Linear Algebra
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Lecture - 09
Gauss elimination

Come back to this lectures on Linear Algebra. Remember in the last lecture we were discussing system of linear equations.

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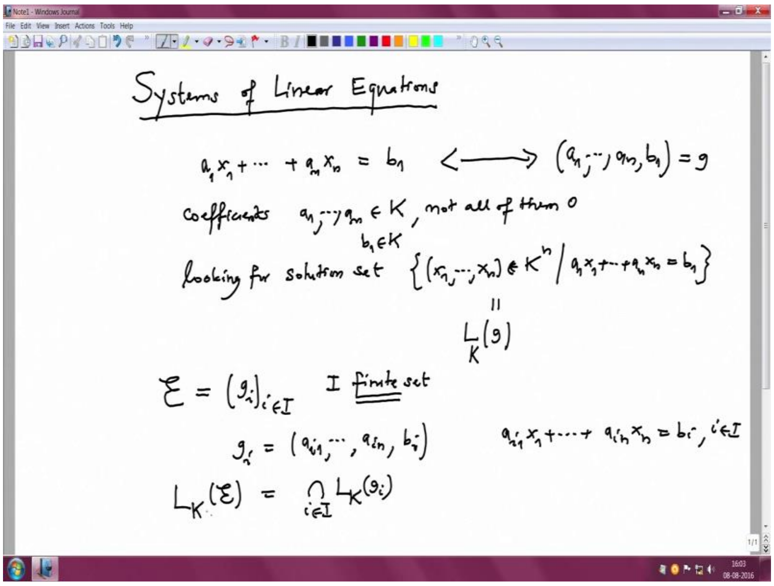

I will quickly recall what we have discussed last time, first of all the linear equation we will denote a 1 x 1 plus plus plus plus a n x n, equal to some b 1 where these coefficients a 1 to a n are element in (Refer Time: 01:19) K and not all of them 0. So, at least one co efficient is non-zero and b 1 is also in K and we are looking for solution set. So, that is we are looking for all the tuples x 1 to x n in K power n such that this equation holds, and we want to find a solution space how big it is and so on, and more importantly the structure on the set of solution set, set of solutions.

I want to improve the notation little bit so that it gives us more possibility to generalize also. So, instead of writing in the equation form I will just identify this with a n plus 1 tuple. So, for us a linear equation we are denoting by tuple a 1 to a n comma b 1, and we call it. So, for example, g and these space solution and we are going to denote L g when also we evolve to insist on the field that also we will write here LK. So, that will just to show that all these coefficients are in the field K, because sometimes it will be necessary to compare solutions with a bigger field and a smaller field etcetera etcetera. So, a general system of equation finitely (Refer Time: 03:40) equations I will denote therefore, E and the equations are g i i in I i is the finite set.

So, each g i looks like a i 1, a i 1, a i n comma b i this is i-th equation. So, if you want to write in equation you should write like this a i 1, x 1 plus plus plus plus a i n x n equal to b i and i varies in I. Actually I will not deal with arbitrary case, but sometime later when I have a enough opportunity I will also tell you there is no necessary to consider only finite set of linear equation we can deal with as many as one wants. So, it is clear that the solution space of the system E is nothing but the intersection in the solutions space of all the g i's this is clear because these means all the equations are satisfied these are the tuples were all the equations are satisfied simultaneous. So, that is intersection.

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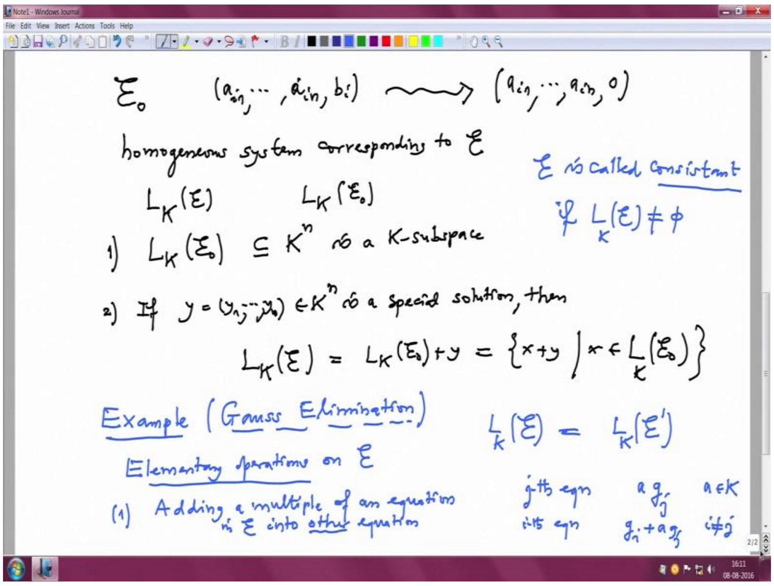

Next is a when we write E 0 that is instead of the equation a i 1, a i n comma b i instead of this we will replace this equation by just put b i equal to 0; a i 1, a i n comma 0. So, this is called a homogeneous system corresponding to E, and we want to relate this 2 solutions say its L K e and L K E 0; first thing we have noted that is LK E 0 which is a subset of K power n is a K sub space.

So, it has a nice structure; that means, if we have 2 solutions of the homogeneous system then their sum is also solutions and their scalar multiple is also solution, this is we have checked last time and. Secondly, somehow if you know one solution, if x prime if y is y 1 to y n in K power n is a special solution I just call it special solution because we have no formal method to find a solution, but somehow if you find a solution and then call it a special solution then we know all then LK E 0 is translation of these solution space of homogeneous system plus y; this means to all this solutions of E 0 add y to that. So, these are all x plus y where x is varying in solutions space of E 0, this is again very easy to check you just plug in this in equation then you check that.

Student: (Refer Time: 08:22).

What happened?

Student: (Refer Time: 08:24).

Ok.

Student: (Refer Time: 08:28).

Ok.

Student: (Refer Time: 08:30).

Sorry, what do what you want to do?

Student: (Refer Time: 08:34) nothing. So, that just mistake LK E 0 instead of LK E 0 its LK e.

Sorry, this is LK E.

Student: (Refer Time: 08:45) yes you can start fresh you can start fresh.

L K E is a translate of LK E 0 and LK E 0 plus y is by definition it is all the vectors in LK E 0 add y to all of them, so that is the solutions to this. So, this is this this structure is this structure will allow us to draw pictures in the in the plane, but this drawing picture I will postpone till I introduce the concept of a fine spaces. Now I want to illustrate this method by one concrete example and which is also known as gauss elimination. So, let us write it as example which is gauss elimination this is the method to find a solution space, this is a due to gauss that is what it is named after the gauss I also want to also introduce language here which say that we will call E sorry we will call I will write it here E is called consistent or E is said to be consistent if one of E is non-empty; that means, all the equations have at least one common solution then we will call the system to be consistent.

So, this process of elimination will tell both the things together namely, it will also decide whether their system is consistent or not. And in that case we will write down all the solutions by a method which is known as gauss elimination, it is like eliminating the variables. So, note that we are going to perform. So, the elementary operations on the system e what does that mean; that means, so there will be 3 kinds of elementary operation, we will perform first one of them is adding a multiple, multiple means scalar multiple of an equation in E into other equation.

So, this means typically I will apply I will multiply j-th equation that is we are writing it g j that we multiply by scalar a and add it to the i-th equation. So, the i-th equation will now become g i plus a g j, and i is not i is different from j that is here I have written other. So, first of all note that if I do this operation and if I call the original system to be e and the mu resulting system to be e prime the solution space will not change this is because if we know the solutions some tuple is in solution space of e then both g j and g i vanish there and a new equation now only the i-th equation is changed it is replaced by g i replaced by g i plus a g j. So, both are 0 then the new equation also will be 0 and the remaining equations are same.

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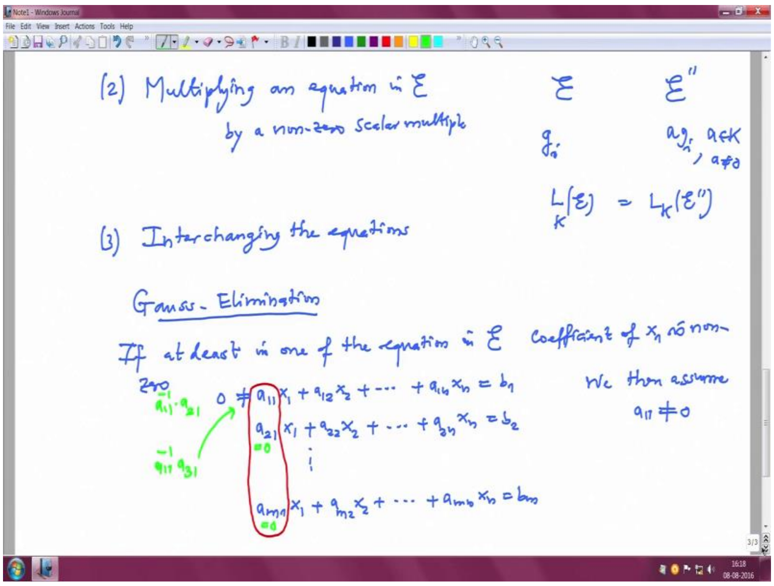

So, therefore, in this process the solution space have not change. So, we are going to do this operation many times this is one of them, second one multiplying an equation in e by a non-zero scalar multiple. So, again if the original system was E and the new system is become E double prime then in this i-th equation g i if you have replaced by a times g i, where a is non-zero then the solution spaces are not changing again note this this is very important to note. Because we are working in a field and therefore, non-zero if I take a solution of E g i the 0 there if I plug in this this equation in g a g i x is zero, but a is non zero. So, g i is 0. So, this means the solution space is not changing.

Third one interchanging the equations; this is the easiest of all because if we interchange the equation interchange the order of the equation then the solution space of this system does not change. So, we are apply we are going to apply this elementary operations and convert the given system of linear equation into a nice one, where we can after looking at the new system we should be able to detect the solutions. So, this is this process is well known, this is the process I am going to explain below is called gauss elimination.

So, at least one of the equation you have bunch of equation and at least one of the equation the co efficient of x 1 will appear to be non-zero, because if it does not appear then we do not need x 1 in the system; that means, we reduce the number of variables. So, at least we will assume or if at least in one of the equation in E coefficient of x 1 is non zero. So, let me write down the equations in E that is a 1 1 x 1, plus a 1 2, x 2 plus plus plus plus a 1 n, x 1 x n equal to b 1 second a 2 1 x 1 a 2 2, x 2 plus a 2 n x n equal to b 2. So, on the m th equation a m 1 x 1, plus a 2 2 x 2 no a 2 n, a m 2 plus plus plus plus a m n x n equal to b m. So, at least one of these coefficient one of these coefficient equation is non-zero. If all are 0 then we do not need x 1 when we go to x 2 and so on. So, even if it is in between I am going to interchange the equation and bring it on the top position.

So, we then assume a 1 1 is non-zero. Once it is a non-zero element in a field I will multiply this equation I leave the first equation to kill all these coefficients I will make this 0 this 0 that is simply because I will multiply the first equation by for example, if I want to make this as 0, I will multiply the first equation by a 1 1 inverse times a 2 1 and subtract it from the second equation, I replace second equation by adding this multiple the second equation and similarly I will multiply a 1 1 inverse a 3 1; third equation or the first equation I will multiply by this and subtract it from the third equation and so on. So, when I do this. So, all the equations will become 0 other than a 1 1.

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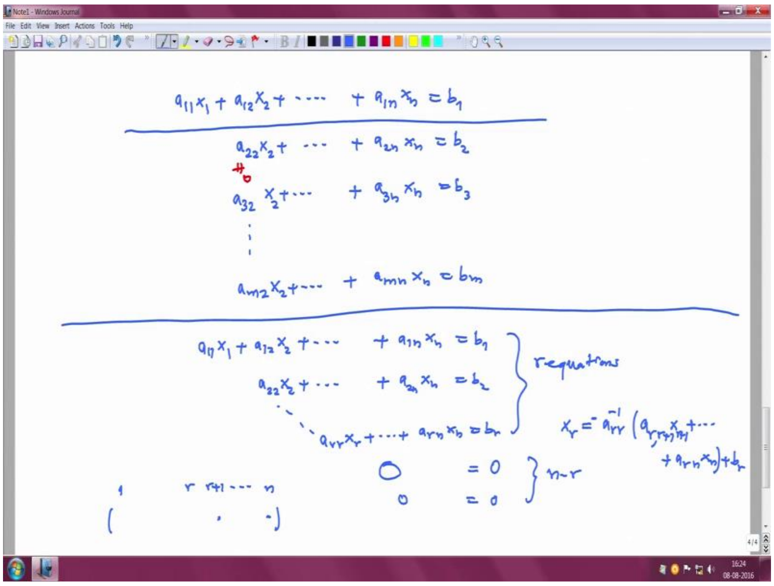

So, the new resulting system will have equations I will just write equations only. So, a 1 1 here x 1 plus a 1 2 x 2 etcetera etcetera etcetera a 1 n, x n equal to b 1 and this second equation now would have changed, but I still call it the coefficient will change, but I will still the new coefficient also I will call it by a i g s. So, this will become a 2 2, x 2 plus plus plus plus a 2 n x n equal to b 2 and a 2 3, x 2 plus a no it is not 2 3 this is a 3 2, a 3 n x n equal to b 3 and so on, a m 2 x 2 plus plus plus plus a m n x n equal to b m; this will be our new system and this is equivalent to the first one, equivalent mean they have the same solution space.

Now, I perform the same I do the same thing I do anything the equation one, and look at the remaining system which a system in a x 2 to x n and I again repeat this repeat means choose I will renaming I will call I will assume that this coefficient is non-zero, if this is 0 i go to the next 1 and so on and I perform this. So, after (Refer Time: 22:13) I am going to arrive at the system which is like this a 1 1 x 1 plus a 1 2 x 2 plus plus plus plus a 1 n x n equal to b 1 and again I will interchange the equation or the rename the variable then I will get a 2 2, x 2 plus plus plus plus a 2 n, x n equal to b 2 it will go on for some time a r r x r plus plus plus plus a r n, x n equal to b r and then I do not have any more non zero coefficient to choose. So, below that this side will be 0 and of course, this b s are also changing.

So, the moment I find a non-zero on the right side then it is immediate conclusion that this system is as is not consistent; that means it has no solution. So, we have to get like this. So, the remaining these are the only r equations and the remaining equation n minus r equations are 0 all the coefficients are zero, but then; that means, in the solutions if i solutions are tuples. So, from first r coordinates 1 to r, and r plus 1 to n these coordinates I can take arbitrary r plus 1 onwards I can take arbitrary coordinates, because when i plug them their coefficients are 0 here.

So, its trivial that any tuple will be solution provided I can compute the first r coordinates; and first r coordinates are computed by successively by this equation row first start with the last equation that is r-th equation from here I will write down what is x r. X r is because a r r is non-zero x r is a r r inverse times minus because I should (Refer Time: 24:40) to the other side this minus a 1 minus we have already taken out.

So, a 1 1 x 1 not a 1 1 a second time here, that is a r r plus 1 x r and so on go on till a r n x n this and this b r as it is plus b r b r no this is x no no this is correct because this one this one sorry I should take a measure this is x r plus 1 and so on. So, now, I got the x r from this equation, earlier equation I will take x r minus 1 and plug in this x r there. So, that way I get these solutions. So, let me do one numerical example to do this process understandable.

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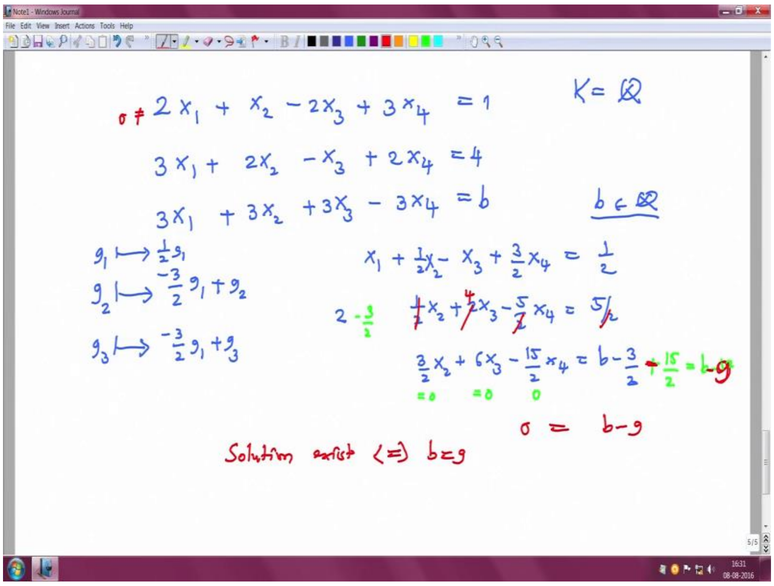

So, suppose i take 3 equations in 4 variables. So, I will written down coefficients here 2 x 1 plus x 2 minus 2 x 3 plus 3 x 4 equal to 1. Second equation is 3 x 1 plus 2 x 2 minus x 3 plus 2 x 4 equal to 4, 3 x 1 plus 3 x 2, plus 3 x 3 minus 3 x 4 equal to 6. The field we are working is that as field of rational numbers so; obviously, this element itself in non-zero. So, I will leave that I will take this element which is non-zero and you this to kill this equations so; that means, I am going to replace to killing this I would to multiply this equation by half and multiply by 3. So, multiply this first equation by. So, I will write here minus 3 by 2 the first equation add it to the second one. So, g 2 we will replace by this this new equation, g 3 we will replace by same minus 3 by 2. g 1 plus g 2 g 3.

So, I will just write the coefficient. So, I will. So, I could multiply the first equation by half first equation g 1 replace by half g 1, to make the coefficient 1 there. So, the first equation will become x 1 plus half, minus half x 2, minus x 3, plus 3 by 2 x 4 that is the first equation equal to half. The second one the first coefficient the coefficient of x 1 is killed by our choice and the second one will become; we are multiplying these by minus 3 by 2 and adding it here. So, half will remain half x 2 plus here it will remain 2 x 3 because we are we are multiplying this by 3 minus 3 by 2. So, this becomes 3 plus adding it here will become 2 x 3, plus here it should become minus 3 by 2 we are multiplying. So, minus 9 by 2 and we are adding it here. So, it will become minus minus 5 by 2, x 4 and this will become 5 by 2, because this is minus 3 by 2 added to 4 this become 5 by 2 plus.

The second one similarly will be 3 by 2, x 2 plus 6 x 3 minus 15 by 2 x 4 equal to; let us let me it do just for I want to do multiple examples in the same. So, I want to not take 6 here, but I want to take constant b, here b is some constant and the problem I want to find is for which b it has a solution. So, this will become b minus 3 by 2; now I can take I can again for the simplicity of calculation, I will multiply the second equation by half by 2 so that the fraction will not be there. So, multiply these by 2. So, I will correct here this will become 1, this will become 4, and this will become minus 5 and this will become 5.

Now I will use to kill this so; that means, I have to multiply this by 3 by 2 minus 3 by 2 and add it here. So, this will go and what will happen here minus 3 by 2 that is what will be there; this will go this will be 0 and this 1 here12 by 4 that is minus 6 and this will also become 0 and this is plus 15 by 2, this is minus 15 by 2. So, this is 0 and this will become what? This will become there was 5 year 5 in to minus 3 by 2 and we are adding it here. So, this will become plus 15 by 2, which is b minus, minus or plus did i make a mistake.

Student: Yes.

No this is minus because this is minus. So, this is minus this is minus. So, this will become minus 18 by 2 that is 9, b minus 9. So, the third equation becomes 0 equal to b minus 9. So, if this system has to be consistent then you should get 0 equal to 0. So, it has the solution exist if and only if b equal to 9; and in that case now we have to forget the third equation and the first 2 equations from the second one you write down x 2 in terms of x 3, x 4 and then you will use this result and the equation 1 to write down x 1. So, that one we will get all the 4 coordinates the third and fourth coordinate are arbitrary and first 2 coordinates are determined by the first equations.

I think we will stop for a break and continue later.