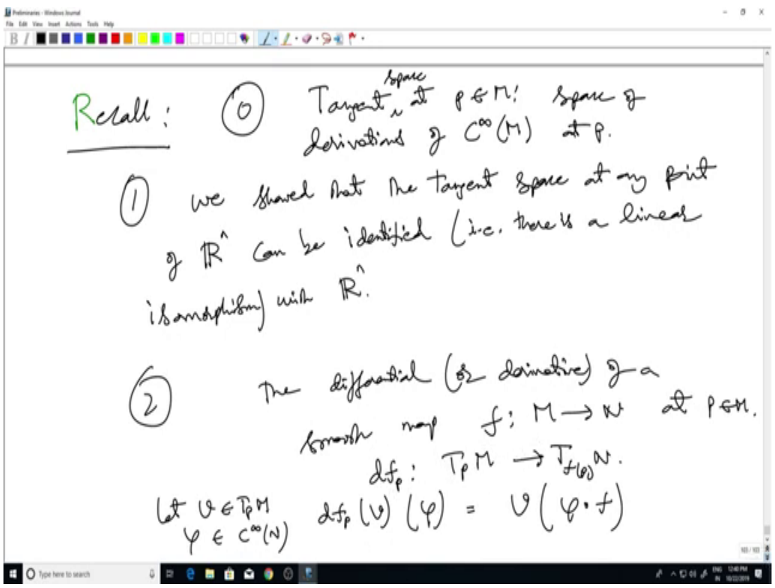
An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 17
Dimension of Tangent Space 1

Hello and welcome to the 17th lecture in the series and in this lecture I am going to show that the tangent space of an n dimensional manifold is a vector space of dimension N. So let me begin by recalling what we have been doing in the last few lectures.

(Refer Slide Time: 0:53)

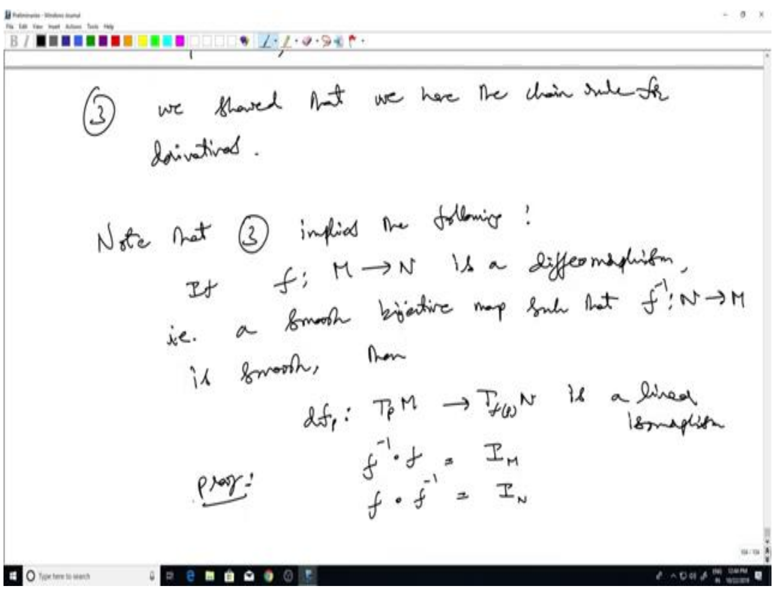


So we started by discussing the case of the manifold being Rn itself. So and in this case we showed that the tangent space at any point of Rn can be identified, in other words identified i.e. there is a linear isomorphism with, so let me put this linear isomorphism with Rn. Here so in this first statement we showed that the tangent space when I say this we are working with so let me call that 0 tangent space at a point P, at point P we define to be the space of derivations of C infinity M at P.

Well in this (definition) so that is and then with this definition in hand, so tangent space with this definition in hand we show of that the tangent space when the manifold is Rn the tangent space can be identified with Rn and in the last class I talked about the differential or derivative of a smooth map F from M to N. This turned out to be a linear map dFP at P in M, dF tangent space to M at P, tangent space to F of P at N and the definition was as follows. So dFP, so let V belong to TPM, I want to see what dFP of V is? dFP is supposed to be an element of TFP of N. In other words, it is a derivation on C infinity N which based at P.

So in order to say what the dFP of V is I have to act it on a function phi where phi belongs to C infinity N and this is supposed to give me a number dFP of V acting on phi and this we define to be V acting on phi composed with F. Phi composed with F will turn out to be a C infinity function on M and then I can act V on that and get a number so this was our definition of derivative. And for to make sense of this one does not really need to know that what the dimension of this TPM is and so on.

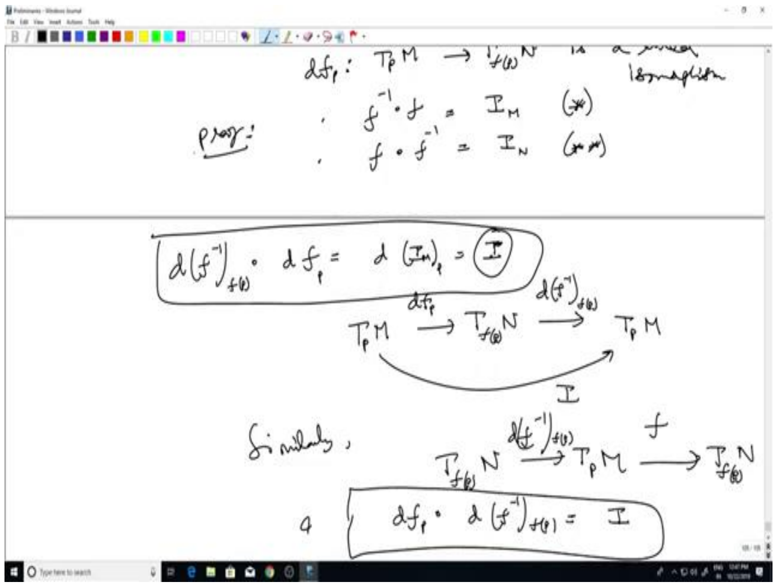
(Refer Slide Time: 5:23)



In fact, we showed that we have the chain rule for derivatives. This was towards the end of the last class and along with that I also showed that the derivative of the identity map is the identity linear map on tangent spaces. So with all this in hand now we can try to work out, what the dimension of the tangent space of our manifold is. Well before I do that let us observe that, note that, note that 3 implies that, implies the following if F from M to N is a diffeomorphism, i.e. a smooth bijective, the definition of a diffeomorphism is a same as what we had for open sets in Rn.

So it is a smooth bijective map such that F inverse from N to M is also smooth. So if we have a diffeomorphism then dFP from TPM to T F of P N is a linear isomorphism. And this follows from 3 because after all we have, the proof for the statement is that F inverse compose with F this should be the identity map on M which I will denote by I sub script M and the other way would be F composed with F inverse that would be the identity map on N.

(Refer Slide Time: 8:51)

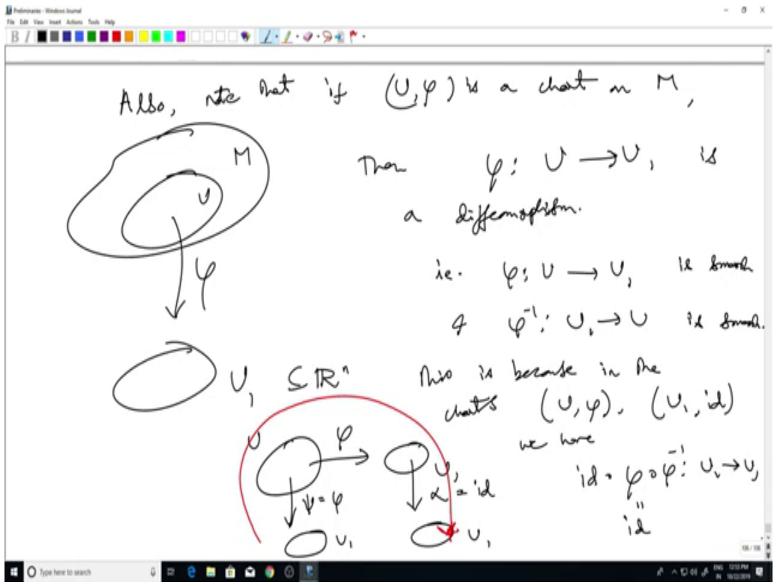


Now property 3 which I wrote above is the chain rule. So let us apply the chain rule to both of these equations and I will get, the first one will give me dF inverse compose with dF equals d of identity which is identity again but this side identity is the identity, so let us see where this maps go. So if I start with P, so I will have to put identity at P and here this would be at F of P what we are working with is F composed with F inverse is after all a map from, so first I have TPM this will dFP will take me to T F of P N and d F inverse at F of P will take me back to TPM.

So this identity here is just this identity from this TPM to TPM. So what I have is that, so this tells me that before I have dFP and d F inverse at F P the composition is identity and if I so that is what I got from this and if I work with this, with the second one if I knew that all these spaces are finite dimensional, just this one equation would be enough but I am not proved that yet so I will use both equations here, this as well as this.

And similarly if I work with the second equation, for working with the second equation I will start at T F of P of N, F inverse will take me back to rather d F inverse at F of P will take me back to TPM and if I apply F now I will go back to T F of P N. So I have this and again the chain rule tells me that dFP composed with d F inverse at F of P equals the identity map, this time identity map would be the identity map on the T F of P N. So If I combine these two equations so this as well as this, the proof is complete. So the dFP would be a linear isomorphism. So we will need this, that is one observation.

(Refer Slide Time: 12:17)



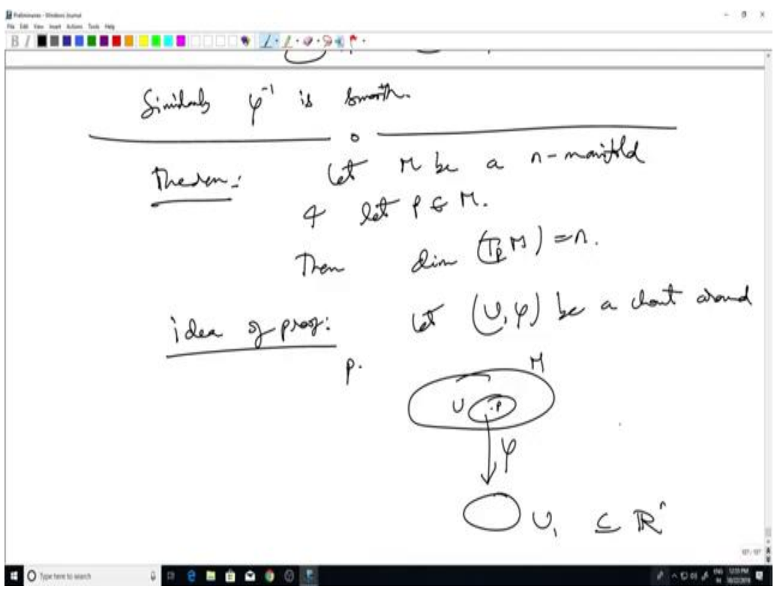
The second thing is that also note that if U phi is a chart on M, as usual I have this is M, this is U, phi is a map from here to what I have been calling U1 inside, this is inside Rn. Then phi from U to U1 is a diffeomorphism in the sense that I define just now, namely, well U itself is a manifold as we saw earlier in open sub sets of a manifold can be regarded as a manifold in a very natural way, namely, the charts for U are essentially the charts for M by (()) (13:22) the size of the open sets just by intersecting with U the chart maps will be the same.

So U itself is a manifold, U1 is a manifold and what we are saying is that then the chart map is a automatically a diffeomorphism. So this is i.e. phi from U to U1, maybe I should change this lightly I should make it seem like U. Phi from U to U1, U to U1 is smooth and phi inverse from U1 to U is also smooth. And phi inverse, phi is since the chart is by definition a homeomorphism of U to U1, phi is automatically bijective what we are worried here, what we are concerned with here is the smoothness of phi.

But it is smooth in a very trivial sense because after all our definition of smooth is that, the map is smooth if in suitable charts after composing with suitable charts it is smooth. Well if we are working with the chart map itself this is because in the chart, so I have to draw the pictures slightly schematically in a different way, so now I will make the map horizontal sort of , U1 and this is U so I am asserting the smoothness of this phi. So according to our definition of smoothness, I have to find a chart here and a chart here, if I can find a chart here and a chart here so that the composition is smooth, so I will need a chart here let us call it C and this is alpha.

Well we choose the obvious charts namely let us take C equal to phi inverse and alpha equals identity. So I will end up back in U1 here I will end up in U1 as well. So in the charts U phi inverse, not quite it is not phi inverse rather U is phi itself, U phi and U1 identity we have so I want to go like this, I want to go in this direction and come here, so I want to check whether this map is smooth here and that is trivially true because I just get the identity. So we have, so the last map is identity, this map is phi middle map and this map is phi inverse and this is supposed to go from U1 to U1 and this is just identity. So this map is identity, similarly one can check that phi inverse.

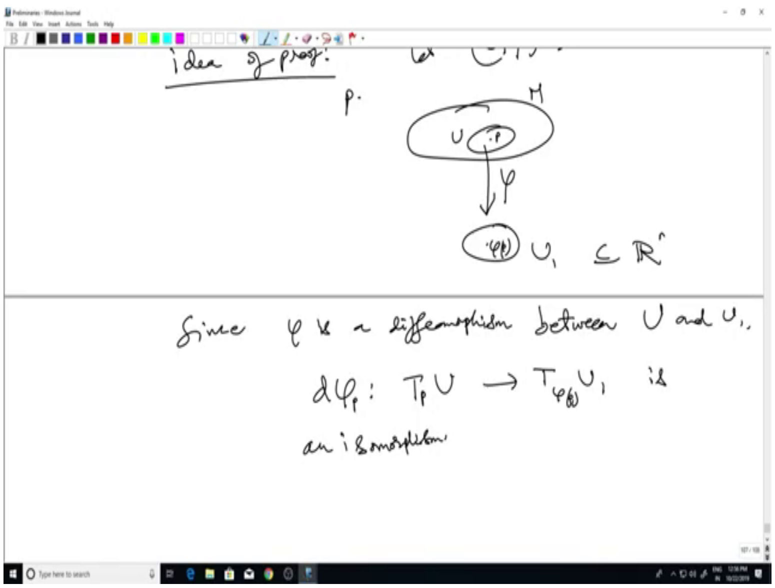
(Refer Slide Time: 17:24)



Similarly, phi inverse is smooth. So the fact that chart maps which are to begin with homeomorphisms are diffeomorphism is just a consequence of the definition of a smooth map. Alright so we have that. Now let me just quickly outline the idea of the statement, proof of the statement that the tangent space of a n dimensional manifold is a vector space of dimension N. So theorem, let M be a n manifold and let P belong to M then dimension of TPM is N.

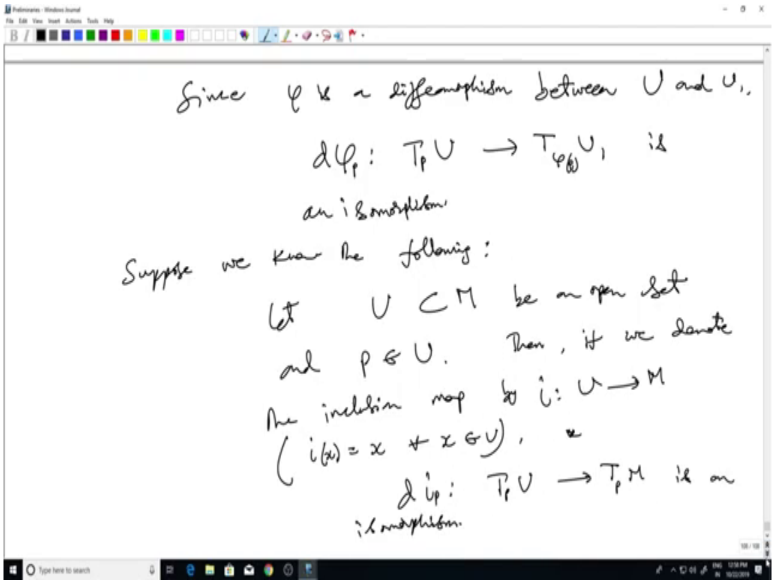
Idea of proof, so let u phi is chart around P so this is P, this is U, the bigger thing is M, so I have a chart map phi to U1 which is in Rn. Now based on the what I said earlier there are two things one is that this phi is a diffeomorphism from this manifold U to the manifold U1 and we have also seen that if we have a diffeomorphism then the derivative of a diffeomorphism at various points will given isomorphism of tangent spaces.

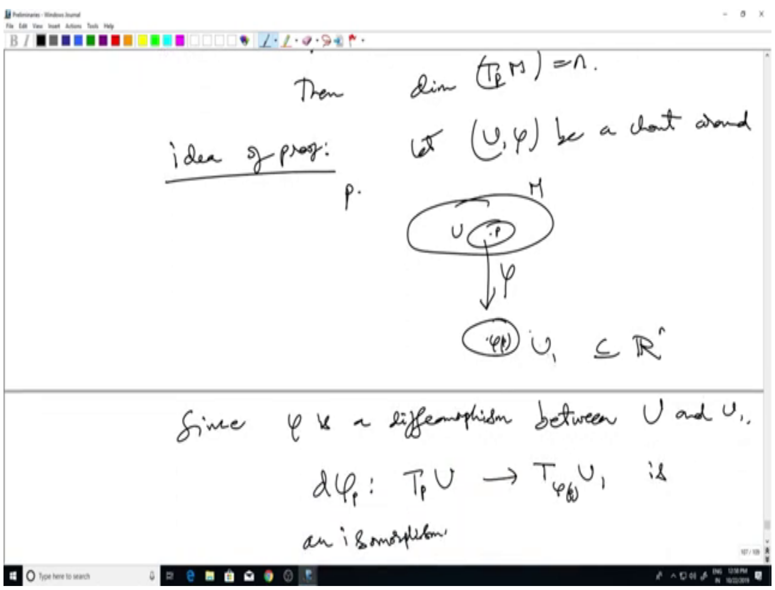
(Refer Slide Time: 19:40)



Since phi is a diffeomorphism between U and U1 d phi P from TPU to T phi of P U1 is in isomorphism, this is what we just observed but finally we are interested in TPM rather than TPU. So we want to relate the tangent space TPU to TPM similarly for this U1 as well I want to relate T phi of P, so phi of P is somewhere here, the tangent space to U1 at phi of P with the tangent space to Rn at phi of P.

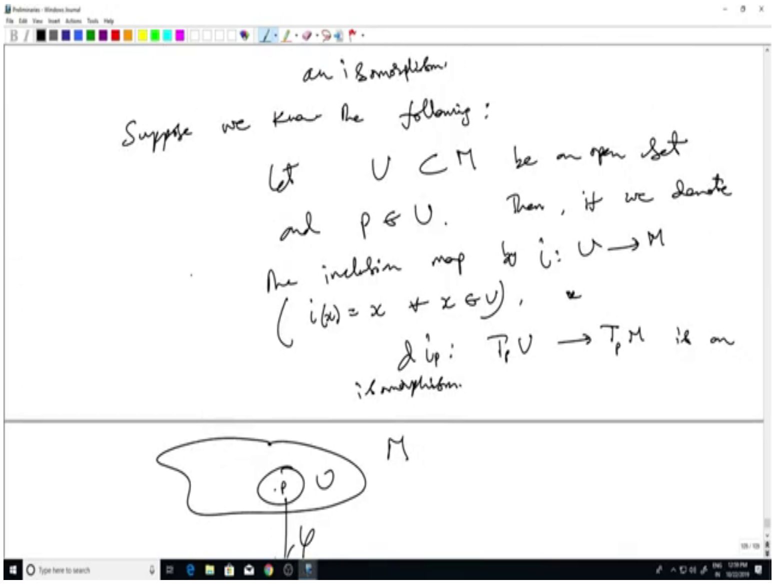
(Refer Slide Time: 20:55)

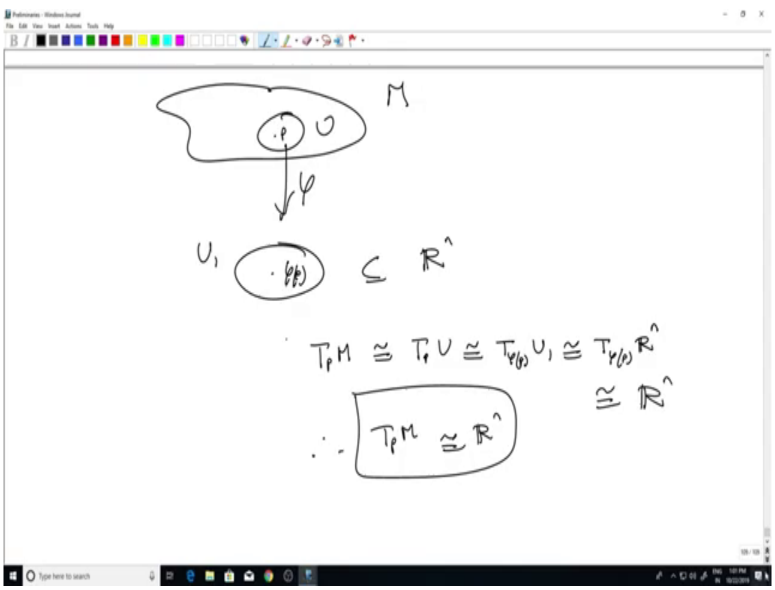




So what I need is, suppose we know the following, let U contained in M be an open set and P in U. Then if we denote the inclusion map I by I from U to M, the inclusion map is just I of x equals x for all x in U. If you denote the inclusion map by this I then the statement is dIP from TPU to TPM is an isomorphism. In other words, if we have an open subset of n manifold and a point inside the open subset then whether you look at the tangent space to the open subset at that point or the full manifold at that point, they are the same. So that is what we are saying here, DIP is an isomorphism. Now I can go back to this, this picture here and use this both for U and M and U1 and Rn as well.

(Refer Slide Time: 22:59)

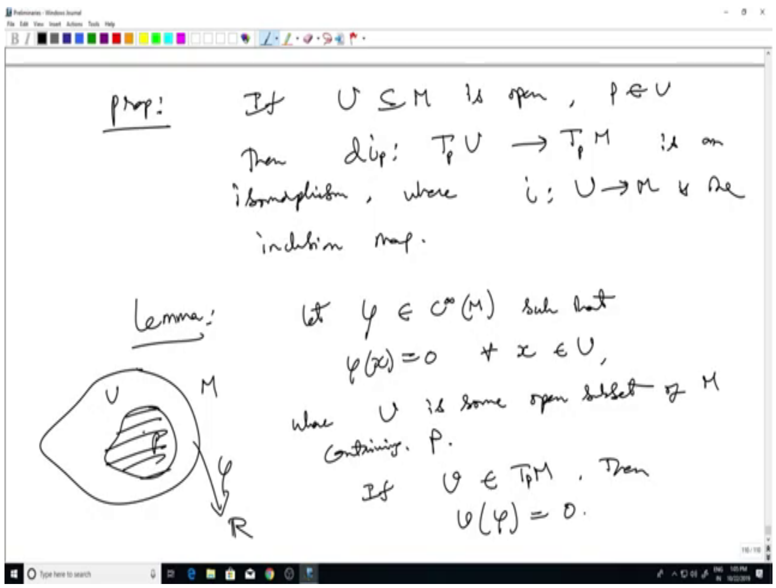




So I have, this was M, this was U and I have a P here and I have phi, this is phi of P and this is in Rn. And I know that the tangent space, this d phi sets up an isomorphism between tangent spaces to U and U1. So but now knowing this statement here, we have the what we want, finally, so what do we well we have two things one is TPM is isomorphic to TPU by this what I just state that the statement here and since phi is a diffeomorphism I also know that this is isomorphic to T phi of P of U1.

Again back to the statement this time I will apply to U1 contained in Rn so I will get, this is isomorphic to T phi of P Rn and which this the last one here is the tangent space to Rn at some point and this we have already seen is isomorphic to Rn itself, which is isomorphic to Rn. So finally I get that, therefore TPM is isomorphic to Rn. So this is the basically the idea of the proof. Now let me, so in this proof I proved everything except when I in this idea, the only thing we need to know is this one that if we have an open subset then the tangent space at any point to the open subset is the same as the tangent space to the full manifold. So let us proof that.

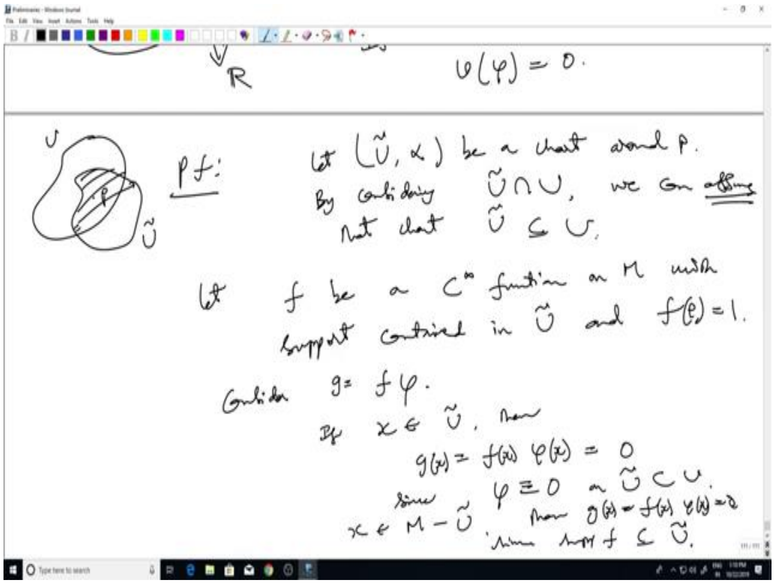
(Refer Slide Time: 25:35)



So proposition, let me write that again if U contained in M is open then dIP, P belongs to U, dIP from TPU to TPM is an isomorphism where i is the inclusion, inclusion map. So for this we need a small lemma, so the lemma is the following, let phi be a C infinity function on M such that phi of x equal to 0 for all x in some U where U is it some open subset of M containing P. So this is M, we have a P and have some open set and what I am, the hypothesis is that the C infinity function phi from M to R is identically 0 on this open set containing P.

Then the claim is that if V is any tangent vector to M at P then V of this map, this function phi is 0. So what we are saying is that even though the function phi may not be 0 on the whole manifold. If it is 0 in a neighborhood of the point P then if we act at tangent vector thought of as a derivation then it is V of phi is 0, and this should not be a surprise if one keeps in mind that after all we think of this action of V on phi as essentially taking directional derivative of phi (along) in the direction V. Now if phi is identically 0 in a neighborhood of P then all derivatives of phi are also 0, therefore V of phi is 0.

(Refer Slide Time: 29:14)

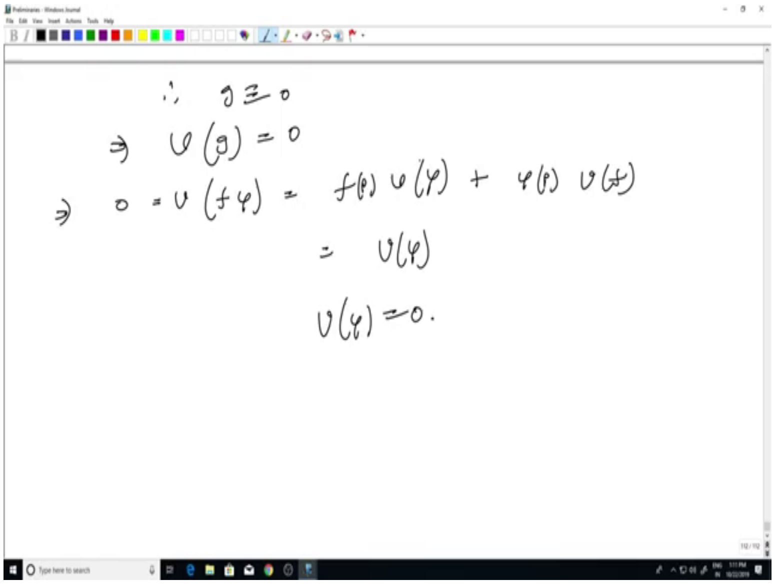


And in fact, one can easily make this precise working in local coordinates but rather let us prove it like this. So what I will do is, so let I have, I am already given an open set U and a P point inside that. So let us take another let me take a chart around the point P, let U tilde then alpha be a chart around P by considering U tilde intersection U after all they already had an open set U but the chart may be, may not lie inside U but that is not a problem. By considering this intersection, we can assume that the chart U tilde is actually contained in U.

Let F be a C infinity function on M with support contained in U tilde and F of P equal to 1. So I will show in my next lecture how to get such a C infinity function. This essentially follows from the discussion that we had about three existence of C infinity function and compact support in Rn. Now consider F times this original function, phi, let us call this G, this is a product of two function F times phi.

Well inside U tilde, if x is inside U tilde then G of x equal to F of x times phi of x is 0 because, simply because since this original function phi is identically 0 on U tilde, actually it is 0 on U so it is 0. And if x is in the compliment of U tilde, so let me put it like this M minus U tilde then G of x equal F of x phi of x is again 0 because this F had support inside U tilde, since support of F is contained in U tilde.

(Refer Slide Time: 33:27)



So basically this G is identically 0, therefore G is identically 0. Now U act V on this which implies V of G is 0, but V of G, now we can use Leibnitz rule, the way we are written V, V of F times phi equal to F of P, V of phi plus phi of P V of F. Well F of P is 1, so this is just V of phi and phi of P is 0 because P is inside U so we get V of phi is 0 which is what we wanted, so let me continue this next time. So we will stop here and I will resume this argument in my next lecture. Thank you.