Linear Algebra
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Lecture - 05
Examples of subspaces

Let us continue our study with subspaces.

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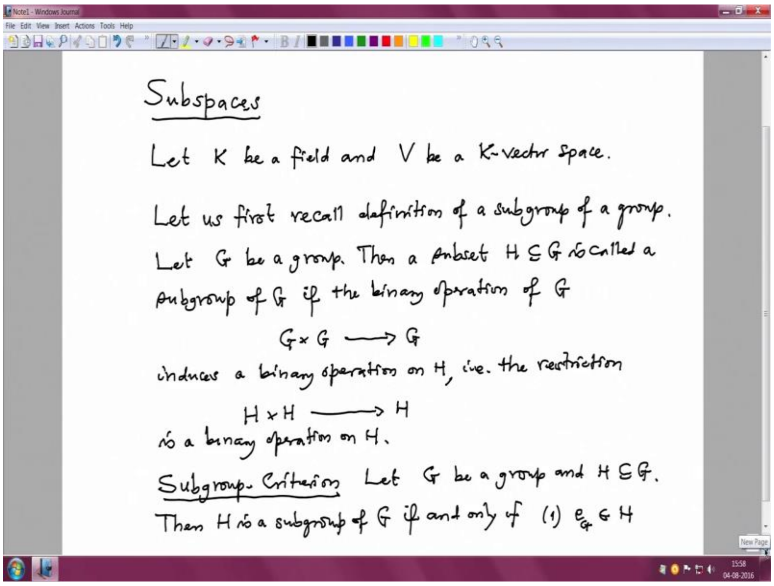

So, let always K denote K be a field, and V be a K vector space. Before I recall a definition of a subspace, I we need to recall let us first recall definition of subgroups of a subgroup of a group. Let G be a group and a subset H of G is called subgroup of G, if in the binary operation G cross G to G of G induces a binary operation on H that is the restriction H cross H, H is a binary operation on H.

This means with the same binary operation H is a subgroup H is a group in its own right. So, easy way to check this is let us recall, this is in this what is called subgroup criterion. Let G be a subgroup G be a group, and H is a substrate of G then H is a subgroup of G if and only one identity element of G is an element in H whenever if a and b are elements in H, then a b in element in H and a inverse also element in H.

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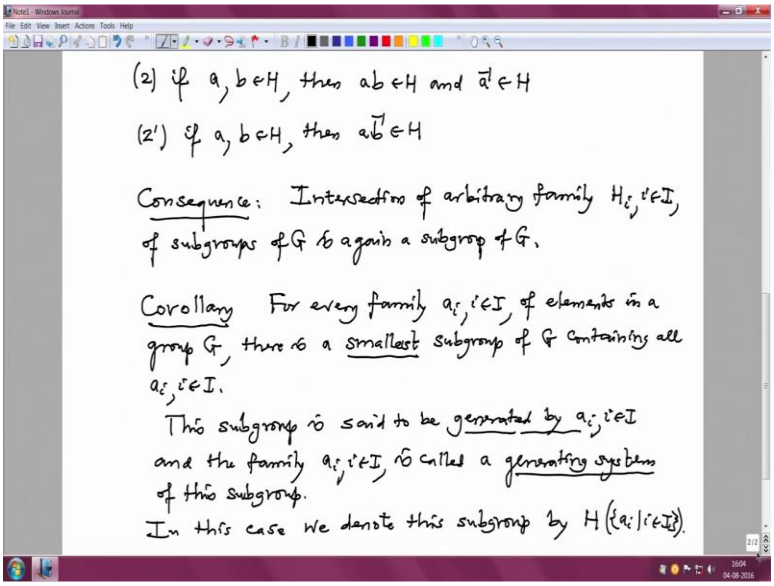

So, in this condition two is also reformulated as two prime, if a comma b is in H, then ab inverse belong to H. Instead of this two parts a b belong to H and a inverse belong to H which in combine this and just write ab inverse belong to H i will leave it for you to check this equivalence which is very easy, but I want to immediately know the following consequence.

Intersection of arbitrary family H i of subgroups of G is again a subgroup of G this is immediate, because condition two is very easy to check and condition one is obvious because identity to belong to each member, so identity to e belong to the intersection. So, also I want to note to the following consequence, so correlate this consequence is the following for each family f elements for every family a i, i in of elements in a group G there is a smallest subgroup of G continuing these elements continuing all a i, i (Refer Time: 07:37) smallest; because we consider all subgroups which contain then all these ai's, which definitely one subgroup which contain all of them in g. So, there is we continue intersection and intersection we will be a subgroup of G by the above consequence. So, for each family we have a subgroup which containing all this elements.

This subgroup is said to be generated by a i and the family a i, i in I is called the generating system a generating system of this subgroup. So, in this case we denote this subgroup by H a i. Let me justify this notation let me first take the simple case to illustrate.

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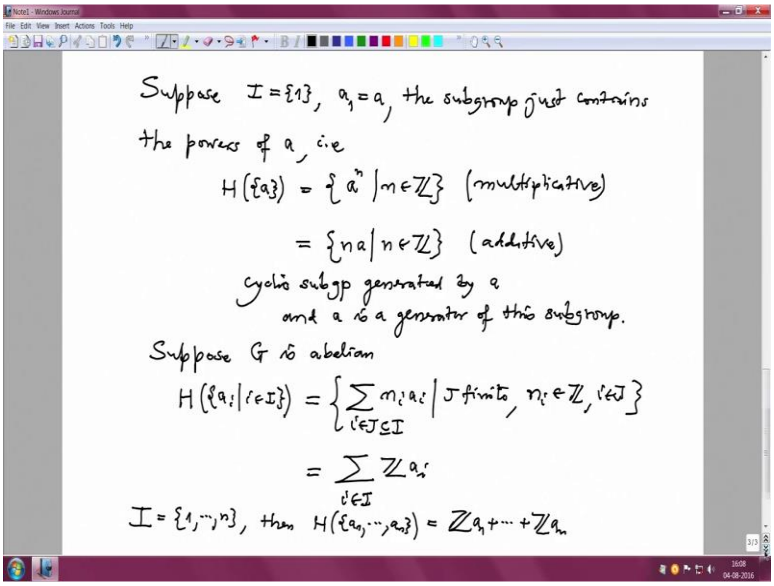

So, suppose the indexing side to adjust single term let us call it one, that mean there will be only one element you have considering let us call a 1 to be a. So, the subgroup just contains the powers of a, that is H written a is nothing, but a power n, n varies in integers. If it is if you are using the if you are using the additive notation, then should write is as n times a, this is this is in a multiplicative notation, this is additive notation.

So, in this case it is a cyclic subgroup generated by a, and a is called a is a generator of this subgroup. If your group is commutative; suppose G is commutative G is Ablian, then in this case the subgroup generated by the family a I you remember we have defined it as a smallest subgroup which contain all this guys, means precise you can describe the elements of this subgroup as the in case additive notation I will use it is the finite sums like this i belong to J, and j is a finite subset. So, J finite and n i's are integers.

So, finite Z linear combinations of in the family a i, especially the set of all elements in the subgroup generated by this you can also use the notation this is nothing, but summation i in I Z a i. So, this justifies also the name the terminology that subgroup generated by a family a i; if we have i is the finite set, if I is the finite set one to n then H a 1 to a n this is we say the Z linear combinations of a 1 to a n, this is the notation for all Z linear combinations of a 1 to a n.

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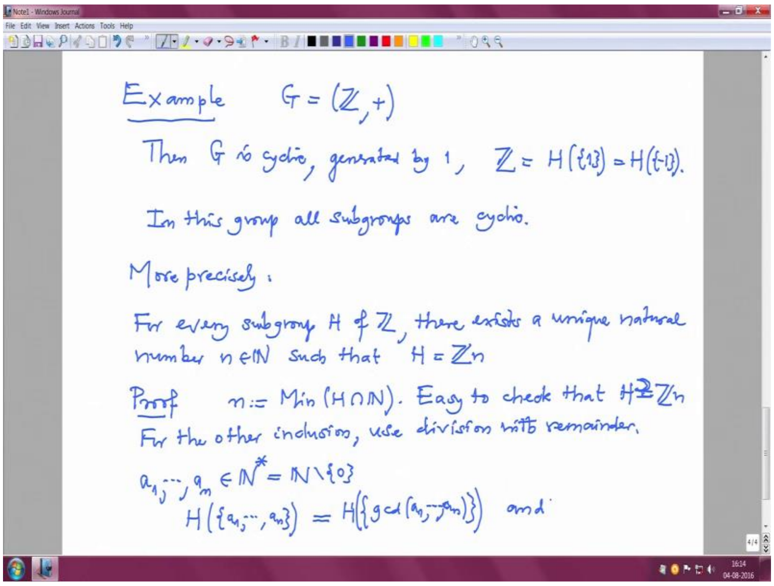

So, I also want to note here one example if you take the simplest group G equal to the additive group of integers; you know that this group is cyclic then G is cyclic generators generated by 1. So, in our notation Z is H of 1 or also this is same as H of minus 1 because minus 1 is also generator in this group. In this group all subgroups are also cyclic not a cyclic.

you can write more precise cyclic more precisely for every subgroup H of Z, there exists a unique natural number n such that H is the multiple integral multiple (Refer Time: 16:55) just one word about the proof the natural choice or n. So, take n equal to a minimum of H intersection in note that this minimum exist because of the a well ordering principle for n and now which is you to check that this n easy to check easy to check that H is precisely multiples of Z. For this equality one inclusion is obvious this inclusion is obvious because n is in H, then for all multiples of n are in H conversely for the converse for the other inclusion use division algorithm division reminder, and I will not write the details here even though you can even though I am not written here some of the details you can find in supplements which will be uploaded here.

So, two more things I want to note for finitely (Refer Time: 18:37) still in the same group of additive group of integers, if I have finitely mean integers a 1 to a n let us assume then they are non zero natural numbers, whenever I rewrite n star it is n minus 0, then this subgroup generated by a 1 to a n, we know we have just seen that every subgroup is finite cyclic therefore, this subgroup is also cyclic then how do you find the generator. Finding the generator is very easy this is almost immediate from the definition of greatest common deviser that this subgroup is nothing, but generated by GCD of a 1 to a n.

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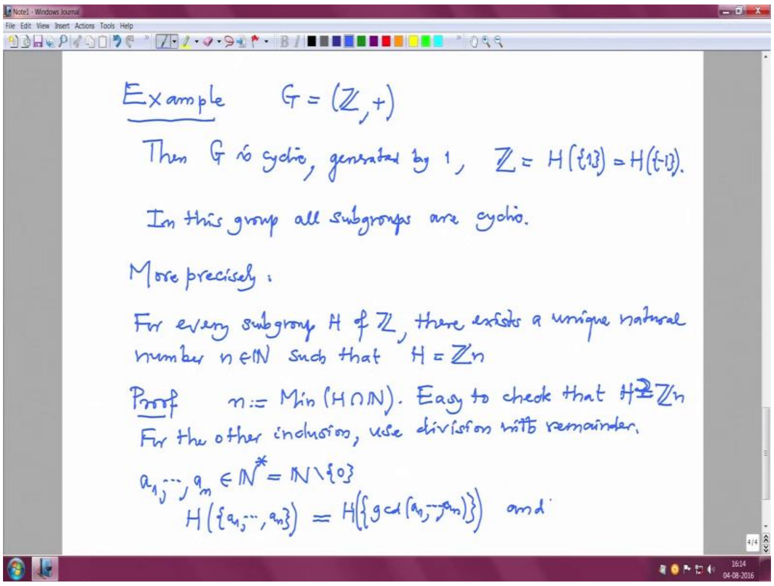

This is almost immediate from the definition of a GCD and the intersection of the; you take the subgroup generated by a 1 that is (Refer Time: 20:04).

Sorry, from where? (Refer Time: 20:23).

Beginning of this (Refer Time: 20:33).

I will repeat this (Refer Time: 20:35). Shall I start?

Yes (Refer Time: 20:50).

Ok.

Start (Refer Time: 20:56).

If you take cyclic subgroup generated by a 1, cyclic subgroup generated by a 2, cyclic subgroup generated by a n and take their intersection, this is also subgroup of Z the generator of this is nothing, but the l c m both. So, both this statements are just immediate from the definition of LCM and GCD.

Now, let us come back to subspaces. So, a subset U of V when you remind you V is a vector space over a field a subset U of subs vector space V over a field is a subspace of V with the same, if with the same addition and scalar multiplication induced from V, U is a K vector space. In this case I will also say that the K vector space structure on U is induced by the given K vector space structure of V. As in the case of subgroup we can also write this subspace criterion.

So, let u be a subset of V then U is a subspace of V, if and only if three conditions: one the identity element in the Ablian group V plus which we were denoting zero that should be in U, two if vectors x y belong to U, then there are some x plus y belongs to U; and third one if a is a scalar and x is a vector in u then the scalar multiple ax of x also belong to U.

So, again I will not write the proof or I will not write the formal proof for this proof easy to write. You can also see the proof in the notes. So, I want to save some time to skipping such groups.

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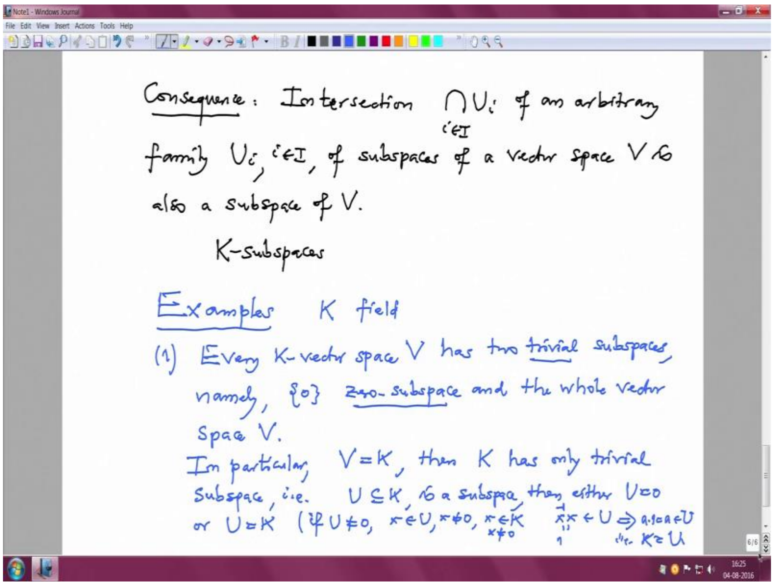

As in the subgroup case we also have the following consequence intersection; intersection U i of an arbitrary family U i, i in I of subspaces of a vector space V is also a subspace of V. I note that i could be infinite index set. So, it could be really infinite index infinite family of subspaces also here I should note that strictly speaking i should also keep that K in notation and write it instead of the subspace K subspace; just to remember the field we are working with, but I have most of the time we have I will drop this K in the notation.

Now, I want to see. So, this is again checked by using the above conditions 1, 2, 3 and all of them are easy to check. Now I want to also discuss many many examples now. So, that (Refer Time: 27:45) ok. Some examples from possibly from different branches of mathematics; number one let the in this examples I will always denote K to be a field when I specify a specific field then I will mention that time. So, every K vector space, V has two trivial we will call them trivial subspaces; namely single time zero, remind you zero is a zero vector which is called zero subspace, and the whole vector space V these are called trivial subs trivial subspaces.

In particular V is K then K has only trivial subspaces so; that means, if U is a subset of K and if U is a subspace, then either U is 0 or U is the whole K. So, just note here if U is non-zero, then it will contain in some non zero vector x, but this x is now vector and also scalar because V equal to K. So, x is in K x is non zero. So, x inverse exist and so x inverse times x it should be U, but this is one; ones one is in U a times one which is also U therefore, every a is in. So therefore, so that is K equal to U.

So, we will have a break and then continue in the next half.