An Introduction to Smooth Manifolds

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Lecture 14

Tangent Spaces

Hello, and welcome to the 14th lecture in this series. So I am going to talk about tangent vectors and tangent spaces, eventually on manifolds, but to begin with. Let us look at the case of Rn as a manifold.

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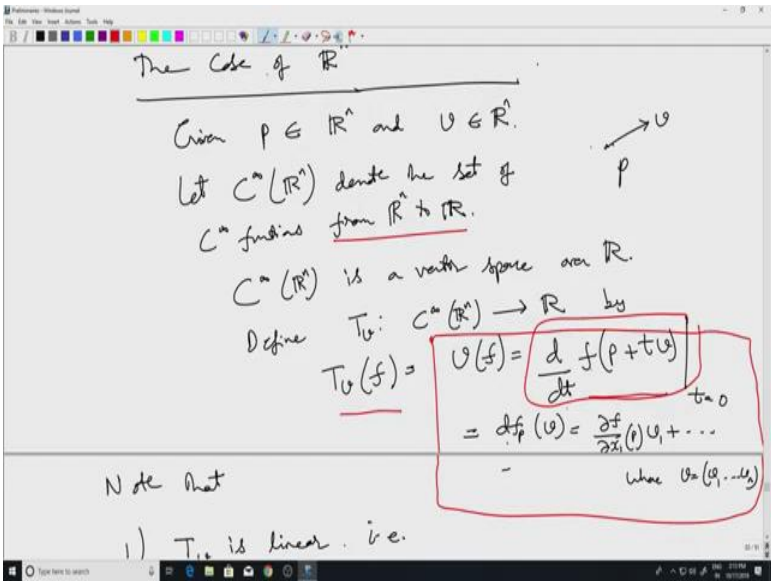

So given a point in Rn, we it is natural to think of a tangent vector at Rn at that point as any direction with the starting at that point. But, so here, let me redraw this picture. So given a point P, tangent vector would be just schematically one would draw it as an arrow based at P. But really, v is just any element of Rn. Of course, the key point here is that this point P is being thought of Rn has the point is that Rn has two kinds of structures.

One is as a manifold, or even more basically a topological space. If you think of Rn in that way, then an element of Rn would be called a point on the other hand Rn is also a vector space and from that point of view an element in Rn would be a vector. So, in this setting I have a point P and a vector v and where v is what we would called a tangent victory at P.

But this is not a I mean this is not a formal definition of a tangent vector in fact, we formally define a tangent vector what we do is the following. So, let us as before let us C infinity Rn denote the set of C infinity functions from Rn to R from Rn to R and C infinity Rn is a vector space over R so given a need to C infinity functions I can take scalar multiples and add them up.

And then we define corresponding to every V, which is now thought of, which is still an element of Rn, but now thought of as a vector, as an element of a vector space. I define Tv from C infinity Rn to R as essentially the directional derivative of f along the direction v at the point P. So, which we have denoted in my earlier lectures by V of f. And by definition, that is equal to the usual derivative of f when I restricted to the straight line that starts at the point P in the direction v namely P plus Tv.

So when I restrict f to that I get a function of one variable, namely T and I take the derivative of that by chain rule. That is also equal to, but not yet, I mean the we saw that this is equal to the derivative the differential of f as a linear transformation acting on v. And that is in fact, equal to the differential is just given it in by partial derivatives. So therefore, I have the last expression here.

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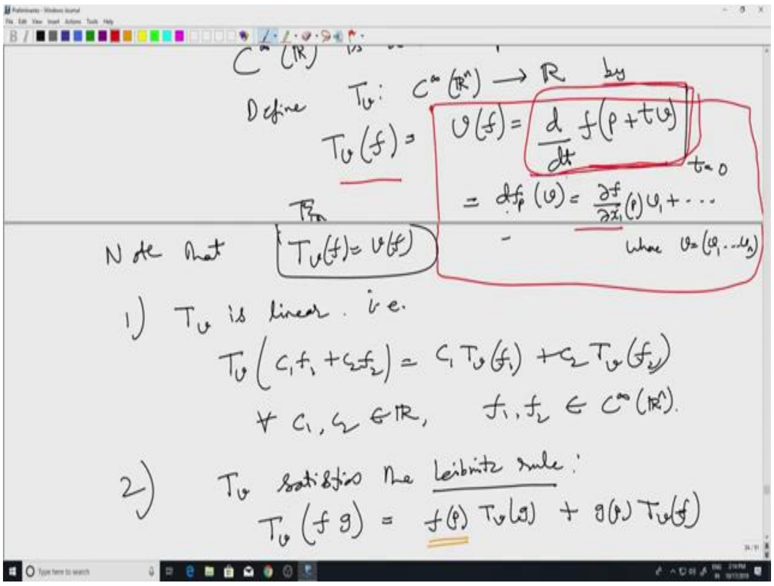

Okay, so the main thing is that I mean, we will need the other expressions but what we have to keep in mind is this T of f is Tv acting on fs the directional derivative of f along the direction v at the point P. Now, if you think of Tv in this fashion as a linear transformation from C infinity Rn as a map from infinity Rn to R then Tv is linear.

In other words, Tv of C1 f1 plus C2 f2 is C1 Tv f1 etcetera. And this follows immediately from this, this thing here the name that the definition of the directional derivative will immediately give the first linearity and that will also tell us that Tv is a, as Tv satisfies the

Leibnitz rule, the product rule for functions in other words, the directional derivative of a product is given in the expected way like this, what I have here.

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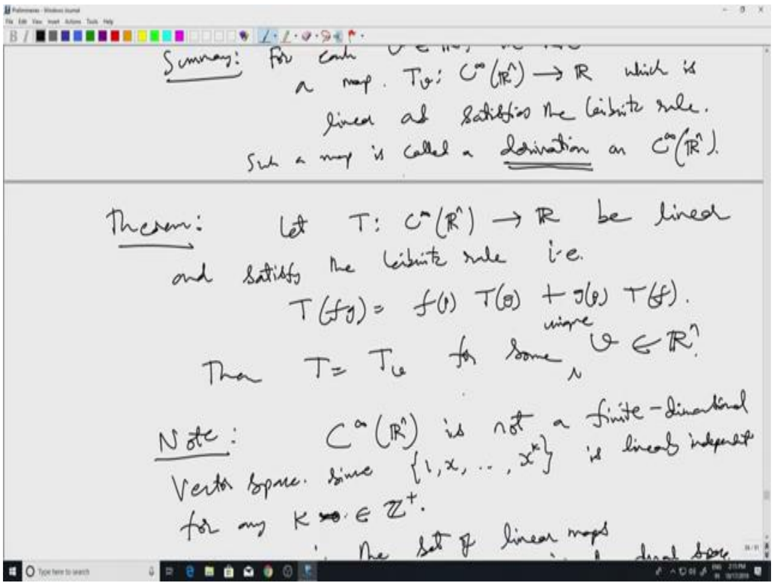

So, in summary, for each v in Rn we have a map Tv C infinity Rn to R which is linear and satisfies the Leibnitz rule let us call such a map as a derivation on C infinity Rn. So, linearity and the Leibnitz rule anything satisfying that will we will call it derivation. So, the main idea here is that every instead of talking about v in Rn we would like to talk about this Tv. So, in other words, we would like to say that corresponding to every derivation is essentially of the form Tv and for a unique v.

So, we would like to regard first, we I mean the, all this property shows that corresponding to every vector I get a derivation, but we would like the converse as well. So that is the theorem here. Let T from C infinity Rn to R here be linear and satisfy the Leibnitz rule which I have again written down then the claim is that T corresponds to the directional T is actually equal to the directional derivative of exactly one vector v in Rn. So this sets up a bijective correspondence between v, between Rn and derivations on C infinity Rn.

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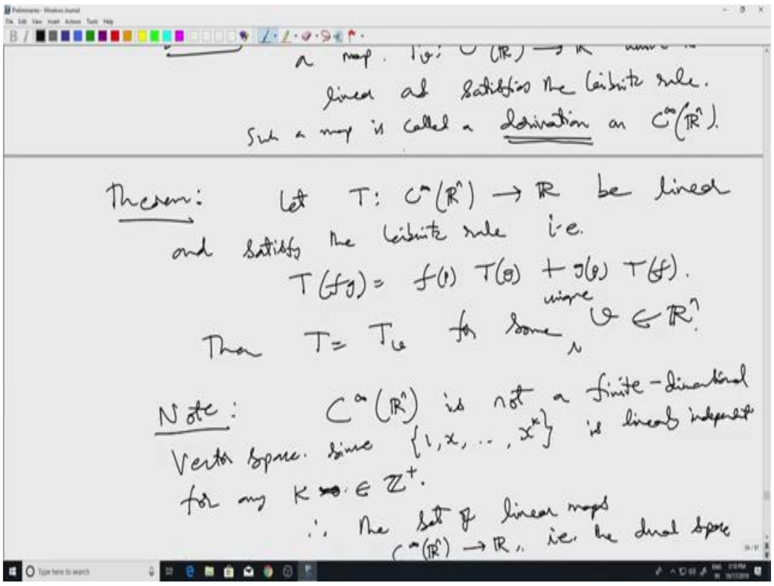

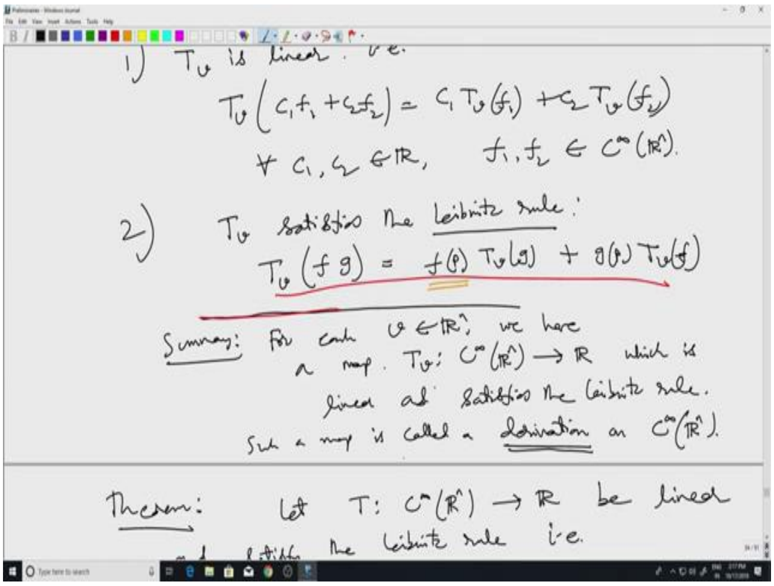

Now, one thing which we have to note here is that the C infinity Rn is not a finite dimensional vector space, since we can find linearly independent sets of arbitrarily large size one can just take monomials. And one knows that if it is not a finite dimensional vector space, if any one starts with something which is a vector space which is not finite dimensional then linear maps on such a vector space is also not finite dimensional.

However, we this condition we are looking at not just linear maps but linear maps satisfying this very special property of Leibnitz rule. This severely cuts down on the number of maps. And as it turns out, that is a content of this theorem that the set of derivations is in bijective correspondence with Rn.

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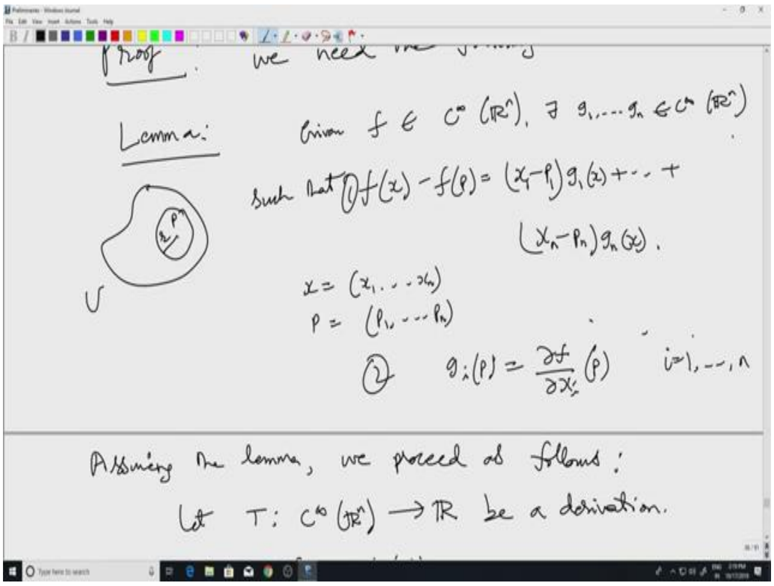

So, let us prove this, the proof is essentially in just on the following crucial lemma. So the lemma is the following. Given any C infinity function on Rn, we can find n C infinity functions g 1 g 2 g n, such that f of x minus f of P has this form x 1 minus p 1 times g 1 x plus x n minus Pn gnx. Here of course, as usual this xi denote the coordinates of x and Pi denote the coordinates of P, such that, this well, there is another such that two things happen.

One is this, the second thing is gi at P as the del f by del x i at P for all i. So, this is sort of like a Taylor's expansion for f around the point P. It is not really the usual notion of Taylor expansion, but something analogous to that. And this is the main property. So, let see why, if we have this lemma, we will get the result that every derivation is actually a directional derivative. So assuming the lemma we proceeds as follows. So I will prove the lemma, once we are done with the proof of main theorem.

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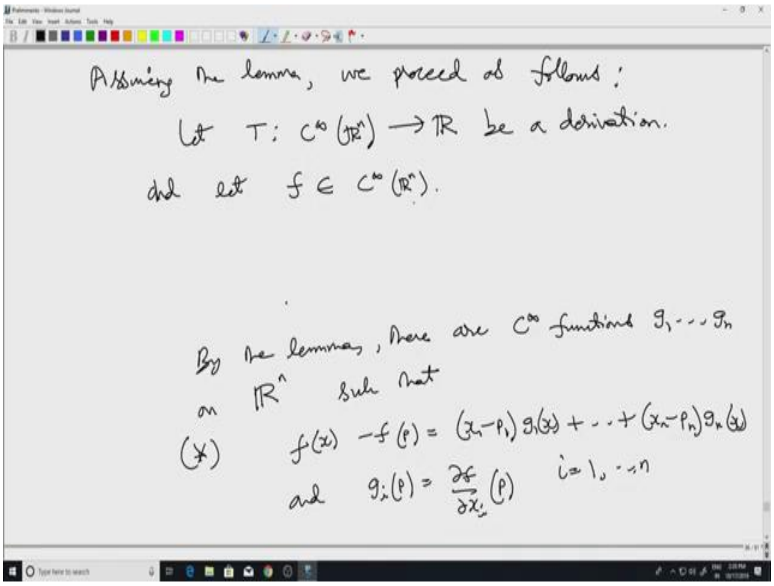

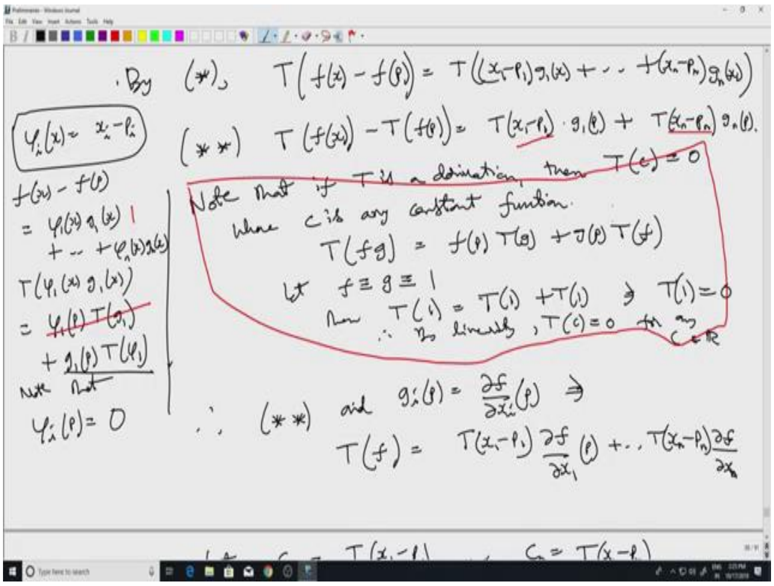

So, we start with a derivation and a function in C infinity, an element of C infinity Rn. So, we apply the lemma to this function f, and we get NC infinity functions g 1 g 2 g n, in Rn such that we have this expansion and the values of the gi at Pi are given by this. So, let us call this star this thing that this equation star and bi star. Now I am going to just act T on both sides, T of the left side equals to T of the right side, here the what we have to this f each of these terms here, this x 1 minus p 1 times g1 x etcetera there it is a product of two functions.

So I can this define a function vi x as xi minus Pi. If I define such a function, then f of x minus f of P as the form phi 1 x g1 x phi n x, gn x act on both sides. So here I have not in this what I have written down already I have not to use the phi notation, or just written it as x 1 minus p 1 x n minus Pn etcetera. So now using linearity, I will get two terms on the left side, where this f of p is regarded as a constant function. It is a constant map. So this minus this, and here I will get n terms when I use the sum. Then I will be using Leibnitz rule on each of these n terms. As I wrote on the left side, I have a product so and then I can use Leibnitz rule t of phi 1 x for instance the first term will look like this.

So it would befe1 P t of g 1, g 1 plus phi 2, well, no g1 P t of phi 1 now the thing about this phi I functions is that, note that, phi i at the point P phi i after all phi i x is x i minus p i just the difference of the ith coordinates. So, when I plug in x equals p, then I just get 0. So, that implies that this term does not even make an appearance. So what I am left with is just this g one P times T of phi 1 and that is what I have written here t of x 1 minus p 1 times g 1 P etcetera. Now, here on the left side, we have in order to get rid of the second term T of constant we have to make a small observation, that if you have any derivation then T of the constant function C is 0 where C is any constant function and this is seen, this follows immediately from the from the Leibnitz rule, so I start with a Leibnitz rule in general.

And let us take f and g to be the constant function 1. That would imply that T of 1 here, I will get 1 here I get one so T of 1 is two times T of 1 therefore T of 1 0. Of course, if T of the constant function one is 0 T have any other constant function is 0 by linearity. Now, so the, the small observation here, this thing here will imply that the second term T of fP is goes away. So T of f would be equal to all this right hand side. We can also simplify the right hand side a bit more namely, we know exactly what g 1 P g 2 P are, the lemmas tells us that g I p is the del f by del x I and P. Therefore, I can write T of f is T of x 1 minus p 1 del f by del x 1 etcetera.

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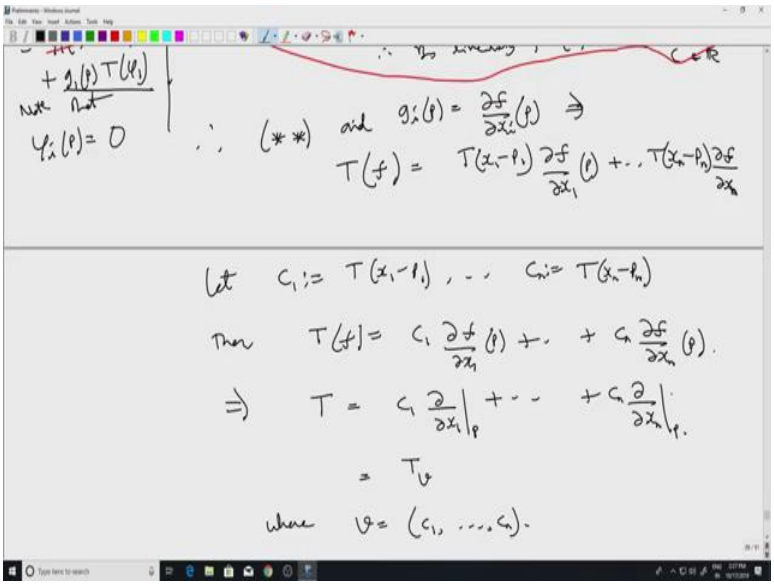

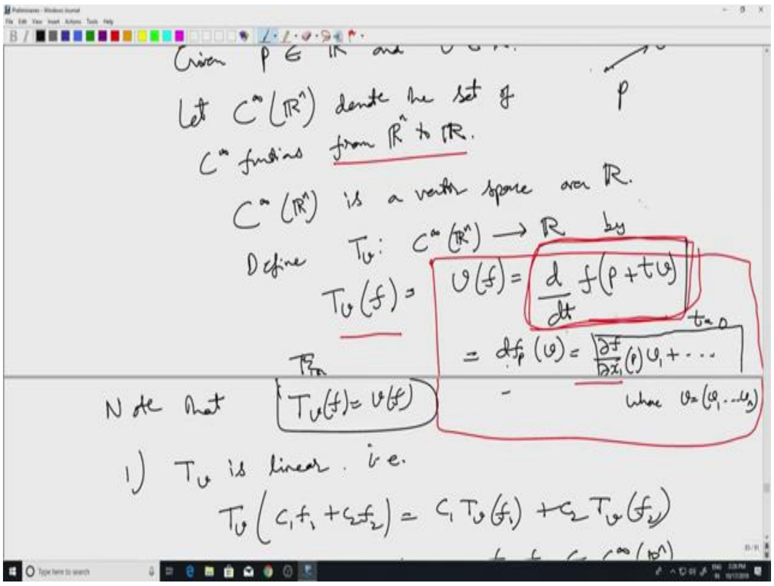

Let know, given the point P, these are just some numbers that we get and of course, given the point P and the derivation T I get some numbers. C 1 equals this Cn equals this. So, let us define so what I am doing here is that I am defining this C 1 C 2 etcetera. Once I have that, then I just rewrite it as T of f is C 1 del f by del x 1. So, what we have done is we have essentially shown that we started with an arbitrary derivation. We have just shown that it is a linear combination of partial derivatives purely from the algebraic properties of linearity and Leibnitz rule.

And once we have this, then we are essentially done because this is nothing but in fact, I do not have to write this at all. So at this point, I can just say that this is this thing on the right hand side is exactly Tv of f where v C1, C2, Cn, because remember that the directional derivative of v of f along the direction v was exactly this expression here, as I have written earlier. So I am talking about this last part here. So this is what I am using.

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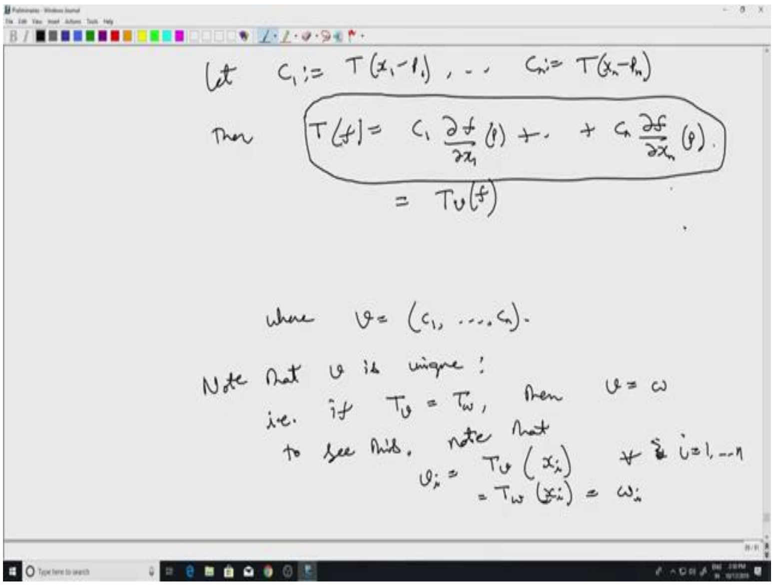

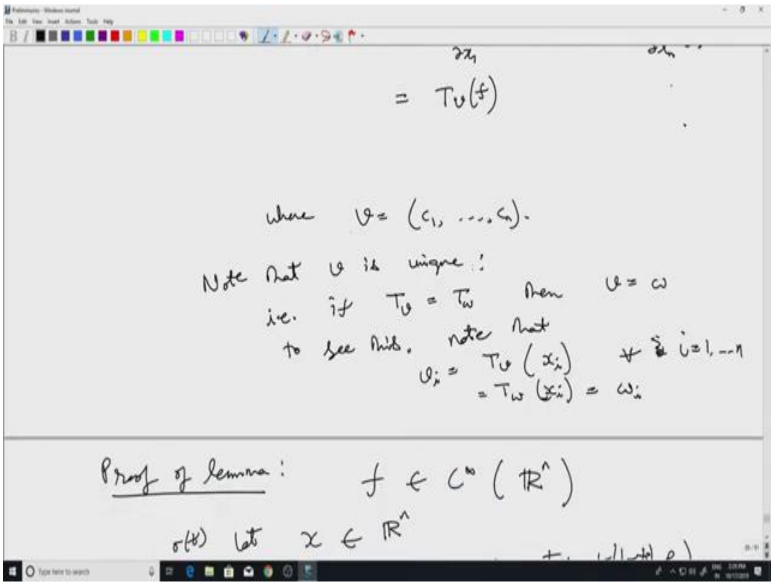

So, we found a v starting with an arbitrary derivation, we found a v such that T f equals t v of F for all f there was no assumption on f. And of course, it should be noted that this V, as I said earlier, is determined once you know what T is once you given T and a point P. f obviously f should not play a role here. And the next thing is to observe that this v is unique, this is almost immediate from what we have discussed so far.

So, I cannot have T f equals Tv as well as equal to Tw for two different vectors. The reason is that if Tv equals Tw for some v and w in Rn, then v necessarily has to be equal to w. And reason for this is that, if you can recover the coefficients of the vector from the directional derivative operator just by acting the directional derivative operator along these coordinate functions xi. So, Tv of xi again, all I am doing is I am just using this, this way of looking at the directional derivative. So if f equals x i, only the ith term will survive, all other partial derivatives will be 0. Therefore, I get the coefficient, the ith term will survive, and it will be just equal to Ci. So here, it is vi equal to T x i equal to T w, xi equal to wi.

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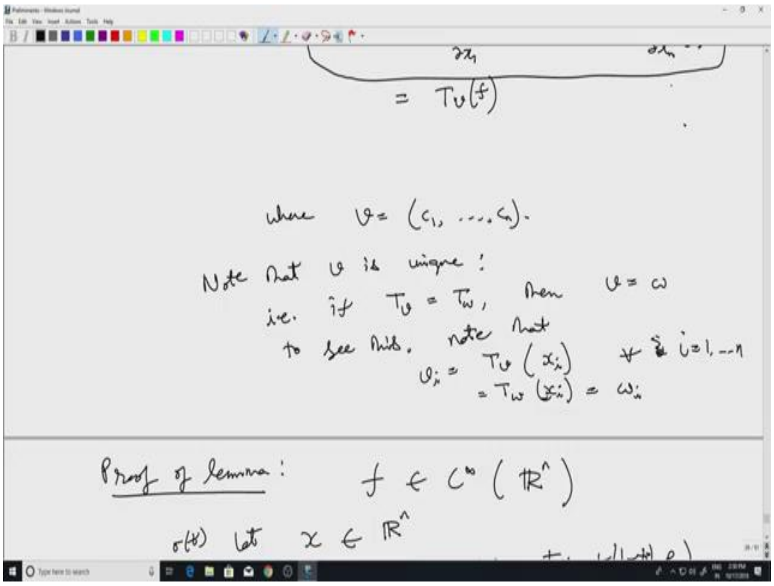

In the previous thing I had put v equal to C1 C2 Cn, here I am using the coordinates of v to v, v1, v2, vn, and I mean it is not that important. The point is, the coordinates of the vector can be recovered by knowing the directional derivative operator, all it so that proves what we wanted. So, that we have set up a correspondence between derivations, bijective correspondence between the derivations and elements of Rn.

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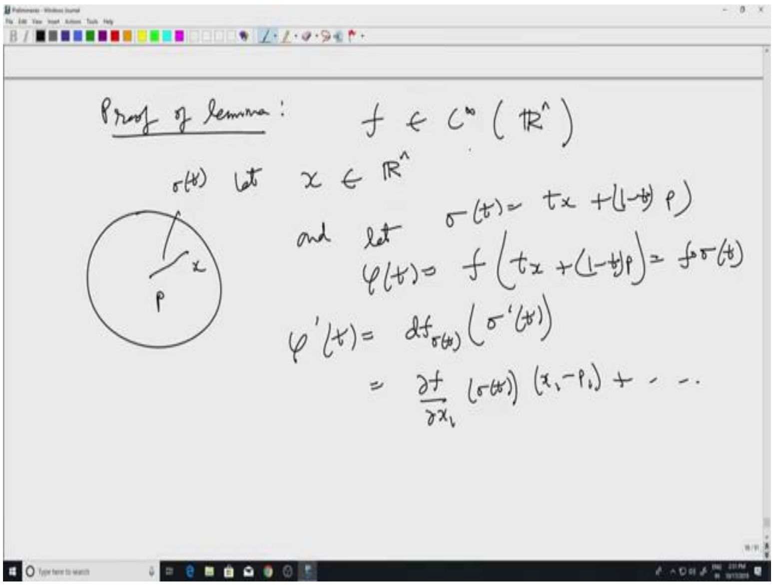

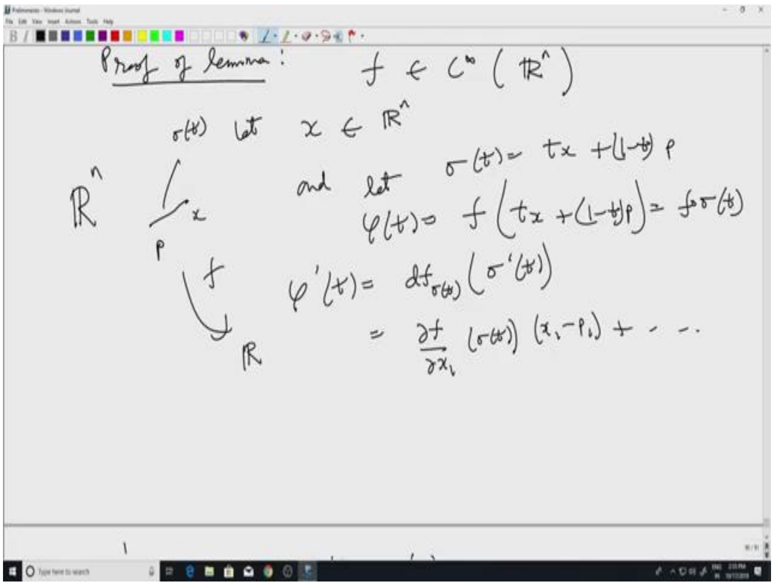

So, but that still leaves the lemma, which I will prove now, so I will us start with C infinity function the goal is to find g1, g2, gn, such that and of course, given the point P, so goal is to find g1, g2, gn with the appropriate equations holding. So, to define this gi functions to start with any point x in Rn and then what I will do is look at the straight line sigma T joining so there is no need for this bracket. Let sigma t be the straight line joining x and P. So it is going to start when t is 0, I will start at p when t is 1, I will end at x.

Let me define in another, so this is a map from, sigma t is a map from the real line to Rn. Now I am going to compose this sigma t with f. So in as, in other words, restrict, f was already a function from here to R. So this entire thing is Rn. I am going to restrict f to this line. So f of sigma t, which has this form. If I take the derivative of phi I will just get chain rule tells me that there is a composition of two functions. So chain rule tells me that it is derivative of f at sigma t acting on sigma prime t. And, again, writing it out in coordinates will tell me that it has this form.

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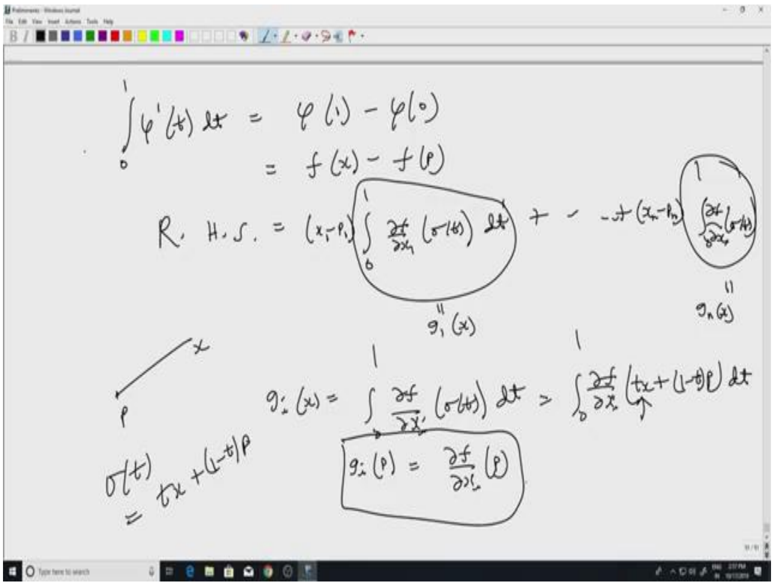

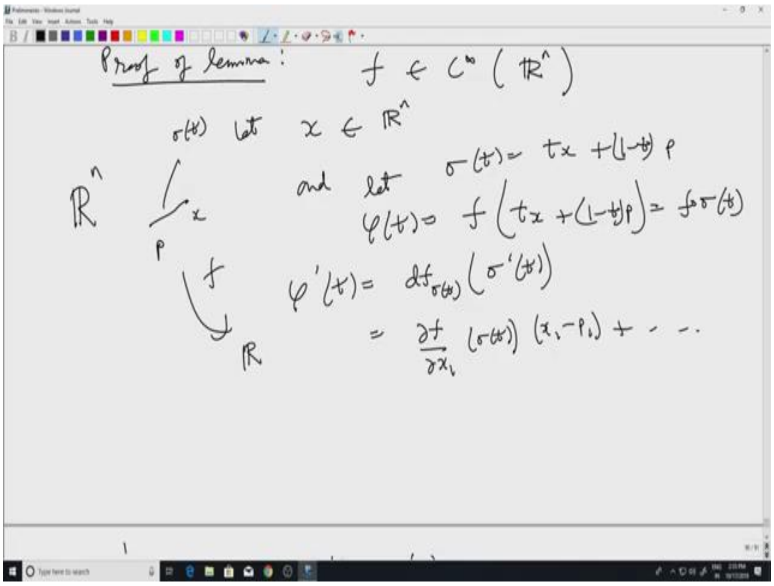

Now let us integrate, now that I know derivative of phi, let us use the fundamental theorem of calculus to and integrate out this. So when I integrate this between 0 and 1, I will get a phi of 1 minus phi of 0. But as we observed, when t is 1, sigma t is P. No, sigma t is x. The line is ending at x. So and when t is 0, I get P, so I just get this f of x minus f of P. On the other hand, no, this is not the LHS, this is the RHS, the right hand side is given by, I just integrate out all the n terms here. And note that this, the each term is a product of two things. One is this partial derivative evaluated at sigma t, the other term is xi minus Pi.

Well, if the xi minus Pi do not depend on T. So I can put it outside the integral sign like this and just integrate out this partial derivative. And I do this for all n of them. And this is what I call g1 x, g2 x, etcetera. These things here, it is not apparent from the way it is written that where the dependence on x is coming from. But it is clear if you remember that sigma t is the straight line joining P and x, or rather more explicitly, sigma t is in fact, given by this formula, tx plus 1 minus t times P.

So here the dependence on x is clear. So that is what I am plugging in, in this. So here it is 0 to 1 del f by del xi at tx plus 1 minus t times P dt. And finally, the last statement about gi was that gi at P equals del f by del xi at P. And that is clear because when I plug in x equals P here, then this I just get, well, tx tp and minus tp cancel out and there is a p here. So, I just get the del f by del x at P. So, the dependence on T goes away. So, I can put it outside the integral sign and integrate 1, integral consider 0 to 1 dt, which is 1. So I end up with this.

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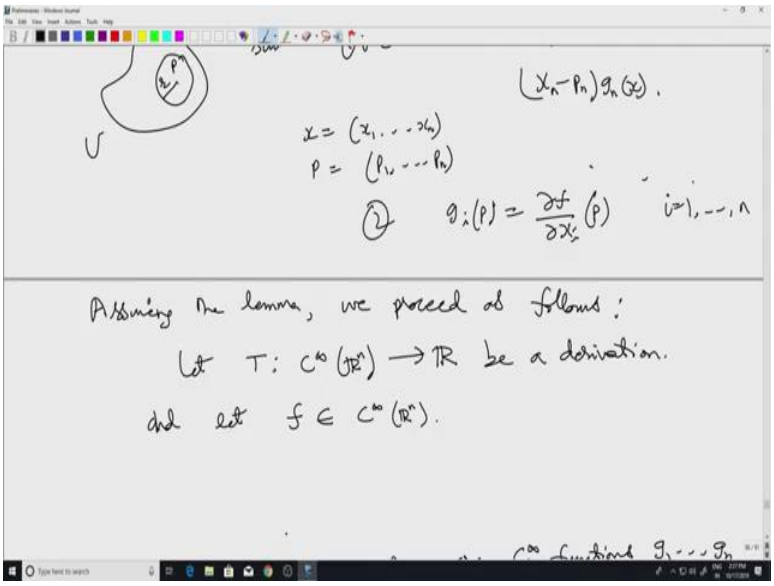

So, that proves the lemma and therefore we have the whole, the theorem as well that every derivation is actually the directional derivative of a unique V in Rn. And we will use this to this is going to be the motivating, sort of motivation behind defining tangent vectors as derivations on manifolds. So, we will stop here in the next class. I will see how all this can be transferred to a manifold. Okay, thank you.