Ordinary Differential Equations
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Lecture - 8
Analysis

Welcome to the new module on this course on differential equations. We will also be concerned in this course to do some theoretical aspects of differential equations, and we will state and prove the existence and uniqueness theorem. For that purpose we need to get familiarized with some of the concepts, basic concepts of real function theory. In this lecture, we will discuss about convergence of sequence of functions; 2 types of convergence we deal with namely, the point wise convergence and uniform convergence.

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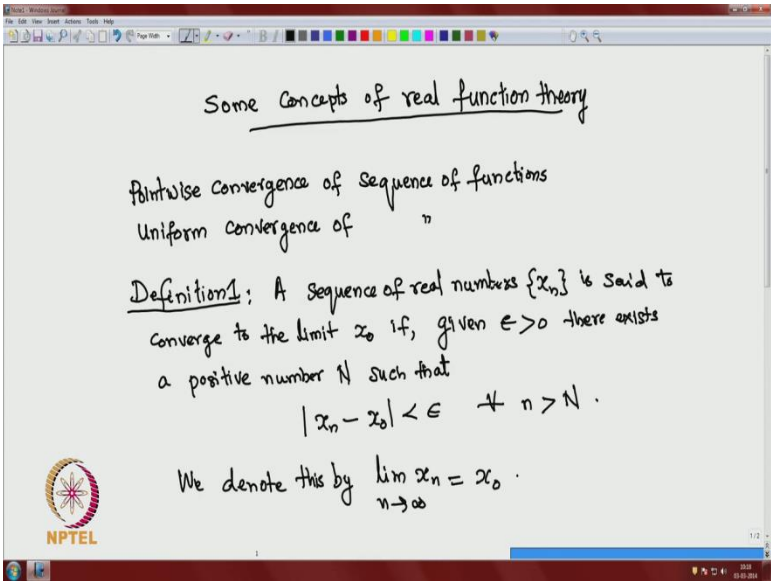

Some concepts of real function theory. So, first we deal with point wise convergence of sequence of functions, and also we will do uniform convergence of sequence of functions. Before we start the point wise convergence and uniform convergence of sequence of functions, let us start with the convergence of sequence of real numbers.

So, definition say, called it definition 1, a sequence of real numbers, let us denotes this by x n, is said to converge to the limit, call it x 0 if, given epsilon greater than 0, a positive number, there exists a positive number denoted by N such that, absolute value of x n minus x 0, is less than epsilon, for all n greater than this number, N. So, a sequence of real numbers x n, is to converts to a limit x 0, if this happens, that for all n greater than N, the difference between x n and x 0 that is made less than epsilon. And, we denote this by limit x n, as n goes to infinity, is equal to x 0.

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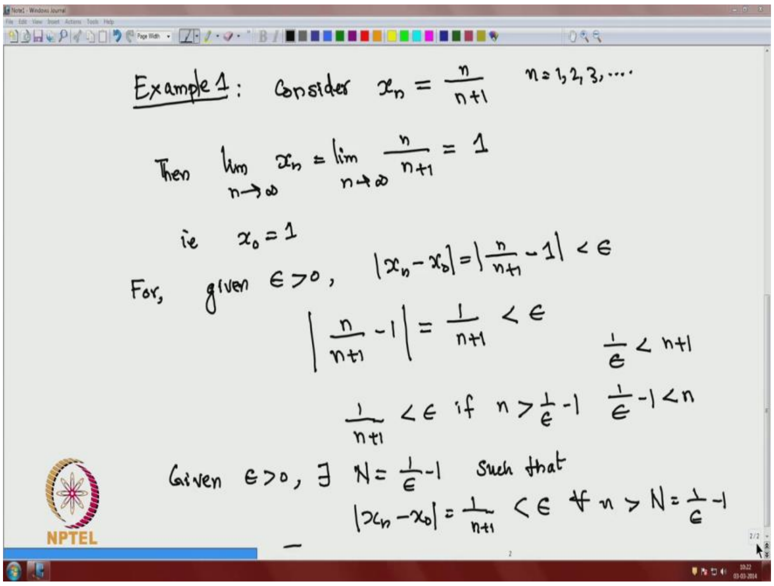

So, let us consider an example. We will consider an example of real numbers, sequence of real numbers. So, example 1, let us consider a sequence of real numbers given by x n is equal to n by n plus 1 where n goes from 1, 2, 3, etcetera. And then, we will see that, limit of the sequence, limit n goes to infinity, x n, which is by definition limit n goes to infinity, n by n plus 1 is equal to 1. So, that is, x n is known by n by n plus 1, and x 0 the limit is 1. So, we will see how this happens.

So, for, so, given epsilon greater than 0. So, what we are looking for? We are looking for x n minus x 0 which in our case n by n plus 1 minus x 0 is 1, you are looking for the situation that this difference is less than epsilon. And, we know that, have for, this less than epsilon for a large N. So, we need to find a N, such that for all n greater than that n this difference is less than epsilon.

Let us consider this n by n plus 1 minus 1. This quantity, just by simplifying, it is, since n is positive, this is 1 by n plus 1. So, we want this to be less than epsilon. And, a simple manipulation shows that 1 by epsilon is less than n plus 1, or 1 by epsilon minus 1 is less than n. So, therefore, this happens, 1 by n plus 1 is less than epsilon, if n is greater than, strictly greater than 1 by epsilon minus 1.

So, what is the moral of the story? Is that, given epsilon greater than 0, there exists an N which in our case, we could compute this as, 1 by epsilon minus 1, such that, x n minus x 0, which in our case it is 1 by n plus 1, is less than epsilon for all strictly greater than n; this N is 1 by epsilon minus 1. So, from the, hence the conclusion is, the sequence x n is equal to n by n plus 1 converges to a limit 1.

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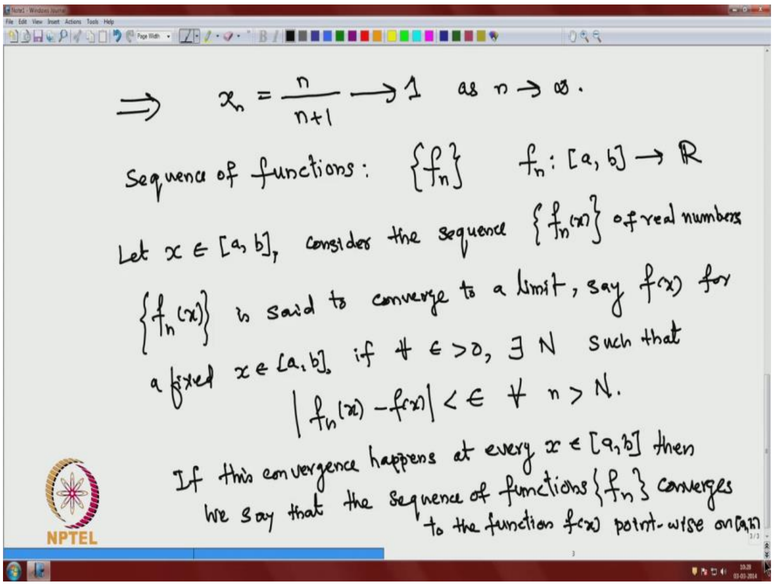

So, this implies that x n is equal to n upon n plus 1, converges to 1, as n goes to infinity. So, this is concerning a sequence of real numbers. Now, we will consider a sequence of functions. So, you know, convergence of sequence of functions, consider a sequence of functions denoted by f n, such that each f n is a functionary from an interval a, b to R. So, real valued function defined on the interval, a, b.

And, we will now discuss about the convergence of this sequence of functions; when we do say that, this sequence of functions converges to some limit function. Now, let x be any fixed real number in the interval a, b. Now, consider the sequence of real numbers, consider the sequence which is value of the functions f n, evaluated at x. So, now, this is a sequence of real numbers.

Now, consider the sequence, f n (x), of real numbers. Now, this sequence of real numbers for every fixed x is set to converge to a limit, call it f (x). If we can find a capital, for a every epsilon greater than 0, we can find N, such that the difference between f n (x) and f (x) is less than epsilon for all n greater than N.

So, now, this f n (x), this sequence of real numbers is said to converge to a limit, say f (x) for a fixed x, in the interval a, b. If, for every epsilon greater than 0, there exists a number n, such that, this difference f n (x) minus f (x), the absolute value of f n (x) minus f (x) is less than epsilon, for all n greater than N. So, this sequence of functions when it is evaluated at point x, it turns out to be a sequence of real numbers. So, therefore, the convergence of sequence of functions at point turns out to be a convergence of a sequence of a real numbers.

So, this sequence is said to be, said to converge to a point, a limit f (x) if this happens. And, if this happens for every x, in the interval a, b; so, if this convergence happens at every x in the interval a, b, then we say that the sequence of functions f n converges to the function f (x) point wise on a, b. So, this defines a point wise convergence of sequence of functions, defined on some interval a, b.

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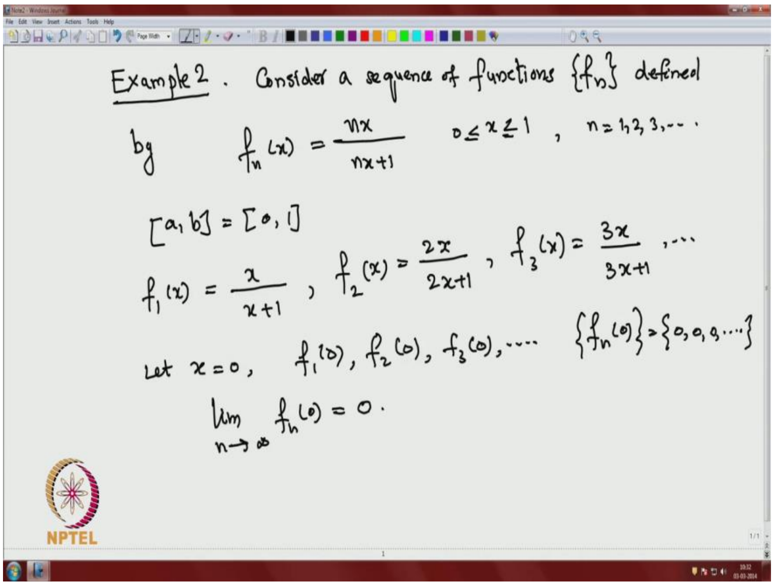

Now, we give an example of a sequence of functions converging to a function point wise. So, example, say called example 2. So, consider a sequence of functions f n, defined by; so, we can see sequence of functions f n defined by f n (x) is equal to n x, divided by n x plus 1, and for all x between 0 and 1, and here n goes from 1, 2, 3, etcetera. This is a sequence of function, infinite sequence of functions defined by n x by n x plus 1; here a, b is intervals 0, 1. So, a, b, you have definition is 0, 1.

Now, if look at the few terms of the sequence, say for example, the first 3 terms, when n is equal to 1, 2 and 3; say, f 1 (x) is x upon x plus 1, and f 2 (x) is 2 x upon 2 x plus 1, and f 3 (x) is 3 x by 3 x plus 1, and so on. If you evaluate this sequence of functions at x is equal to 0, say let x is equal to 0, which is point inside our interval, then f 1 (0), f 2 (0), f 3 (0), and so on, will get the sequence f n (0) which is a sequence of 0s. So, obviously, at x is equal to 0, this as values, the 0 sequence, that converges to 0. So, limit n goes to infinity f n (0) this goes to 0, say by it is a limit, when x is equal to 0.

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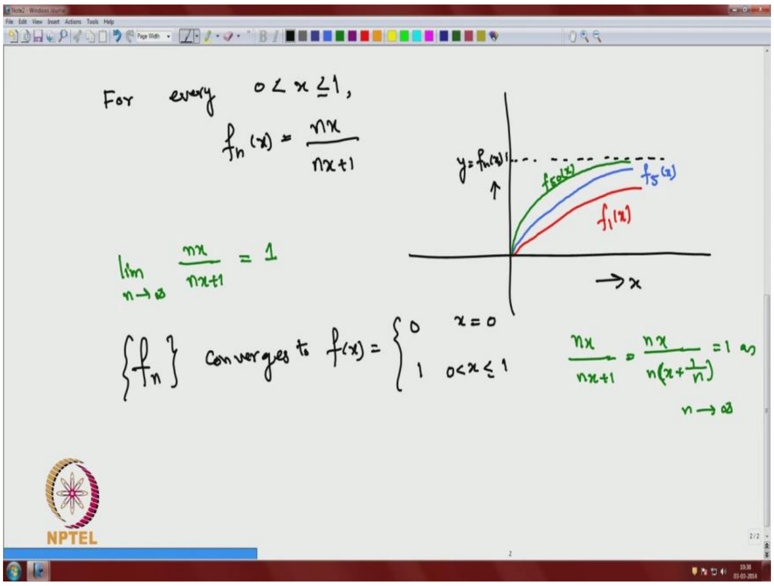

Now, we consider for the case when x is no 0, but value in the interval. So, for every x which is strictly greater than 0 and less or equal to 1, the sequence is the same, f n (x) is n x by n x plus 1. And, if you look at the graph of the sequence, so, this is x and this is y is equal to f x (r), the function, y is equal to f n x. And, if you look at the graph of the sequence, say this point is 1, and you see that the graph may be f n (x) will have f 1 of this type, and is say f 5 (x), and if look at say f 50 (x), so, this will be f 50 x.

And, we can show it, the limit n goes to infinity for x between strictly greater than 0 and less than or equal to 1, n goes to infinity, n x by n x plus 1, this is equal to 1. So, this goes to 1, and 1 is the constant function. That can be seen just by simplify this, n x by n x plus 1, which if I take n common, n of x plus 1 by n, and as n goes to infinity, 1 by goes to 0; and, n and n can be cancelled; and, you get this is equal to 1 as n goes to infinity.

So, therefore, for the values of x strictly greater than 0 and less than equal to 1, the sequence of function f n (x), that converges to 1, the constant function 1. So, in short, we write the sequence of function f n. So, this sequence of function converges to a function f

(x) which is 0 when x is equal to 0, and which is 1 when x is strictly greater than 0 and less than or equal to 1.

So, this sequence of functions in the example converges point wise to a function f (x); and, the limit function, look at the limit function, it is a discontinuous function. So, and also, if you look at each function, each f n (x) in the sequence; each, f n (x), is a continuous function on the interval 0, 1; and, this sequence of function converges to a function f (x), the limiting function is not a continuous function, which has a discontinuity at 0.

So, one thing we observe that even if we have a sequence of continuous functions which converges to a function f (x) point wise, is not necessary that the limit function f (x) will be continuous on the given interval. But, in the existence unique sequence theorem, we want to have a situation where, the limit function also must be continuous. So, what are, what is a guarantee, or what makes it sure or ensure, that if you have a sequence of continuous functions, and that converges to a function of x, what is the guarantee that the limit function is also continuous?

That is the stronger requirement that is ensured, if we assume with some different kind of convergence rather than point wise convergence. So, point wise convergence does not guarantee that the limit function is continuous even if each individual function in the sequence itself is continuous. So, we, to guarantee that the limit function is also continuous, we need a strongness of convergence which is known as a uniform convergence. So, we now introduce what we call as uniform convergence of sequence of functions.

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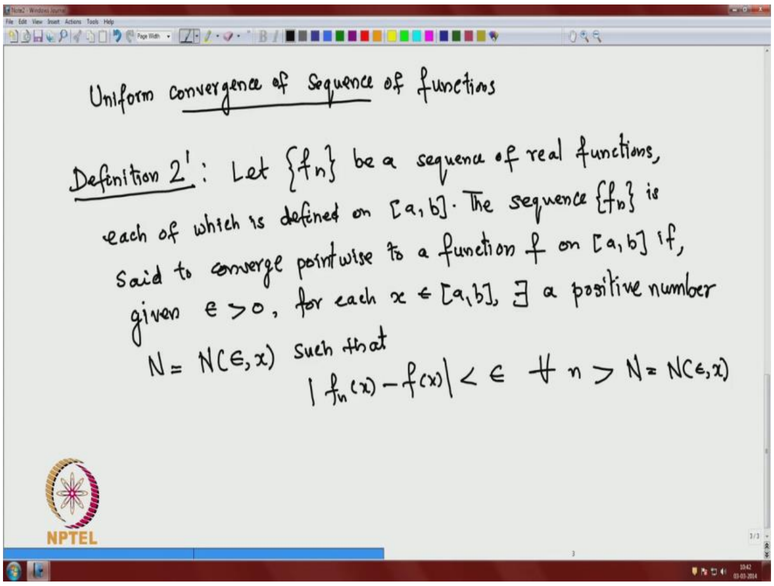

So, uniform convergence of sequence of functions. So, let us recast the pointwise convergence once again in a different a way of putting. So, definition, we say definition 2 bar, which is another way of defining the point wise convergence; point wise convergence we have already defined. Let, f n, be a sequence of real functions, each of which is defined on an interval called a, b. Now, the sequence of function, f n, is said to converge point wise to a function f on a, b, if given epsilon is greater than 0, for each x in the interval a, b, there exists a positive number called it N.

Now, this N is going to depend on epsilon and also on x, for each x, and for a given epsilon, if there exists a positive number n, that depends upon both epsilon and x, such that the absolute value of f n (x) minus f (x) is less than epsilon, for all n greater than this n; this n is basically that depends upon epsilon and x. So, observe that interval of the number n, depends not only on epsilon, it also depends on the point x.

So, for a given epsilon and for the given the numbers n epsilon will be different from different point x. And, if you can find a single n which is available for all x in the interval a, b, then we say that the convergence is uniform convergence. So, uniform convergence is a case where for a given epsilon greater than 0, there exists a positive number n, that number n will work for all x in the interval a, b, then the point wise convergence turn to be a uniform convergence. So, defined right as a definition.

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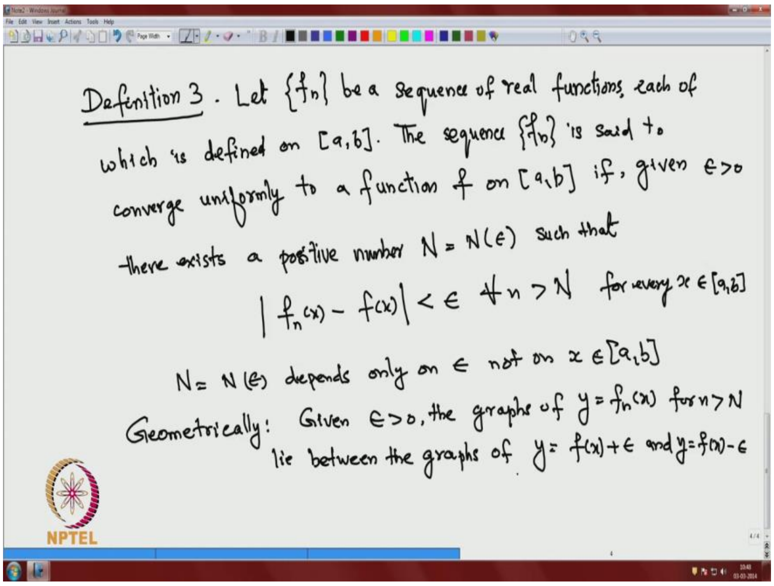

So, call it definition 3, let, f n, be a sequence of real functions, each of which is defined on a given interval a, b. Then, the sequence, f n, is said to be, or is said to converge uniformly to a function f on a, b, if, given epsilon greater than 0, there exists a positive number N. And, now, in this case N depends only on epsilon, such that the absolute value of f n (x) minus f (x) is less than epsilon, for all n greater than this N, and for every x in the interval a, b.

So, here the N, that positive number N depends only on epsilon, not on x. So, that is true for all x in the interval, a, b. So, geometrically, what does it mean? See, geometrically, so this means that, so given epsilon greater than 0, the graphs of y is equal to f n (x) for n greater than N, the graphs of y is equal to f n (x) for n greater than N, lie between the graphs between the graphs of y is equal to f (x) plus epsilon, and y is equal to f (x) minus epsilon.

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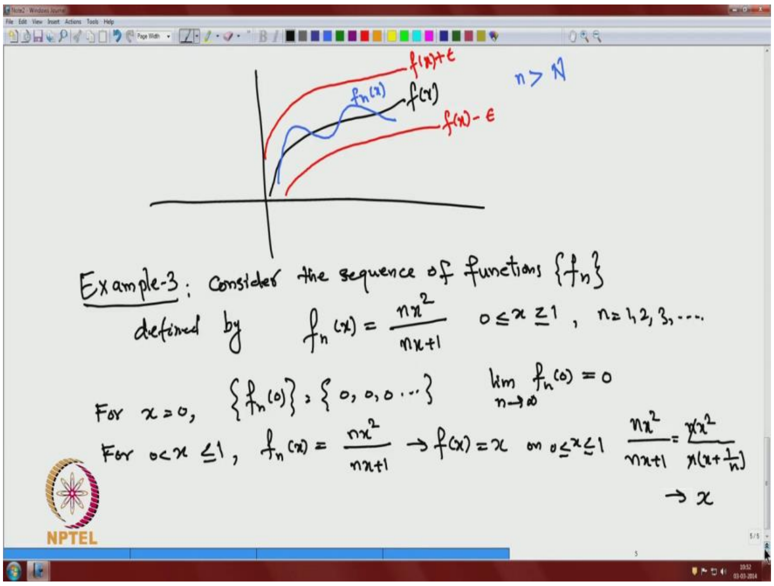

So, that is, if we have the interval, so, this is the limit function, f (x); then say, this is f (x) plus epsilon, and this is f (x) minus epsilon, then uniform convergence ensures that for all n greater than the N, the function f n lies between f (x) plus epsilon and f (x) minus epsilon; so, n greater than N. So, we did not see an example of uniform convergence.

So, an example; so, this is a third example. So, consider the sequence of functions f n defined by f n (x) is equal to n x square divided by n x plus 1, where x is between 0 and 1, and n is equal to 1, 2, 3, etcetera. So, obviously, for x is equal to 0, when f (x) is equal to 0, f n (0) is a 0 sequence, and that converges to 0, n goes to infinity, f n (0) is 0.

So, at x is equal to 0, the given sequence of function converges to 0, point wise. And also, for x strictly greater than 0 and less than or equal to 1, f n (x) which is given by n x square by n x plus 1; and we can show it easily by a manipulation n x square by n x plus 1, which is n x square by, n we take common, x plus 1 by n; and this is equal to; so, n and n get cancelled; and as n goes to infinity, denominator becomes x and the numerator is x square.

So, x, x square by x, that is x. So, this sequence, this converges to a function f (x) which is equal to x, on x, if this is true, including 0. So, therefore, it converges point wise to the function f (x) is equal to x.

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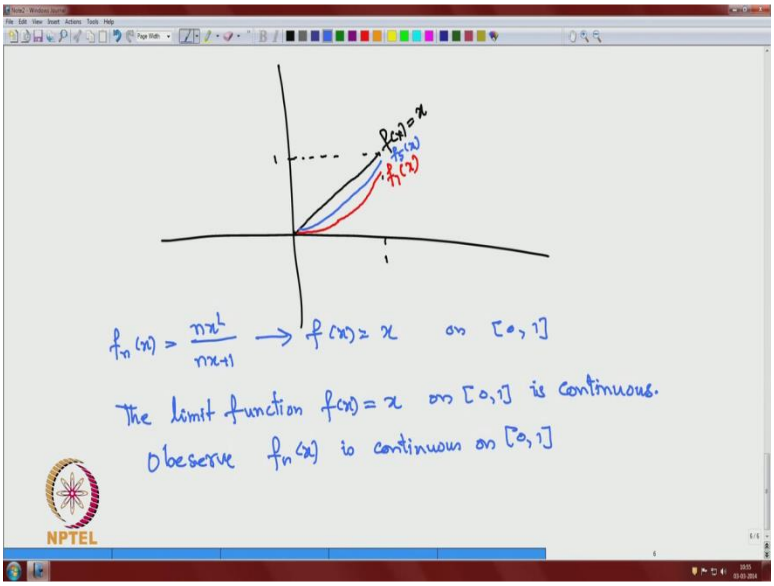

So, graphically we can see that the graph of this functions, for this is 1, this is by is 1, then f (x) is equal to x, is a limit function. And, you see, that for f 1 looks like a continuous function f, and f may be f 5. And, as n is becoming larger and larger, it is crossing towards the function f (x) is equal to x. So, what is the conclusion? The conclusion is, the sequence f n (x) which is n x square by n x plus 1, this converges to a function f (x) which is equal to x, on the interval 0, 1.

And, one thing we can observe that this time the limit function is also continuous. The limit function f (x) which is equal to x on 0, 1, is continuous. And, observe, that each of the function with the sequence f n (x) is continuous on 0, 1. So, each function in the sequence is continuous, and this sequence converges to a function f (x) which is also continuous.

So, this time we brought a better result that the limit of the sequence of continuous functions is also continuous. And, this happens because the convergence in this situation is uniform convergence; we will show that the convergence is uniform.

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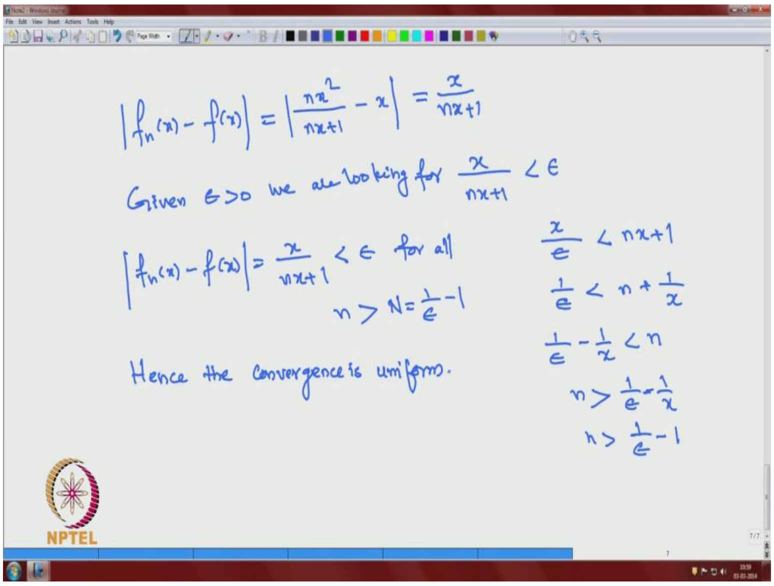

So, how do we show? So, for a given epsilon greater than 0 we should be able to find an n which is independent of x, such that f n, the difference between f n (x) and x is less than epsilon, for all n greater than that N which is independent of x. So, we will show for this example. So, f n (x) minus f (x), by definition of the sequence, is n x square by n x plus 1 minus, the limit function is x. Just doing the algebraic manipulation, we see that this nothing but x upon n x plus 1.

And, what we want is, we want; so, given epsilon greater than 0, we are looking for x upon n x plus 1, is less than epsilon, for all n greater than a given N, which is independent of x. So, we are looking for this one, x; a symbol manipulation if we apply here, this one, this can also be written as x upon epsilon is less than n x plus 1. And, if you divide throughout by x, then 1 by epsilon is less than n plus 1 by x.

So, in other words, 1 by epsilon minus 1 by x, is less than n; or if n is greater than 1 by epsilon, minus 1 by x; and 1 by x, that there you can have a bound that n is greater than 1 by epsilon, so, x varies in the interval 0, 1. So, the greatest value it can assume is 1; so, 1 by epsilon minus 1. So, therefore, it is obvious that this difference f n (x) minus f (x) which is x by n x plus 1, and which is less than epsilon, for all n strictly greater than N that N is 1 by epsilon minus 1.

So, this n is independent of the point x. So, that is true, this n will work for all x. So, hence, the convergence is uniform. And, we have a uniform convergence of sequence of functions, and it converges to f. And also, we got an additional property that the limit function is also continuous when each individual function, the sequence is continuous. So, we state an important theorem in this regard, when we have uniform convergence.

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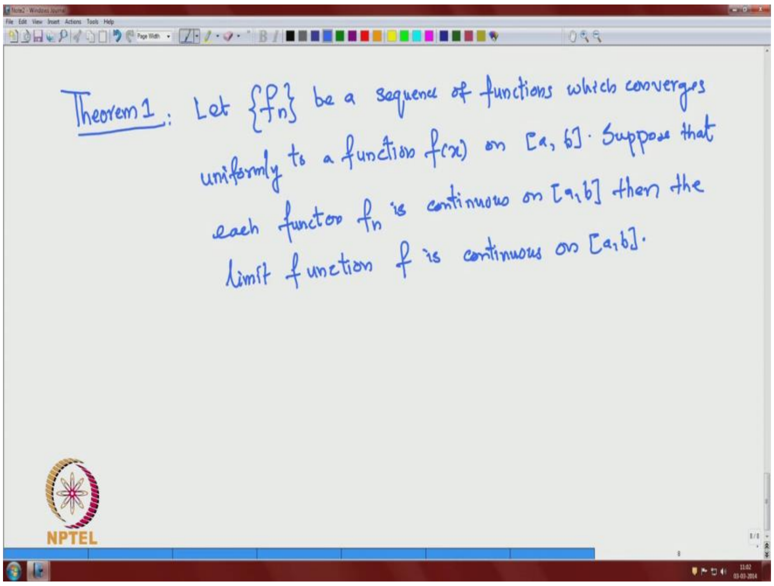

So, theorem, called theorem 1. Let, f n, be a sequence of functions which converges uniformly to a function f (x) on an interval a, b. And, suppose that each function f n (x) is continuous on a, b. So, this sequence converges uniformly to a function f (x) on a, b; and, we assume that, each function f n is continuous on a, b, then the conclusion is then the limit function f (x). So, limit function f is continuous on a, b.

So, this is an inversing result, that guaranties that if a sequence of function converges to a function, and each of the function the sequence is continuous then the limit function is also continuous, provided the converge is uniform. So, therefore, the sequence of continuous functions converging uniformly to a function f, then the function limit function is f itself is continuous. In the existence and uniqueness theorem, we will also use the idea of interchange of limit and integration of sequences of functions.

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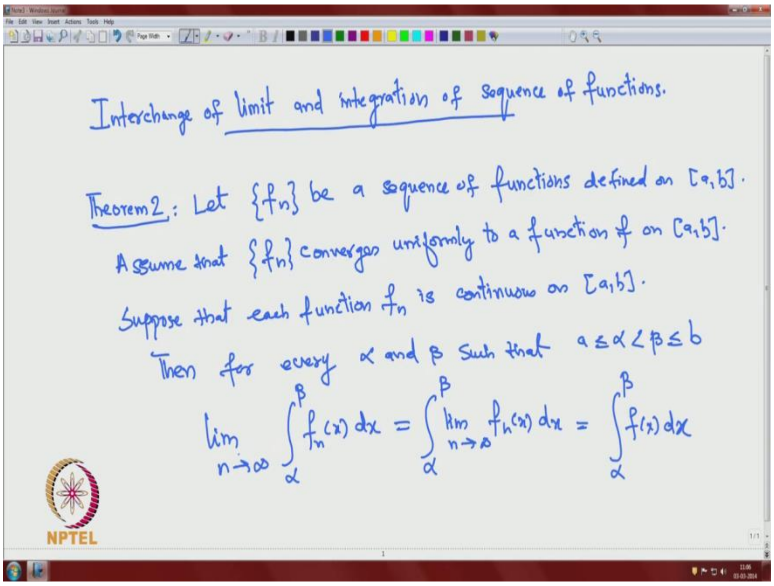

So, we now see a result, that deals with interchange of unique and integration. So, interchange of limit and integration of sequence of functions. So, we state a theorem with a proof. Proof can be seen in any one of the standard text on real answers or calculus. So, theorem, call it theorem 2.

So, let, f n, be a sequence of functions defined on some interval a, b. So, assume that, f n, converges uniformly to a function f on a, b. So, f n is the sequence of functions on a, b, and that converges uniformly to a function f on a, b. Also, suppose, that each function f n is continuous on a, b, then the conclusion is, so then for every alpha and beta, such that a is less than or equal to alpha, strictly less than beta, less than or equal to b.

Limit n goes to infinity, integral alpha to beta, f n (x) d x is equal to, limit integral alpha to beta, you can take the limit inside, n goes to infinity, f n (x) d x, which is nothing but integral alpha to beta, f (x) d x. So, what does it say? If you have a sequence of functions that converges uniformly, and each of the function in the sequence is continuous, then you can interchange the limit and the integrations. So, you can take the limit inside the integral, and, which is same as, you know taking the limit of integral or integrating the limit of the function.

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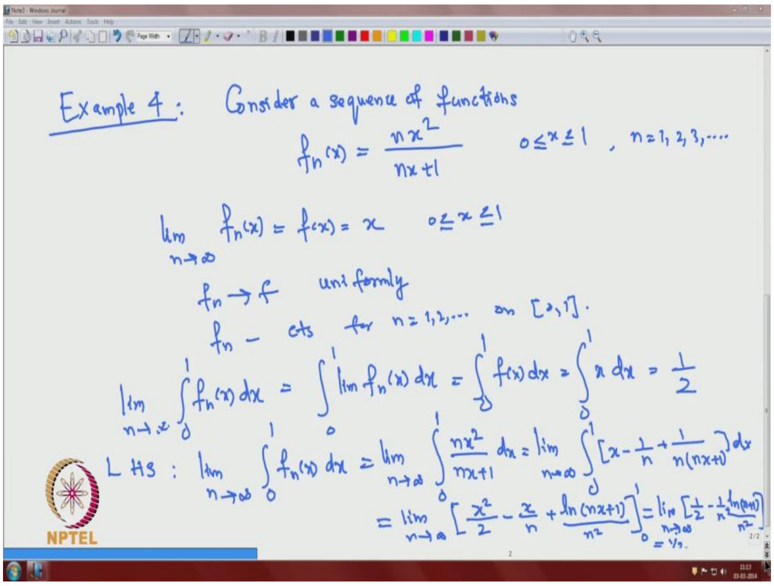

So, we will see 1 example in this regard; so, example 4. So, consider a sequence of functions which we have already considered. So, f n (x) which is equal to n x square by n x plus 1, in the interval x is less than or equal to 1, greater than or equal to 0, n is equal to 1, 2, 3, etcetera. So, we have already seen that this sequence of functions converge uniformly to a function, limit function f (x) is equal to x.

So, limit n goes to infinity, f n (x) is equal to f (x) which is x, we have seen from the previous example, for x between 0 and 1. And, this convergence is uniform convergence. So, f n converges to f uniformly, and also f n is continuous for n is equal to 1, 2, on the intervals 0, 1. Now, let us see what will be the value of the limit? So, limit n goes to infinity, integral 0 to 1, f n (x) d x which is equal to, theorem says that, this can be the limit can be taken inside. So, limit be, integral 0 to 1, limit f n (x) d x, which is integral 0 to 1, the limit is f x d x, that limit is f (x) is x; so, which is 1 by 2.

Now, if we take the LHS, so, if we take the left hand side and treated separately, so, LHS we take the limit n goes to infinity, integrals 0 to 1, f n (x) d x which is limit n goes to infinity, integral 0 to 1, n x square by n x plus 1 d x, which is equal to, limit n goes to infinity, integrals 0 to 1, if you do simple algebraic, suppose if you divide this polynomial by the denominator, this can be shown that this is nothing but, x minus, 1 by n plus, 1 by n into n x plus 1 d x.

So, this is equal to, if you do the integrations, so, limit n goes to infinity. So, if we integrate it, it is x square by 2 minus, integral of 1 by n is x by n, and integral of 1 by n into n x plus 1 is natural logarithm of n x plus 1 divided by n square. And, if you evaluate the integral at 0 and 1, you get this is equal to, limit n goes to infinity, 1 by 2 minus, 1 by n into l n of n plus 1, divided by n square, this plus. And now, if you take the limit, that limit happens to be, so, this is equal to 1 by 2; ensure that this is equal to 1 by 2. So, this limit is equal to half.

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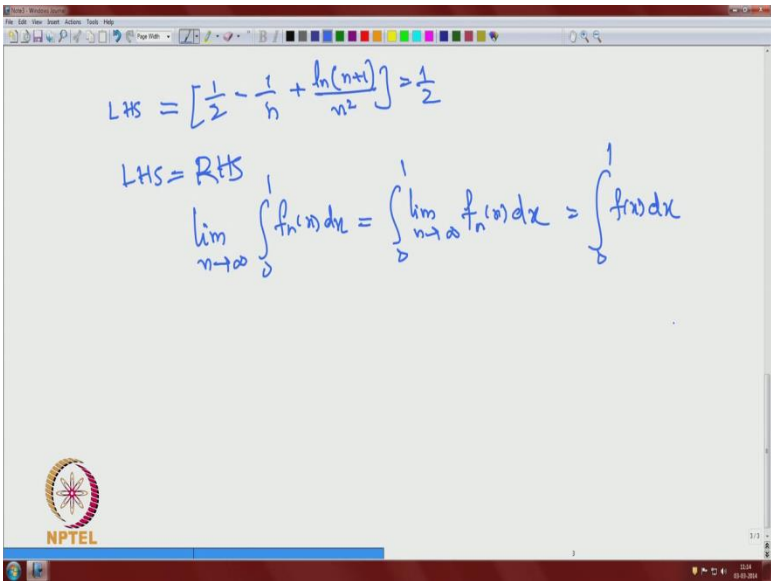

So, the limit, the LHS is, 1 by 2 minus, 1 by n plus, l n of n plus 1 by n square, which is half. So, the RHS is equal to, LHS is equal to RHS. So, therefore, this we, this verifies at, as limit n goes to infinity, integral 0 to 1, f n (x) d x is equal to, integral 0 to 1, limit n goes to infinity, f n (x) d x, which is of course, is going to integral 0 to 1 f (x) d x. So, conclusion of the theorem, the conclusion is then the theorem is verified.

So, in this lecture, what we have seen is, 1 say, sequence of functions are given; there are 2 types of convergences we discussed, one is a point wise convergence, and other one is the uniform convergence. The advantage of uniform convergence is that it preserves the continuity of the terms of the sequence to the limit function, if each of f n is continuous, and f n converges uniformly to f, then the limit function f itself is continuous on the given interval a, b.