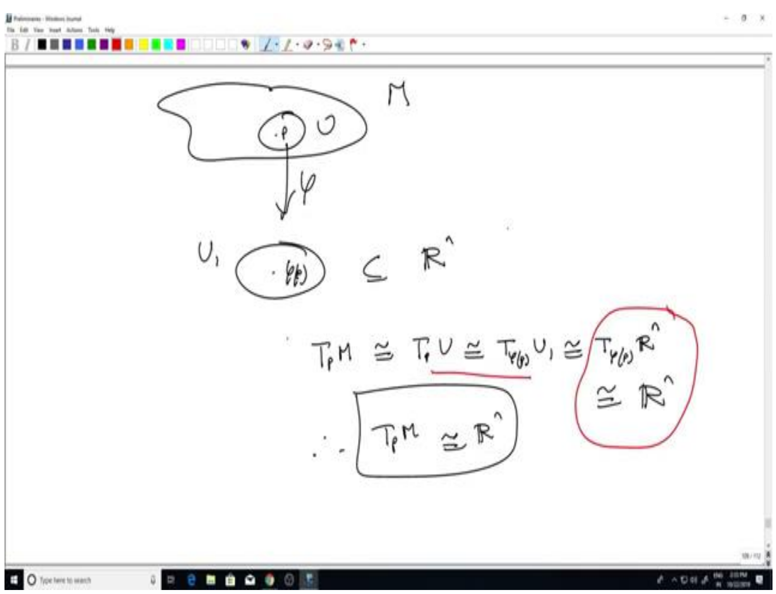

So we have to prove that if you have an open sub set of a manifold then, and a point inside that, then the tangent space to the open set at that point is the same as the tangent space to the manifold at that point. Assuming one had this then one has the series of isomorphisms. TPM is isomorphic to TPU and because this chart by almost by definition of smoothness is a smooth map is a diffeomorphism, it sets up the derivative of phi, sets up an isomorphism between TPU and TP, T phi of P U1 which is what I have here.

Then again U1 is an open sub set of Rn therefore its tangent space, the tangent space to U1 at phi of P same as this. Finally, one had this, this was what I had started with a couple of lectures ago and in fact this is the, the last state of isomorphism is the most the one with nontrivial content, the remaining these other three are easier in some sense because the last isomorphism one had to use a little bit of analysis to get this. And well so I was in the process, so in this chain of isomorphisms all we have be done if we show that for any open subset for manifold TPU is isomorphic to TPM that is what we are aiming for.

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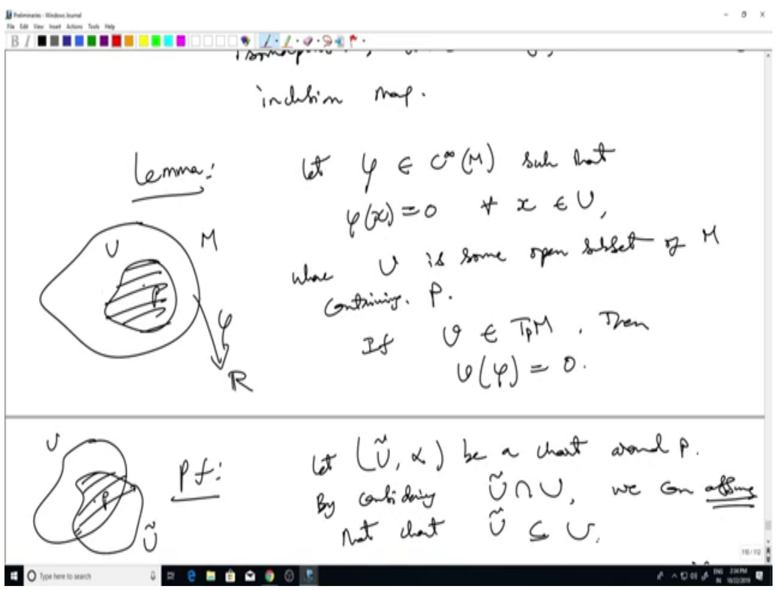

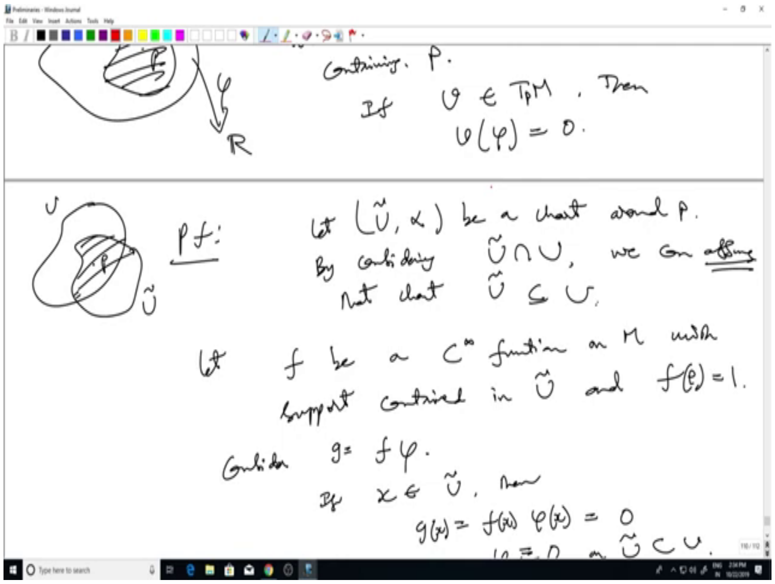

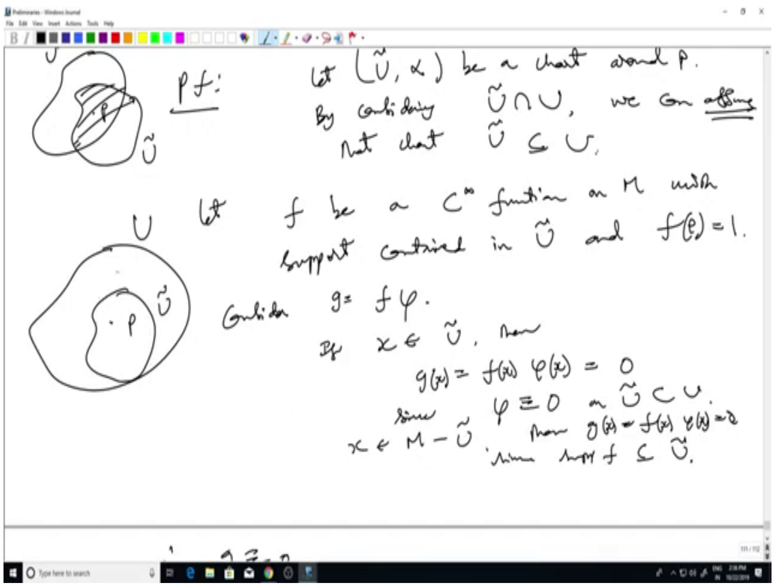

And the we needed a small lemma for this so the lemma is that if phi belongs to C infinity in this, this identically 0 on some open set containing P and if V is any tangent vector at P, tangent vector to M at P then V of phi is 0. So, in other words even if the function phi is not globally 0 but 0 near in neighborhood of P, V of phi should be 0 and again the interpretation of this is that the derivative of a function, if a function vanishes in an open subset then the derivative of the function at any point inside the open subset is 0. So in order to prove this what I did was, I took a chart around this point P then I looked at the, then we can assume that the chart is actually contained in U.

Now here is the main idea, so if let F be a C infinity function on M with support contained in this. So now the picture is, this is the point P, this is a big open set and I can assume that U tilde has actually contained in U and this function is supported, the support of the function is contained in U tilde and moreover its value at P is 1. So to say that the support is contained in U tilde and particular implies that its identically 0 outside U tilde. So if I look at this product function F times the original function phi then its actually just the 0 function because if X is in inside U tilde then phi is 0 on the other hand if X is outside U tilde this F is 0.

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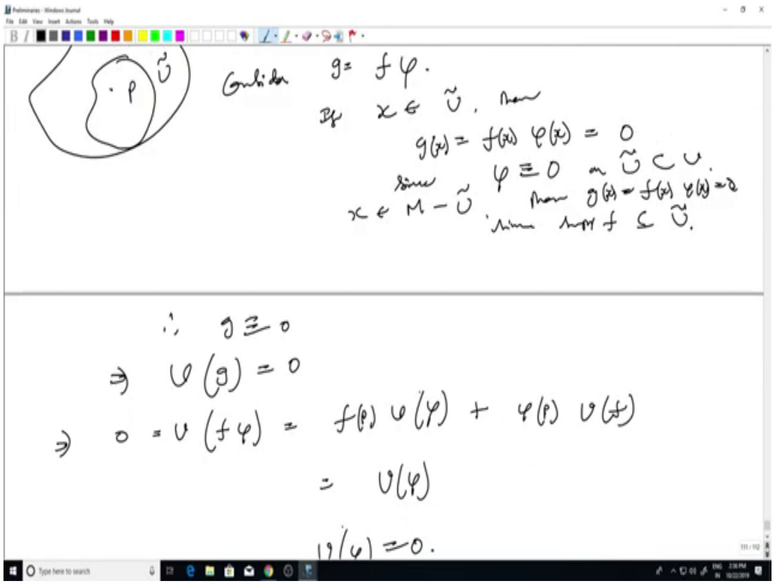

G is identically 0 and then but if take V of G and apply Leibnitz rule I get this, then finally I get V of phi 0. So let me quickly point out how we can get a C infinity function with support contained in U tilde and F of P is 1.

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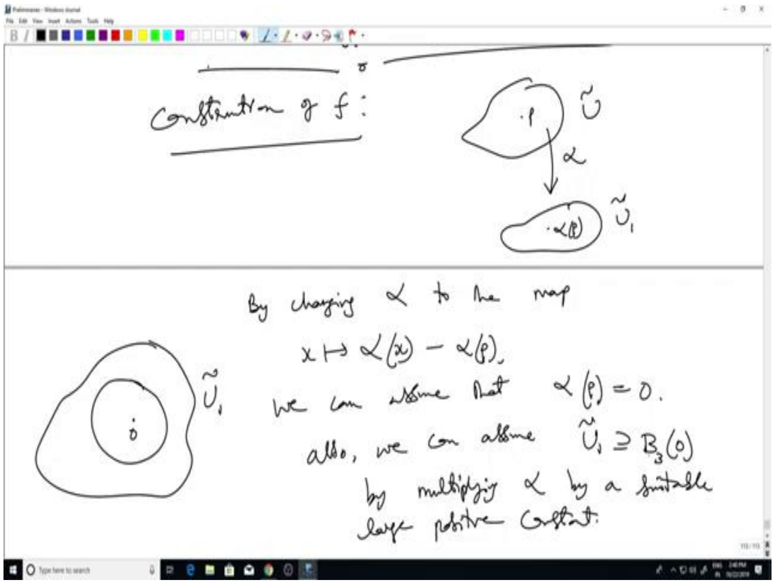

Construction of F well we had a chart U tilde and a point P and now I have used phi in a different sense here so let for a chart let me use a map alpha and it goes to some U tilde1 alpha of P. By changing alpha to the map alpha of X, X going to alpha of X minus alpha of P we can assume that alpha of P is actually 0, just by translating this map alpha in Rn, I can assume alpha of P is 0 and once we have alpha of P is 0, also we can assume U1 tilde contains the ball of radius let us say 3 centered at the origin.

This is set of a, so we can assume, also we can assume this by multiplying alpha by a suitable large positive constant. In other words, multiply, so consider C times alpha of X, where C is some large positive number. Then we can assume that the image contains the ball. So this is U1 tilde and now we have 0 and we have the ball here, so this is 0 and we have ball of radius 3 in the image.

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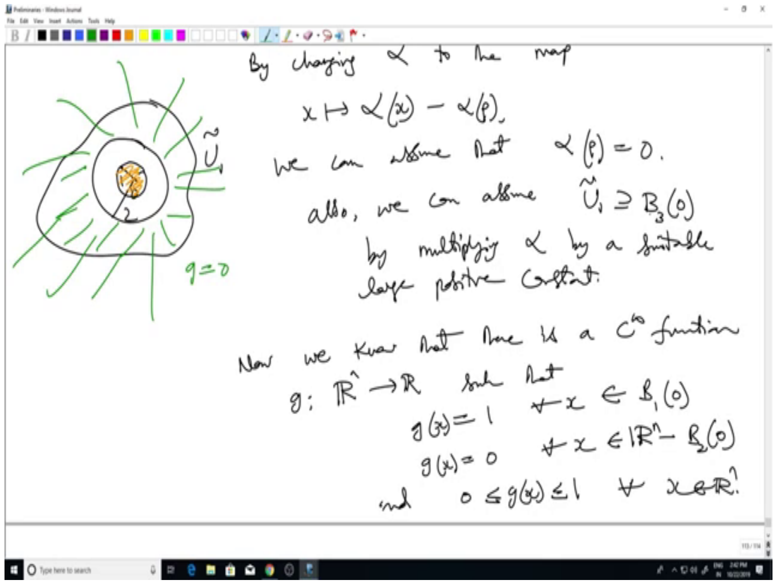

Well we have seen already that we can construct a C infinity function, now we have a, we know that there is a C infinity function, C infinity function G on Rn such that G of X is identically 1 for all X in the ball of radius 1 centered at the origin. G of X is identically 0 for all X in Rn minus the ball of radius 2 and everywhere, and even though we do not this last thing I will just write it down and G of X lies between 0 and 1 for all X in Rn.

So I really do not need the ball of radius 3, I just have to work with the ball of radius 1 and 2 so this is 1, radius 1 and this is radius 2 and my function G is identically 1 inside the smaller ball, inside this orange ball the function is 1 and outside everywhere it is 0. So G is 0 here and G is 1 inside and in this and gap between these two balls it varies between 0 and 1.

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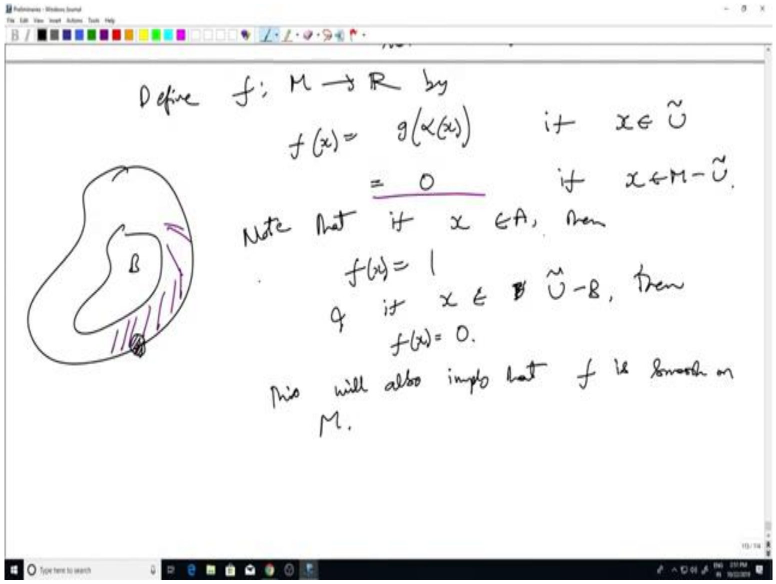

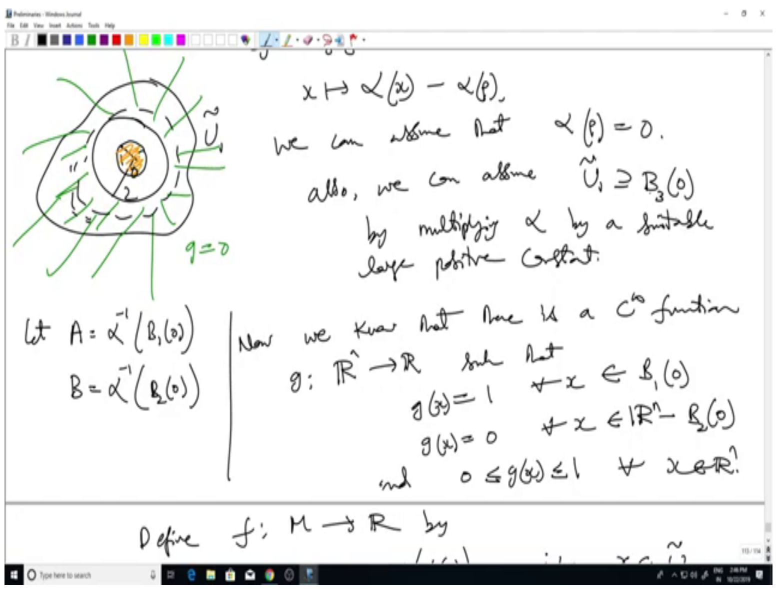

So what we can do is, use this this G to get a similar function on the manifold which is my F. Define F from M to R by, well so we have to work with so before I write this let us see here I hd a ball of radius 2 and ball of radius 1, this is all this is in U tilde 1which is in Rn. Now using this alpha which was the homeomorphism, diffeomorphism from U tilde to U tilde 1 I will get corresponding sets in the manifold. Let A equal to phi inverse of this first ball B1 0 and B equal to phi inverse the second ball B2 0.

Then I define f from M to R by f of X equals. So the idea is simple so I just, I am going to essentially use the if my X belongs to this U tilde then I will just use this function G via the map alpha. So f of X equals G of alpha of X, if X is in U tilde and then is equal to 0 if X belongs to M minus U tilde, so yeah this is being defined in a piecewise way sort of. So essentially note that for the movement if we assume that f is smooth, note the following, note that if X belongs to what I call this A then f of X, oops there was a small correction here it is not phi rather I called the chart map as alpha now so alpha.

So note that if X belongs to A then alpha X by definition will belong to B1 0 therefore G of that will be 1 and if X belongs to, and if X belongs to M minus, rather U tilde minus B then f of X is, well if X belongs to U tilde minus B then when I go via this chart map I will end up in this green part in U1 tilde and we know that G is 0 there, so f of X would be 0. So in short, now in fact this second thing that f of X is identically 0 outside U tilde minus B will also imply that, this will also imply that f is smooth on U tilde, is smooth on M.

Whenever we define functions in a peace vice fashion like this one has to be careful that smoothness holds everywhere. One has to check that for instance in this simple case if X belongs to U tilde anyway it is a composition of smooth function therefore it is smooth and if X belongs to M minus U tilde as well, well here, this is the constant function which is smooth though ideally actually when anyone talks about smoothness the domain should be in an open set so here one has to be change it slightly and regarded as function on M minus U tilde closure then the constant function is smooth.

Then the task becomes what happens, so here to be very precise I would want to define it as if X belongs to this and if X belongs to M minus this U tilde right. No this is fine however when checking smoothness you will, one will have to segregate into two cases, so one is X belongs to M minus U tilde closure which is an open set there f of X is identically 0 similarly if X belongs to U which is an open set, f is smooth.

The question becomes what happens if X belongs to the boundary of U tilde but it is not an issue because if X belongs to U tilde minus B, already f of X is 0 so therefore the point is that this is our B rather this is, it might be a distorted ball so this is our B and the point is that U tilde minus B in this entire portion the function f of X is 0 anyway. And even outside this U tilde we had define it to be 0 by this here. So the point is even if you are on the boundary of U tilde it is not an issue in an ease we can find a small neighborhood such that in that neighborhood the function is identically 0.

So it is a constant therefore it is smooth there as well. So this proves that F is smooth everywhere and it has a required property well the thing is that since it is identically 1 on B and a point P, certainly belongs to this B because point P corresponds the way we set up things, P actually corresponds to the origin in Rn which belongs to the ball of radius 2 certainly ball of radius 1. Therefore, f of X will be identically 1 at P. And the support of this function by definition is inside B which is inside U tilde for sure that is what we wanted. So this completes the proof that if a function vanishes in an open set then.

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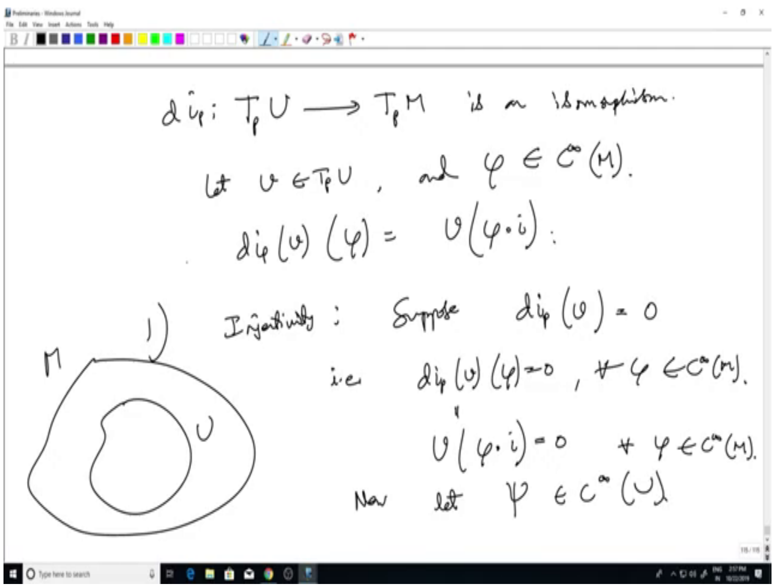

Now let us see how this is going to help us in overall scheme of things. So we were trying to prove that so we had this TPU and then we wanted to claim that this map dIP, the derivative of the inclusion map is an isomorphism. So let us, so what does this map do fsirst before we prove injectivity and subjectivity let us just check what this map does. So if I start with a V here TPU then dIP of V should give me an element of TPM so in other words it should act on a C infinity function on and C infinity function on M.

Then by definition dIP of V acting on phi is supposed to be V of, well first I go via I and then phi of I, so this is my, in other words this map phi composed with I is nothing but the restriction of phi, phi is defined on all of M but I am restricting it to this open set U and then I act it, then I take the directional derivative along V. Well, let us see why this is a injectivity. Suppose dIP of

V1 equal to dIP of V2 for some tangent V1, V2 in TPU, then actually right, so rather than this let me just, these are linear map, we already know that dIP is linear.

So when we are in the realm of linear maps to check injectivity I just have to check that the kernel is 0. Suppose I have this I would like to say that V is 0 i.e. dIP V of phi is 0 for all phi in C infinity M. So now this is by definition what I have written here so this is equal to V of phi composed with I is equal to 0 for all phi in C infinity M. So in other words what we are saying is that if I take a C infinity function on, so here is U and it is sitting inside some big manifold M.

I am starting with a C infinity function on the big manifold (restrict) restricting it to U and then I want to say, then I have the property that V acting on such a function is 0.Well if I can say that every C infinity function on this U arises as the restriction of a some bigger C infinity function then I would immediately conclude that V is identically there but it will not be true that every C infinity function on U is a restriction on some bigger function, on bigger C infinity function on M. So but we really do not need that fact so what we do need at this stage is that, so here at the stage V composed with this is 0 for every C infinity in this thing. So now let f belong to, may be not f some other thing, si, si belong to C infinity U.

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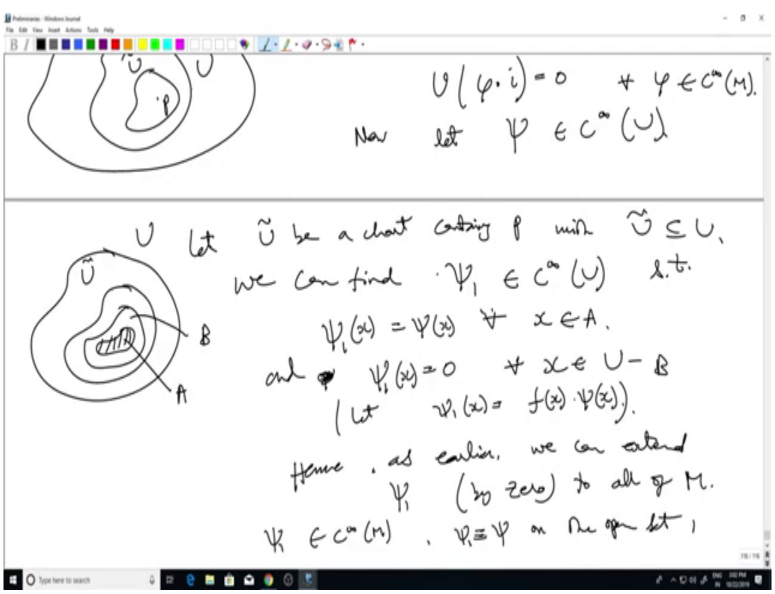

So instead I am going to use this fact that previous statement about two functions agree then the directional derivative should agree so let me start with some function F and C infinity U. Now let U tilde be a chart containing P with as before you just take put it in some chart U tilde. So what I will do now is I will find, we can find si 1 in C infinity U such that si of X equals to si 1 X for all X in U tilde. Actually let me use the same notation that I used earlier so recall that I had this A and B so let me just expand this picture a bit so I have this U this U tilde then two more sets, the innermost set was A and containing A was B.

So what I want is, we can find new functions such that si of X equal to si 1 X for all X in A, for all X in A and si 1 X is 0 for all X in outside this B essentially U minus B. And the way one would to do it as just we do whatever I did last time, just let si 1 X equal to that function f of X that I had the same f of X times si of X, recall that f of X was identically 1 inside A so when X is inside A, si 1 X will be equal to si of X and may be, okay let me just in this maybe it is better to write this as si 1 X.

Yeah when X is inside A, si 1 X is si of X and when X is outside B, well when X is outside B then this f of X is 0. So I will get si 1 X equal to this. So this will be 0 and that is what is I claiming here, hence as earlier we can extend this new function si 1 extend by 0 in other words declare all value outside U to be 0 to all of M. So what we have now is, we have a function si 1 defined in all of M which agrees with si on this small open set A containing P. So main thing is that si 1 is defined on all of M and si 1 is identically equal to si on the open set A. We will have to stop somewhat abruptly in this lecture we will stop here next time I will take off from this point. Thank you.