An Introduction to Smooth Manifolds

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Lecture 16

Chain Rule On Manifolds

Hello and welcome to the 16th lecture in the series. In my last lecture, I had stopped at, I had towards end of the lecture we had discussed the derivative or differential of a smooth map between manifolds and this turned out to be, if we have a smooth map between manifolds then the differential at P turned out to be a linear map between their tangent spaces which we still have not proved a finite dimensional but that is not needed to talk about the derivative. Now, in fact to talk about to prove that the tangent space is finite dimensional I will need to talk a bit more about the differentials of smooth maps then we will all put it all lot of the various things together and get the finite dimensionality.

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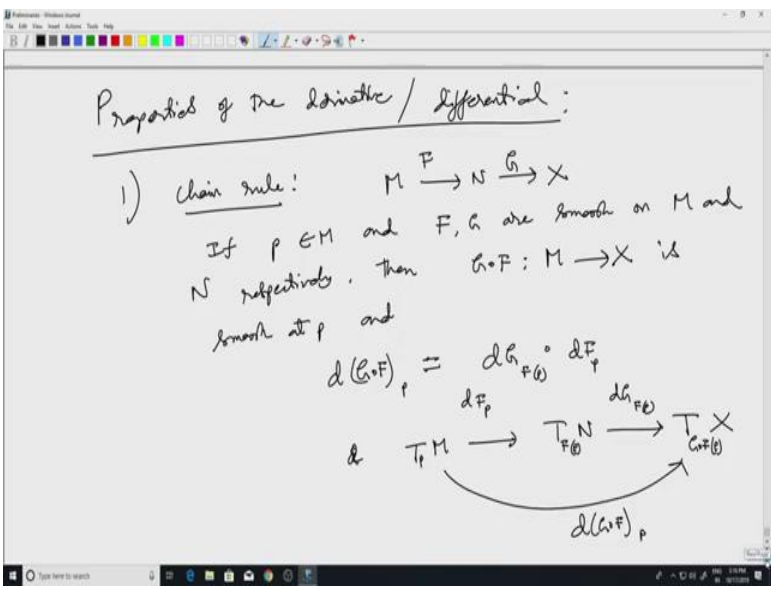

Properties of the derivative slash sometimes I will say it is the derivatives and sometimes it is the differential, it is the same thing. The first thing is that the chain rule, so I have three manifolds M, N and X. So let us say I have G and F two smooth maps. So if P belongs to M and F, G are smooth on M and N respectively then G composed with F from M to X is smooth at P. Once we know this map is smooth I can talk about its derivative, dG composed with F at the point P would be a linear map from TPM to T G composed with F of P at X.

So let me write it like this dG at F of P composed with dF at P. So there are three linear maps here this left hand side and two maps on the right side. Now such that G, so there are three tangent spaces as well TPM, T F of P of N and T G composed with F of P of X and this derivative is dG F of P, this derivative is dFP and what we are saying is composing this dG with dF at the appropriate points is equal to dG composed with F at the point P which is a linear map from here to here.

So I can go on, so this is a usual chain rule essentially except that now we have to keep the big difference between this and the case of Rn is that we have to keep track of the tangent spaces where this maps are defined the domains and the images of these linear maps are important. Even though it is going to turn out that all this tangent spaces have the expected dimensions and so we know from linear algebra that they are all isomorphic to Euclidean spaces of the appropriate dimensions, there is no what is called canonical isomorphism, natural isomorphism from different tangent spaces to Rn. So in other words the (iso) there is a choice involved in an (iso) identifying a tangent space with the right Euclidean space.

So this choice complicates things and the thing is that if one regards Rn as a manifold then however, then at every point there is a natural way of identifying the unique not unique there is a natural way of identifying the tangent space at every point with Rn itself which is not the case with an arbitrary manifold. So one has to keep track of this whenever we talk about differential we have to keep track of the domain and range vector spaces. So this is the chain rule.

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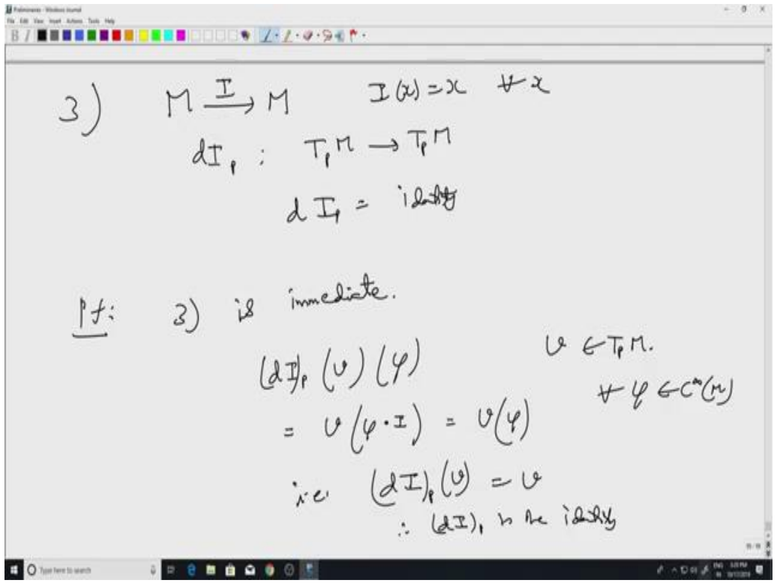

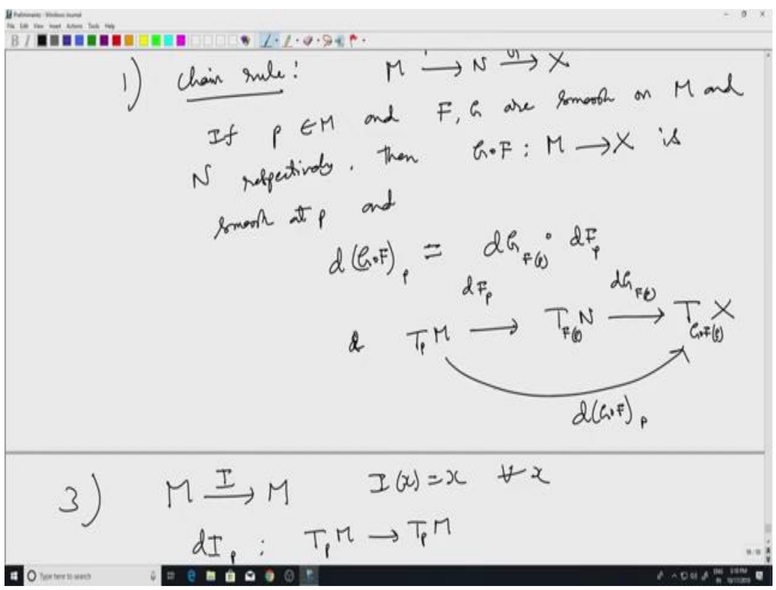

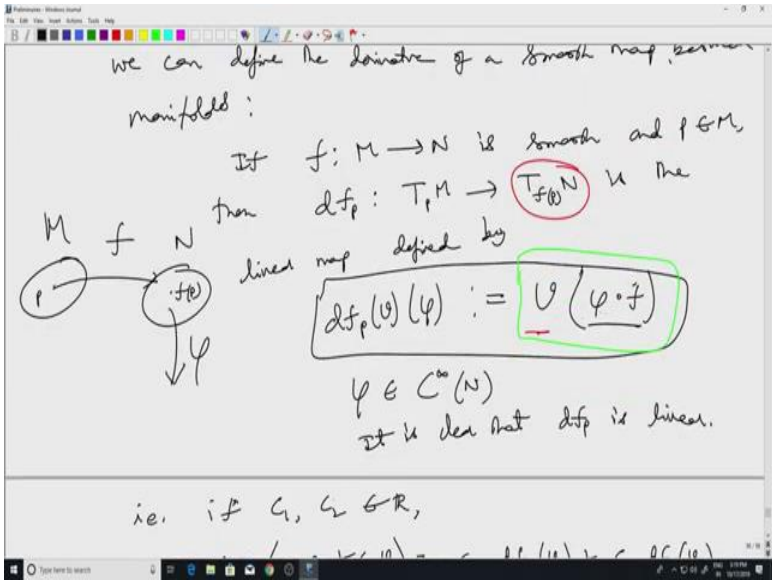

The next thing that I need is that if I look at the identity map from M to M so I is the identity I of x equals X for all x. And if I look at the derivative of this at any point P which is a map from TPM to TPM this is the identity map again. Thus if we start with the identity map between manifolds from the manifolds to itself, the derivative of that will be the identity map of the corresponding tangent space. And then so I need this, well this is not exactly a property, these two properties.

Usually these are, these two properties that come the behavior of a composition and what happens to the identity if the expected things happen then that it refers to as a functorial association. So associating a map, a smooth map with its derivative has this two functorial properties namely that the derivative of a, composition behaves in the right way and derivative of identity is identity. The three is immediate but let us do it any way so this is the definition of a derivative is about abstracts. So what does it, right, so what do we have to check? Well we look at the derivative dIP is an element of TPM not quite.

So dIP is a linear map from TPM to TPM, so it should act on V an element of TPM and which that will act on as a infinity function and by definition this is V acting on, so I just have to go back this notion of what I defined here V acting on phi composed with F. So here F is identity for me, V composed with identity. Of course anything composed with identity itself so I will just get this. And so in other words i.e. d the derivation dIP of V is the same as V itself, so since their action on in for all c infinity, since their action on any c infinity function is the same dIP of V is equal to V. So therefore dIP is the identity map on TPM. So here V is any T element of TPM.

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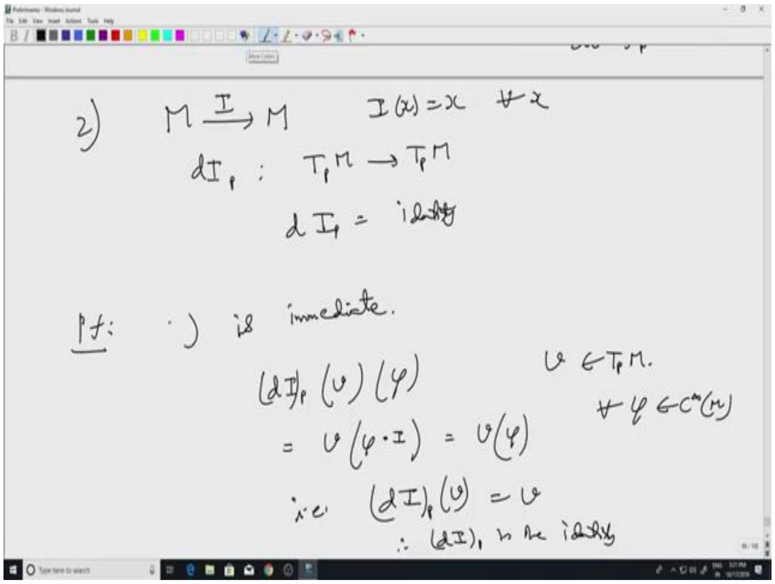

Now as for the chain rule, I do not know why I a small correction I do not know why I put 3 here this should have been 2 let say 2, so here 2.

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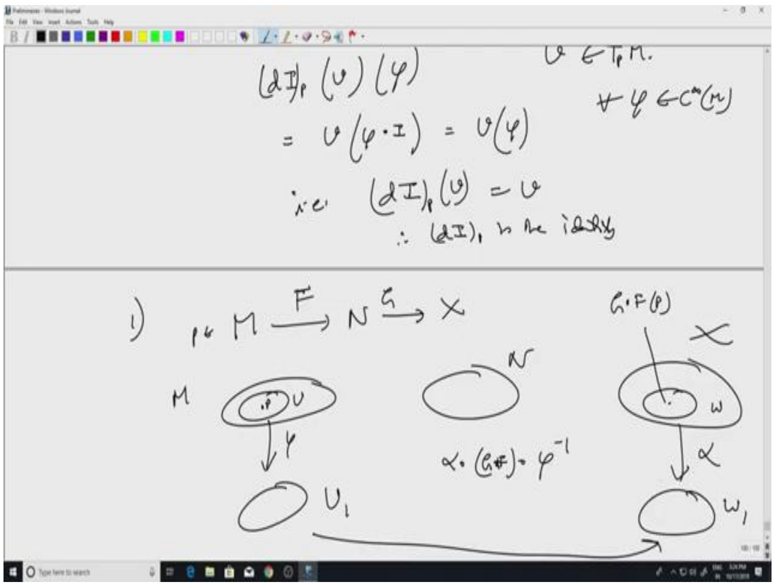

So now 1 is the chain rule, first let us see that composition of the smooth maps is smooth. So I have a F here and G here and I have a point P here. Now according to our definition of smooth maps of three spaces M, N and X, so I will start with a point P here according to our definition of smooth maps, smoothness amounts to saying that if I take, if I start with a point P to say that G composed with F is smooth I will have to construct a chart around P and a chart around so this point here G composed with F of P.

So I need a chart here and a chart here so that the map G composed with F in these local charts should be smooth as a map between open sets in Euclidean space. So now the way I construct these two charts is of course I will just look at, I have to use the hypothesis on the smoothness of G and F. Since F is smooth I am assured of, I just have to write this a bit clearly, so this is P. Since F is smooth I know that for any chart here and any chart here I can, the corresponding map is smooth and similarly for G.

Essentially so I will just start with, let us just start with some chart here U this phi will go to U1 and let us say this is W and the map is alpha, alpha W1. So I will just start with some charts here and I want to look at alpha composed with G composed with F composed with phi inverse that would be a map from here to here. So I am interested in checking that this map is smooth. So what want does is something that we have seen earlier namely you just sort of in this composition you put some, so let me move on to the next page it is getting bit crowded here.

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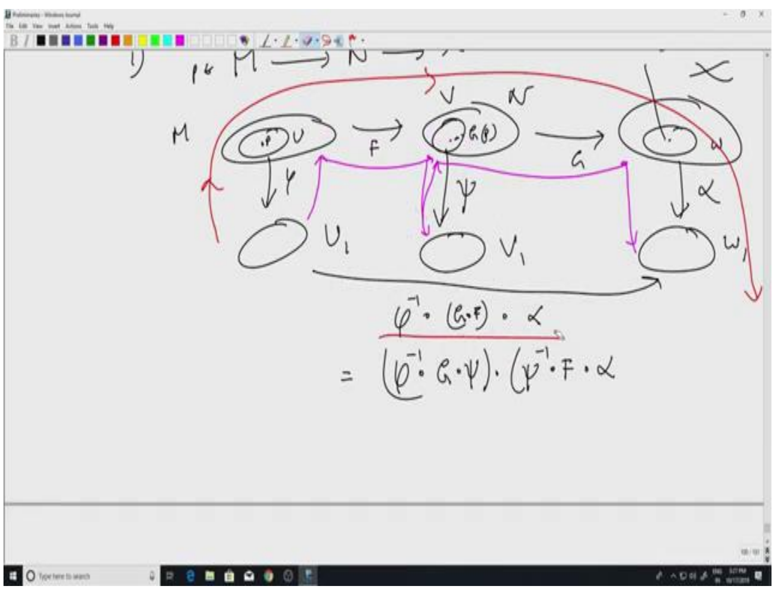

And let me move this one a bit. So what I have is alpha composed with G composed with F and then phi inverse. So the idea is just that, now there is a one missing point here which I have to now write down namely this is G of P. So let me take any chart I might have to modify it for now I will just take some chart V around G of P and then a map, chart map C to V1. U1, V1, W1 are all open subsets of appropriate Euclidean spaces. Now what one does is so one if in order to check that this is smooth, you just write this as phi inverse composed with, so instead of going, so this was G and this was F.

Instead of starting here, going here and ending up here, well actually not quite so I am starting here, going here go all the way to the right side so let me show it like this so instead of going like this which is what I am doing here, I sort of take a detour. So I will start here, go here, come here then go down and go back up again then follow G and then the rest is the same. So in other words I throw in this C and C inverse in between, so let me write it like this. So G composed with C composed with C inverse composed with F composed with, oh I need a C, yeah this is fine G composed with F.

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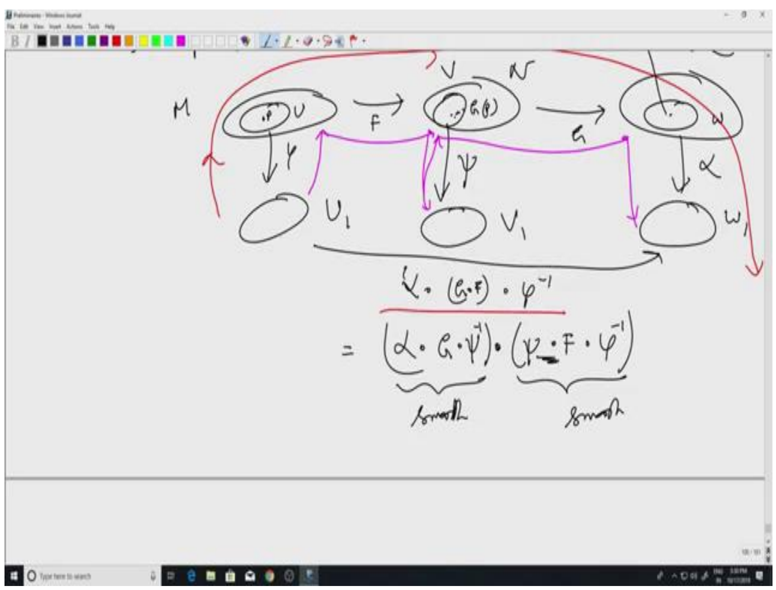

So I have to start with, oh I put, sorry I think I put it in the wrong direction. So it is actually I have to start, phi inverse should come on this side and alpha should come on the other side. So to start with phi inverse and then alpha, alpha is the last map. So I have to make a small change here as well. So to start with phi inverse and then this does not come here, start with phi inverse then go via F and then go via C, that is okay.

For the next one I start with C inverse go via G and the come via alpha. So this is essentially the idea and the thing is that I know that F and G are given to be smooth, so therefore this is smooth, this is smooth and the main point is that this notion of smoothness is just smooth in the usual Euclidean sense, after all this is a map between open sets in Euclidean space, so is this, all three maps, the left hand side and the two maps on the right hand side are naturally (Euclidean), open sets of Euclidean space, therefore we know that composition of maps is smooth.

Now the only thing which remains to be done is that one has to ensure as usual when I want to do this C of, so F is going to land here, F of, F is going to take this open sets somewhere inside N. But in order to do C of, F of, in order to do this composition here I need to ensure that this open sets, this charts are chosen so that F of U is contained in V, similarly in the next stage G of V is contained in W. So that can be done just by continuity considerations.

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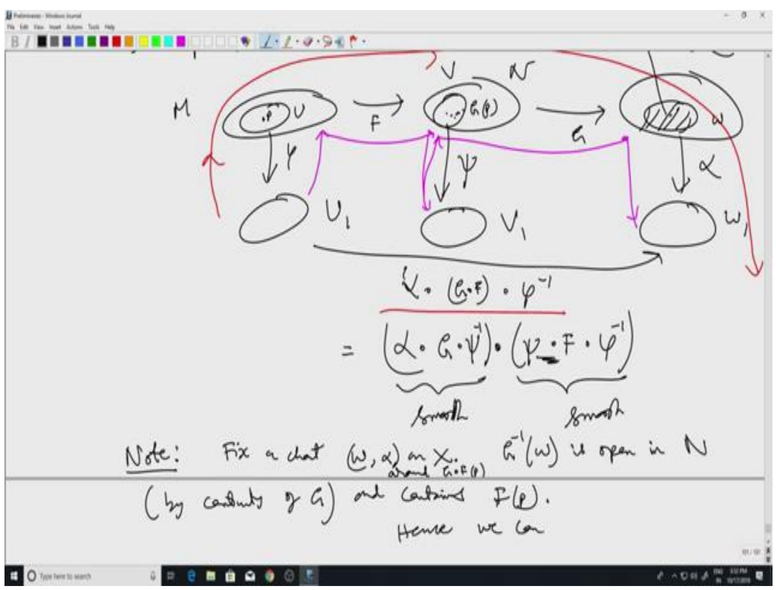

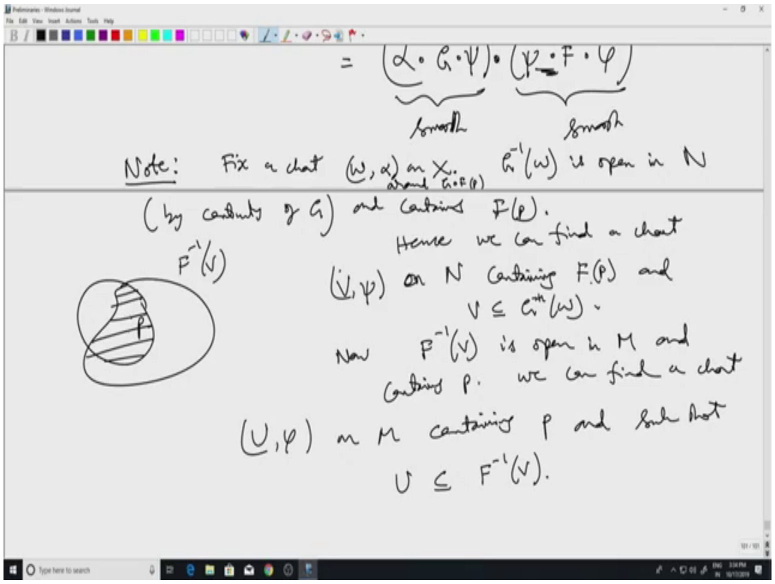

So note, so what one can start with in fact the starting we can start with W, fix a chart W alpha on x then you look at G inverse W the inverse image of that is open in N by continuity of G, by continuity of G. Fix a chart on x, G inverse W is open in N, what else, right and contains, fix a chart of course this chart around the given point G of F of P and contains, so when I take G inverse it will contain F of P this inverse image of this set is something here in N.

Since I get an open set hence we can find a chart V si on N containing F of P and such that this we can assume that this is contained in G inverse W. So when I take G of V, I will safely end up in W and once we have such a V then I repeat the same thing with F. Now F inverse V is open in M and again by continuity and contains P. Since V itself contains F of P which I remarked here we can find a chart U phi on M containing P and such that U is contained in F inverse V.

So F inverse V is some open set actually the same process is repeated twice here I will just deal with F inverse V, we have to do the same thing for G inverse W as well. F inverse V is contain in P we know that. Now if we want a chart which is contained in this open set we take any old chart around P that might go, that might not be contained in V but all we have to do is intersect that chart with the open set V and we still get an open set which has the, so in other words we still have a chart but now that chart will be contained in.

It is a small thing that one has to do in order to check that these maps that we have they are all well-defined. But the main idea is you just throw in this Si inverse and Si and that proves the. Now let me conclude by proving the chain rule, this has just regarding smoothness now the chain rule is, it is a purely, since our definition of derivative is abstract, the proof of chain rule also is kind of quite different from the way one would proof chain rule in multi variable calculus. In my next lecture I will show that after I show that this new notion of derivative coincides with the old one, one can regard this proof as give me a new proof of the classical chain rule.

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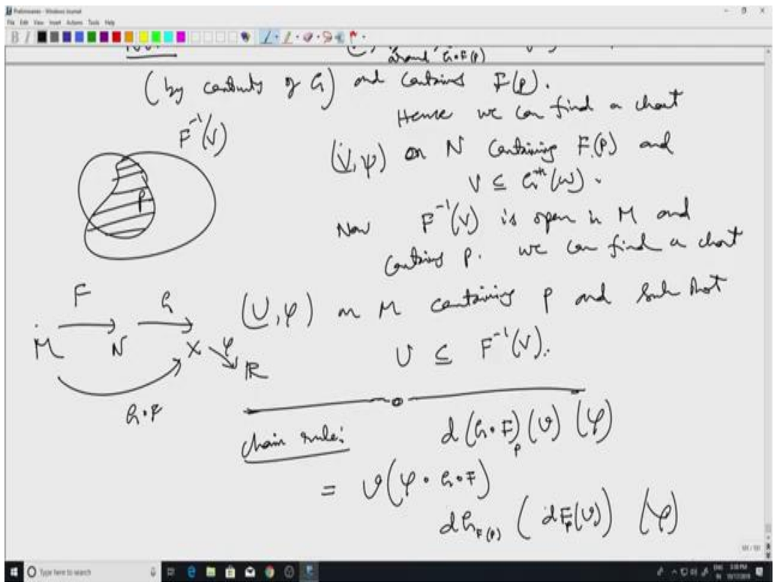

So the proof goes like this of chain rule so I want to do dG composed with F acting on V action on a function phi. This is, this derivative is at the point P, well by definition this is equal to, so again that picture so M, N, X. So this is G this is F and G composed with F and this function phi is a function on X and V is a derivation the tangent vector at M, so V acting on. So first I look at this function this is just the definition of the derivative of G composed with F now what I will do is, so I will just write it as, okay so this is the left hand side. Now let us look at the right hand side, now let me look at the right hand side, the right hand side is in the chain rule would be dG at F of P acting on dF at no dG at, yeah dG d F at P acting on V whole thing acting on this function phi, so this is RHS.

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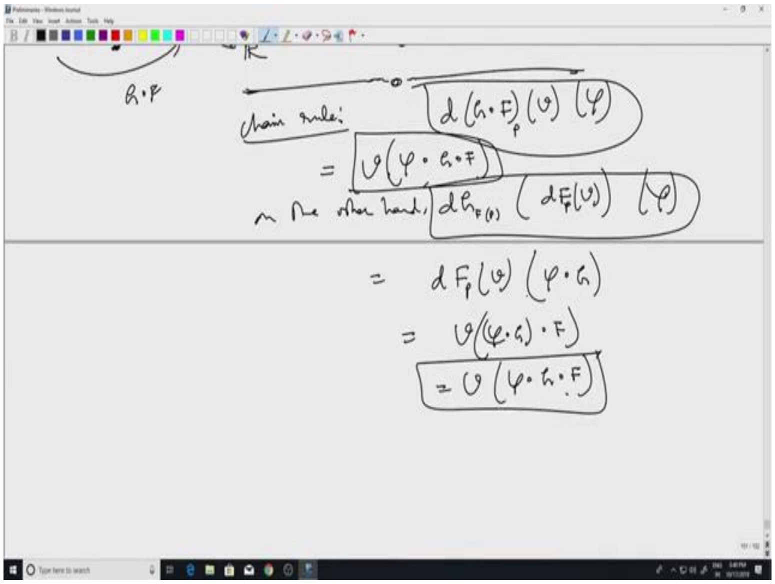

So on the other hand this stuff here, again what is this by definition this thing acting on this is dFPV acting on phi composed with G. Well phi composed with G is this map and dFPV is something here, a tangent vector here so I do apply the definition of derivative once more and then I get so this is equal to V of dFPV, so dFPV is something here. V composed with G is a map on this so I will be doing phi composed with G at composed with F which is the same thing as, so phi composed, I can just composition is associative it does not matter in which order you write, so I end up with this which is the same as what I started with.

So this is the derivative of the composed map that was this and the composition of the derivative is also gives the same answer. So it is look like we have not done anything at all it is just matter of algebraically playing around with symbols but this is indeed the classical chain rule as I will show next time, so we will stop here. Thank you.