Ordinary Differential Equations
Prof. Raju K George
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture - 9
Analysis Continued

Welcome back to the basic concepts of real function theory. In my previous lecture, I have given the properties of convergence of sequence of functions; we dealt with two types of convergences. One is point wise convergence of sequence of functions and also a uniform convergence of sequence of functions. And further in the proof of existence and uniqueness theorem, we deal with a series of real functions. Now we will discuss a uniform convergence of series of real functions.

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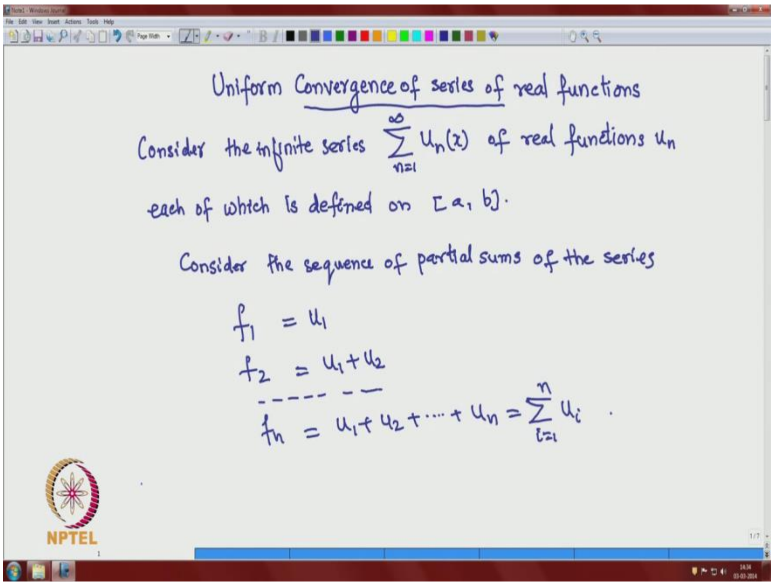

So, consider the infinite series summation n goes from 1 to infinity u n x. So, an infinite series of real functions u n's, each of which is defined on some real interval, say, a, b. Now to talk about the convergence of the series, we first form a sequence of partial sums of the series. So, consider the sequence of partial sums of the series, call it f 1 is u 1, f 2 is sum of u 1 and u 2 and so on; f n is u 1 plus u 2 plus etcetera plus u n. So, that is summation I goes from 1 to n u i.

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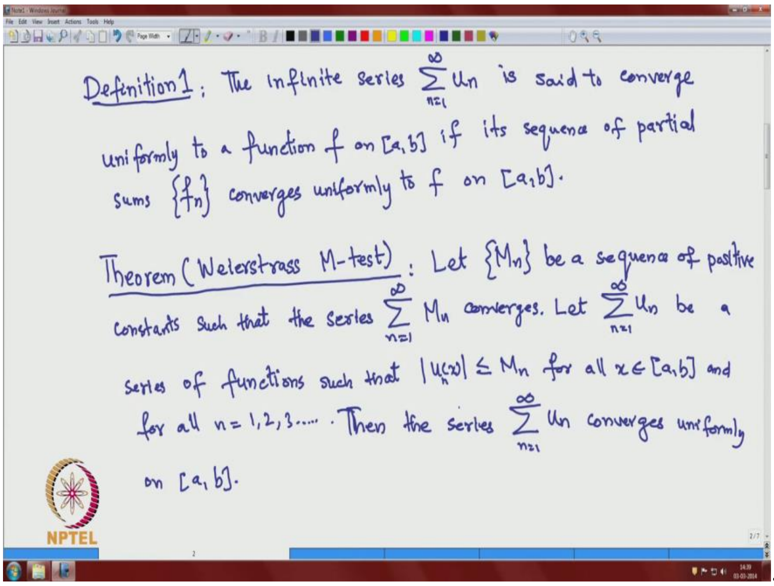

Now we define definition. So, let this be definition one. The infinite series summation u n, n goes from 1 to infinity is said to converge uniformly to a function, say, f on a, b; if it is a sequence of partial sums which we denote it by f n converges uniformly to f on the interval a, b. Now to make sure that an infinite series converges uniformly to a function f, we have the following theorem which is known as a Weierstrass M-test, so theorem Weierstrass M-test. So, let m n be a sequence of positive constants such that the series of this positive constants m n, n goes from 1 to infinity converges to some number.

Now let the series of functions u n, n goes from 1 to infinity be a series of functions such that the absolute value of u x is less than or equal to absolute value of u n x is less than or equal to m n for all x in the interval a, b and for all n is equal to 1, 2, 3, etcetera. Then the conclusion is Weierstrass M-test says then the series of functions then the series u n, n goes from 1 to infinity converges uniformly on the interval a, b. So, each term of the series is bounded by a constant and if the series found by that constants is a convergent series, then the series of functions converges uniformly on the interval a, b. So, that is a test; we will a see an example where you find this fact.

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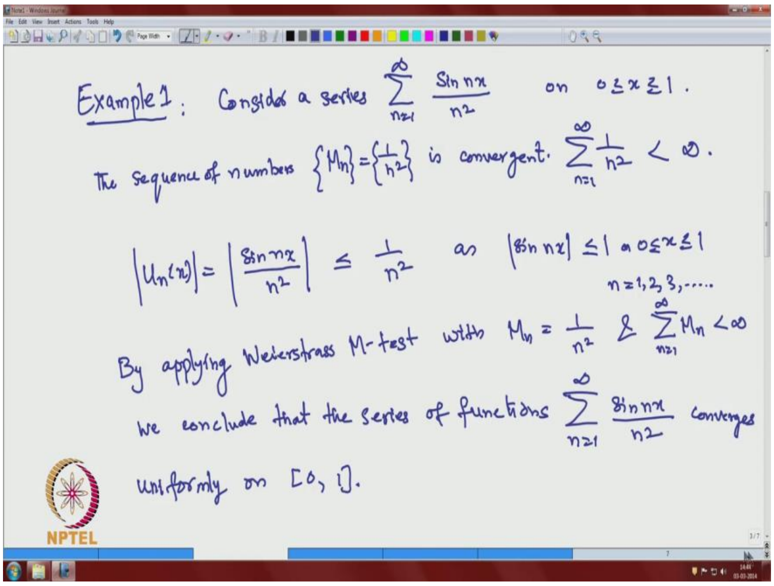

So, example, say, example one; so consider a series, there is summation n goes from 1 to infinity sin n x divide by n square, and this is defined on the intervals 0 less than or equal to x less than or equal to 1 . So, now the sequence of numbers m n which is 1 by n square, the sequence of numbers is convergent. So, we know that summation 1 by n square, n goes from 1 to infinity is a convergent series.

So, if you consider u n x which is by definition sin n x divide by n square, and so, therefore, if I take the absolute value of this, this is less than or equal to 1 by n square; as sin n x is less than or equal to 1 its bounded by 1 on the interval 0 to 1, and this is also true for n is equal to 1, 2, 3, etcetera. So, therefore, by applying Weierstrass M-test with m n is equal to 1 by n square and m n, n goes from 1 to infinity is finite. We conclude that thus series of function sin n x divided by n square for n goes from 1 to infinity, the series of function converges uniformly on the given intervals 0, 1.

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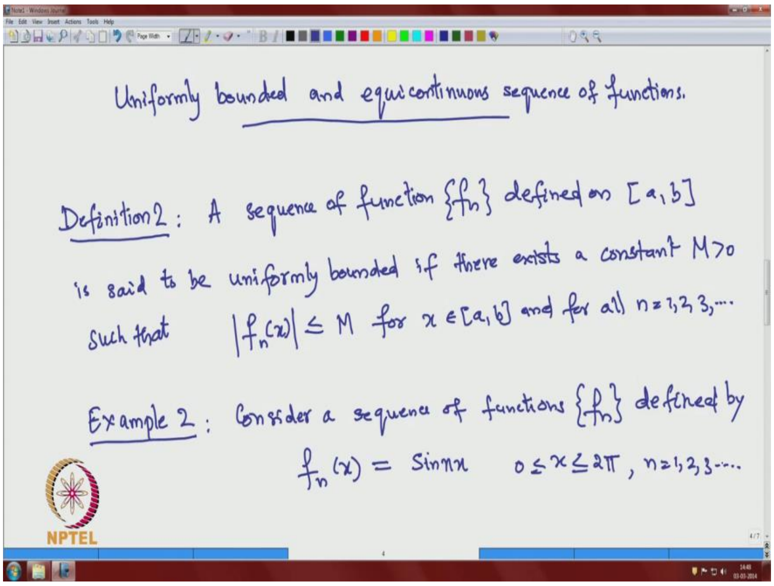

Now another important tool which will be using in the existences and uniqueness theorem is Arzela-Ascoli theorem. Arzela-Ascoli theorem says that if you have a sequence of functions which are uniformly bounded and equicontinuous, then that sequence has a convergence of sequence. So, let me introduce what is uniformly bounded and equicontinuous sequence.

So, uniformly bounded sequences and equicontinuous, uniformly bounded and equicontinuous sequence of functions. So, my definition is, say, definition two. A sequence of functions, say, f n defined on an interval a, b is said to be uniformly bounded if there exists a constant call it m greater than 0, such that f n of x, the absolute value of f n of x is less than or equal to m for all x in the interval a, b and for all n is equal to 1, 2, 3, etcetera.

So, there exists a constant m which is independent of n. So, uniformly there is a bound for each of the function in the sequence, then we say that the sequence of function is uniformly bounded. An example of this example, say, 2. So, consider a sequence of functions f n defined by, say, f n x is equal to sin n x where x is varying in the interval 0 to 2 pi and n is equal to 1, 2, 3, etcetera is a sequence of functions.

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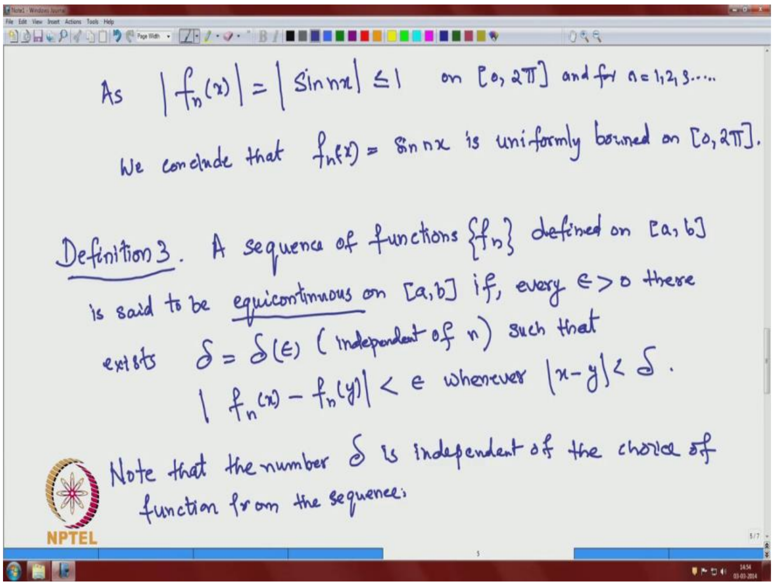

So, as absolute value of f n x which is the absolute value of sin n x which is less than equal to 1 on the intervals 0, 2 pi and for all n is equal to 1, 2, 3. So, we conclude that f n x is equal to sin n x is uniformly bounded. So, this is uniformly bounded on the given interval 0 to 2 pi. Now we define what is known as an equicontinuity of a sequence of function. So, definition, say, three; a sequence of functions f n defined on an interval a, b is said to be equicontinuous on the interval a, b if; so, a sequence of functions f n defined on an interval a, b is said to be equicontinuous on the same interval a, b, if for every epsilon greater than 0, there exists a delta that delta is a function of epsilon for every epsilon greater than 0, there exists a delta that delta depends on epsilon, but it is a independent of n.

So, it does not matter from which function n it is come, so independent of n, such that the absolute value of f n x minus f n y is less than epsilon whenever absolute value of x minus y is less than delta. See sequence of function f n defined on an interval a, b is said to be equicontinuous if for every epsilon greater than 0, there exists a delta. So, we should be able to find a delta which is independent of n; it does not matter from which function f n it is coming, such that f n x minus f n y is less than epsilon whenever x minus y is less than this delta. So, the equicontinuity is continuity in the uniform since for all n. So, note that the number delta is independent of the choice of function from the sequence.

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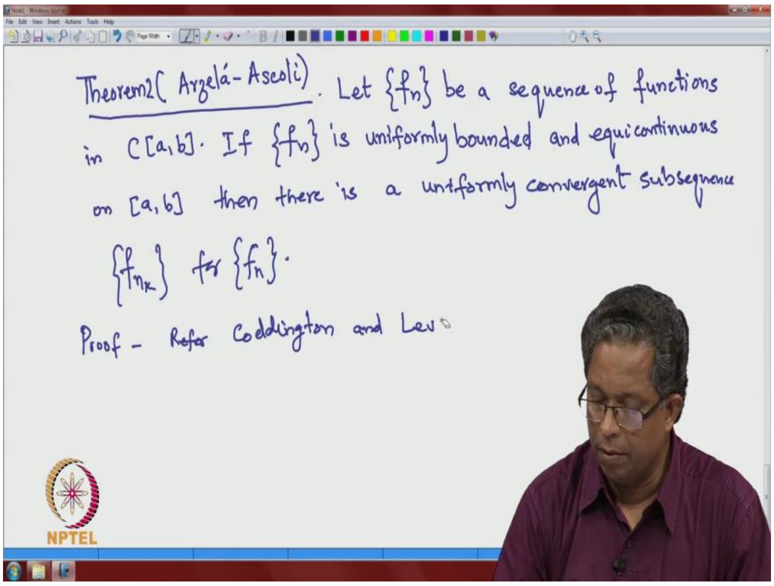

So, theorem known as Arzela-Ascoli theorem, so theorem number two; so, it states that let f n be a sequence of functions in the set of all continuous functions defined on the interval a, b. So, let f n be a sequence of functions in the space c a, b; c a b is the space of all continuous functions defined on a, b and if f n the sequence f n is uniformly bounded and equicontinuous on the interval a, b.

Then the conclusion of the theorem is then there is a uniformly convergent subsequence, call it f n k for f n. If f n is a sequence which are functions from the space c a, b and if f n is a uniformly bounded and equicontinuous set of a functions on a, b, then there is a uniformly convergence subsequence f n k for the original sequence f n. So, this result is known as Arzela-Ascoli theorem, which we will be using for proving the existences theorem for the initial value problem.

For the proof, one may refer to any standard book on analysis or it is also given in Coddington Levinson, refer Coddington and Levinson, book on ‘Theory of Differential Equations.’

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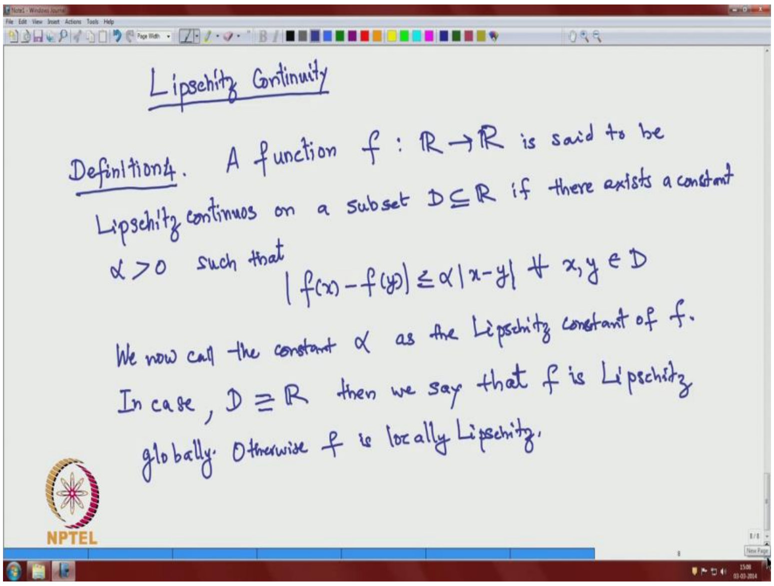

Now we move on to next topic which will be used in proving the uniqueness of solution of initial value problem namely the Lipschitz continuity, so Lipschitz continuity of functions. So, initially for the sake of simplicity we will define Lipschitz continuity for a one variable function. So, this is a definition four in this lecture. A function f from r to r a real valued function from the domain as a real space is said to be Lipschitz is said to be continuous, also is known as f satisfies a Lipschitz condition; we refer to this notion as lipschitz continuity.

A function f from r to r is said to be Lipschitz continuous on a subset, say, d which is a subset of r if. So, if there exists a constant call it alpha strictly positive, such that a function f from r to r is said to be Lipschitz continuous on a subset d of r if there exists a constant alpha greater than 0, such that the absolute value of f of x minus f of y is less than or equal to alpha times x minus y for all x, y in the subset d of r. So, in this case, we now call the constant alpha as the Lipschitz constant of f.

In fact, if alpha is a number satisfying this inequality, then any number larger than that will also satisfy this inequality; we take alpha the least upper bound of all such alphas, and so that alpha we call as a Lipschitz constant. The least number which is the smallest number which is satisfying these inequalities is known as a Lipschitz constant of f and if in case d is the odd real line. So, we here assume the definition that d is a subset of r; in case a d is a odd real line, then we say that f is Lipschitz globally.

So, then f is a globally Lipschitz continuous; otherwise, f is locally Lipschitz. So, if the condition is true is satisfied in the odd space, then the Lipschitz continuity is a global Lipschitz continuity or it is restricted to a subset of the real line, then it is a local Lipschitz continuity. Now we take a few examples showing the Lipschitz continuity.

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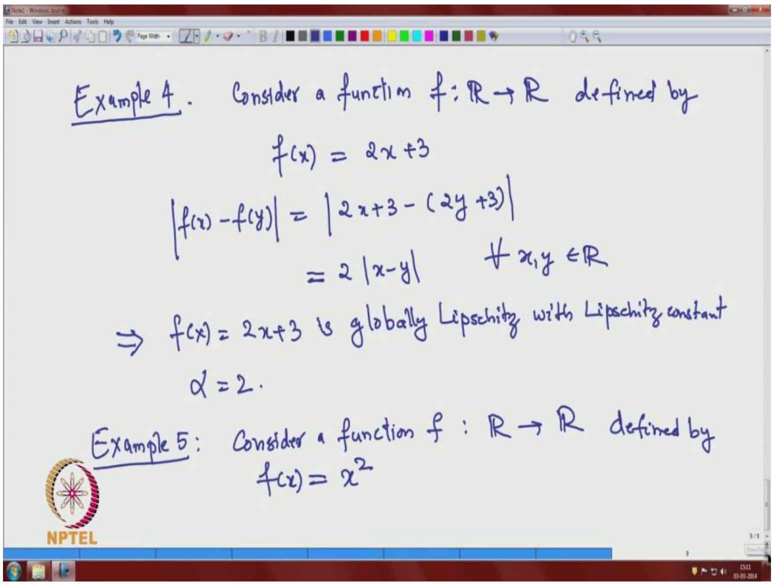
SS 

So, let us take examples. So, example, call it four. So, consider a function f from r to r defined by f of x is equal to 2 x plus 3. So, this function f of x is equal to 2 x plus 3; obviously, this function is not a linear function. We check the Lipschtiz continuity of this function f of x minus f of y is 2 x plus 3 minus 2 y plus 3. So, if you take the absolute value. So, this is less than or equal to 2 times or equal to 2 times x minus y. So, this implies that is true for all x and y in r. So, f x is equal to 2 x plus 3 is globally Lipschitz with a Lipschitz constant, alpha is given by 2. Now consider another example, so example five. So, define or consider a function f from r to r defined by f of x is equal to x square.

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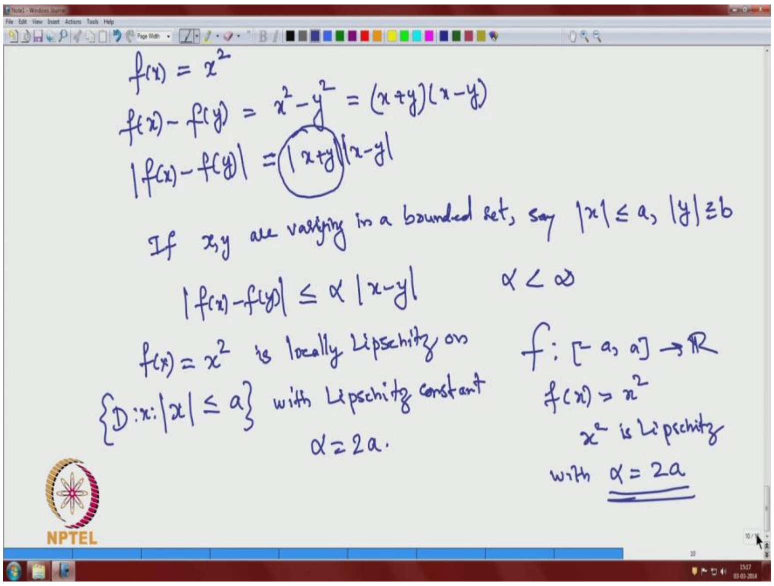

So, f of x is equal to x square. Now to check the Lipschitz continuity f of x minus f of y which is equal to x square minus y square which is equal to x plus y into x minus y. So, the absolute value of f of x minus f of y is equal to x plus y into x minus y. Now consider this quantity absolute value for x plus y; this is not a bounded quantity if x and y are allowed to vary in the entire real line, okay, but if x and y are varying in a bounded set, then this modulus absolute value of x plus y is a bounded quantity.

So, if x and y are varying in a bounded set, say, x is less than equal to a and y is less than equal to b, in that case f of x minus f of y is less than or equal to. So, this value x plus y could be alpha times x minus y, where alpha is a bound for x plus y where x and y are bounded by these two constants. So, therefore, alpha is a finite quantity. So, therefore, f x is equal to x square is locally Lipschitz. So, if this is locally Lipschitz.

So, for example, if f is defined from a set minus a to a to r f x is equal to x square, then f x is equal to x square is Lipschitz with alpha is equal to 2 a, where x can take maximum value of 2 a, and x varies from minus a to a, and y is also varying from minus a to a. So, therefore, absolute value of x plus y that can take maximum up to a plus a that is 2 a. So, for that Lipschitz constant is 2 a. So, therefore, f x is equal to x square is locally Lipschitz on domain, say, d which is set of all x such that x is less than or equal to a, with Lipschitz constant alpha is equal to 2 a. Now, one can provide sufficient condition to ensure that a function is Lipschitz.

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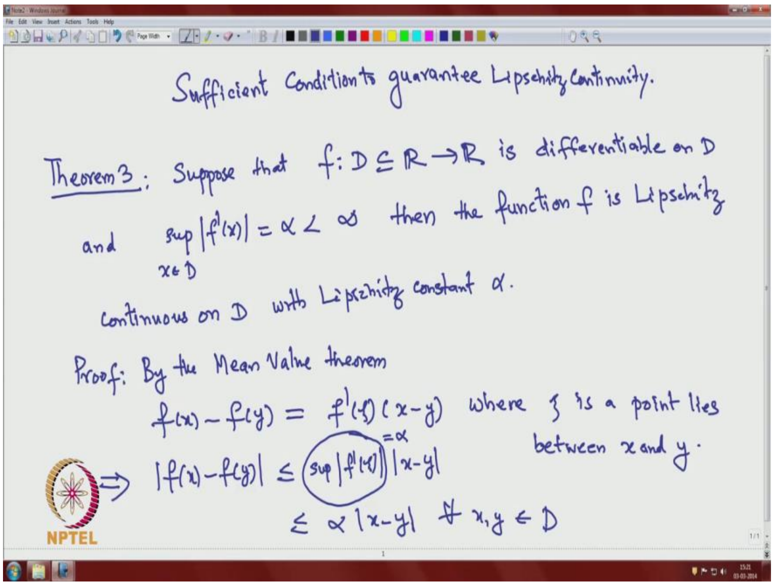

See sufficient condition to ensure that a function is Lipschitz, sufficient condition to guarantee Lipschitz continuity. So, I state, therefore, the theorem, so theorem three. Suppose that f is a function from d which is a subset of r and mapping to r is differentiable on d and the supremum of the bound of the derivative when x varies on d is, say, alpha; it is a finite quantity. Then the conclusion is then the function f is Lipschitz continuous on d with Lipschitz constant alpha.

The proof is very simple just by using the mean value theorem f of x minus f of y is equal to the derivative of the function evaluated at some point psi times x minus y, where psi is a point lies between x and y . So, therefore, this implies that the absolute value of f x minus f y which is less than or equal to sup of f prime psi into x minus y, and this quantity is our alpha; by hypotheses this sup exists and it is bounded by alpha. So, therefore, this is less than or equal to alpha times x minus y for all x, y in the domain d.

And if d happens to be the all real line, then the Lipschitz continuity we obtain it is a global Lipschitz continuity. So, global Lipschitz continuity can also be given in terms of the bound of the derivative. If the derivative of a function it is a slope of a function is bounded globally, then the function is Lipchitz continuous globally. If the slope is or te derivative is bounded on a bounded set, then the function is Lipschitz continuous on that bounded set.

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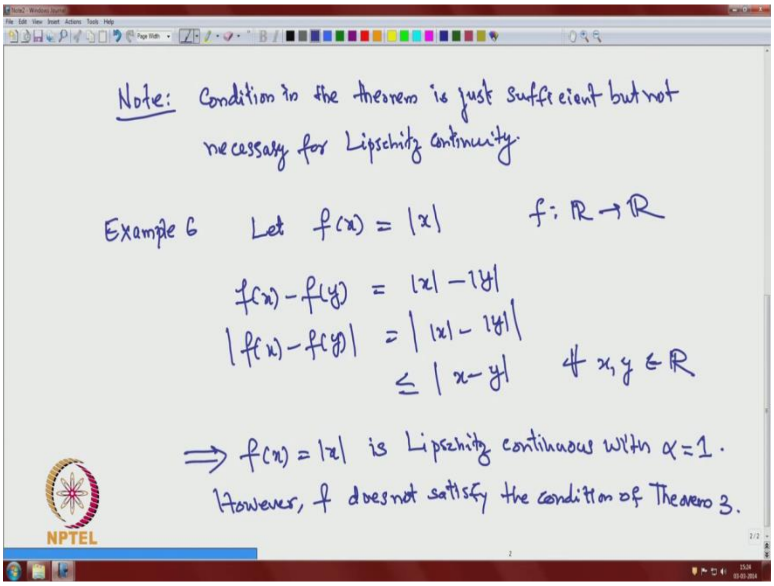

And we note that the condition in this theorem is just sufficient. So, we note that. So, condition in the theorem is just sufficient but not necessary for Lipschitz continuity. So, that means we can produce example where a function is Lipschitz continuous, at the same time the conditions of the theorem is violated. So, example, say six; so, you take a function f x is equal to mod x and f is defined from r to r.

So, obviously, f x minus f y is mod x minus mod y if you take the absolute value f of x minus f of y which is absolute value of mod x minus mod y. So, it can be shown easily that this is less than equal to absolute value of x minus y for all x, y in r. So, what does it say? So, this implies that f x is equal to mod x is Lipschitz continuous with a Lipschitz constant alpha as 1. However f does not satisfy the condition of theorem three. Theorem three we stated up; the reason is f is not differentiable at 0 to verify the condition stated in the theorem.

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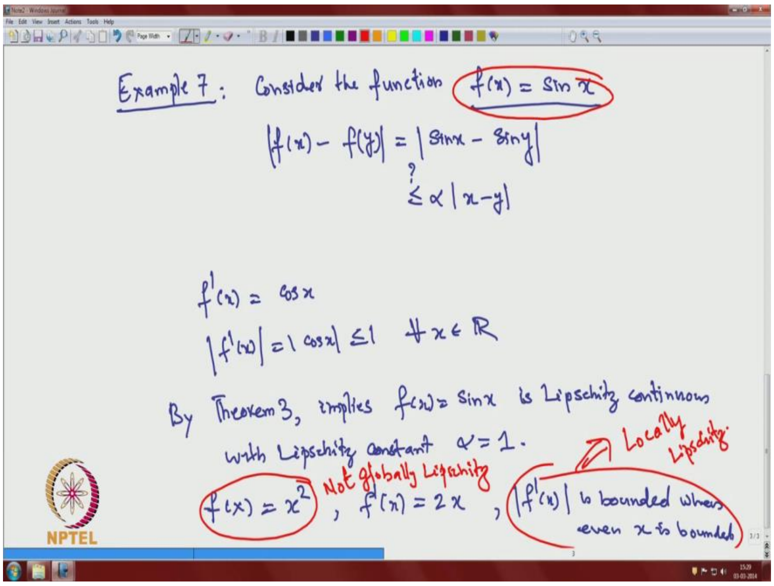

If you take another example, so example 7; so, consider the function f x is equal to sin x and to verify whether this function is Lipschitz or not f x minus f y which is sin x minus sin y. And if I want to take and I want to show if f is Lipschitz, then there exists a constant alpha such that it is less than or equal to that constant alpha times absolute value of x minus y.

So, how to show this? It is not that straightforward provided if you use some trigonometric identities you may be able to, but if you just apply the theorem three, the sufficient condition for a Lipschitz continuity that ensures that the function f x is equal to sin x is Lipschitz continuous. So, how because f prime x is cos x and it is bounded; so, cos x is bounded by 1 for all x. So, therefore, by theorem three there is a sufficient condition for Lipschitz continuity implies f x is equal to sin x is Lipschitz continuous with Lipschitz constant alpha is equal to 1.

So, it is a very good test; if we just recall the function which we considered f x is equal to x square, the derivative is 2 x, f prime x is 2 x and 2 x is not bounded globally, but whenever x is bounded, 2 x is bounded. So, f prime x is bounded. So, whenever x is bounded. So, in a bounded domain f x is equal x square is Lipschitz or that is locally Lipschitz.

And if you look at the function f x is equal to sin, x f x is equal to sin x is globally Lipschitz; f x is equal to x square is not globally Lipschitz, but it is locally Lipschitz, so not globally Lipschitz. So, because of this, this says that it is locally Lipschitz. Now we require this Lipschitz condition for functions of two variables that we will deal with in the next session.

So, therefore, in this session we have seen, we have analyzed the uniform convergence of a series of functions, and by using the Weierstrass M-test we can make sure we can test whether a given series of function is convergent uniformly or not. And also we have defined what is an equicontinuous function and uniformly bounded functions of sequence of functions. And finally, we stated the Arzela-Ascoli theorem which says that in every uniformly bounded and equicontinuous function defined on a bounded interval a, b has a convergent subsequence. We will deal with Lipschitz continuity for functions of two variables in the next session. Bye.