Ordinary Differential Equations
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Module - 2
Lecture - 7
Linear Algebra Continued

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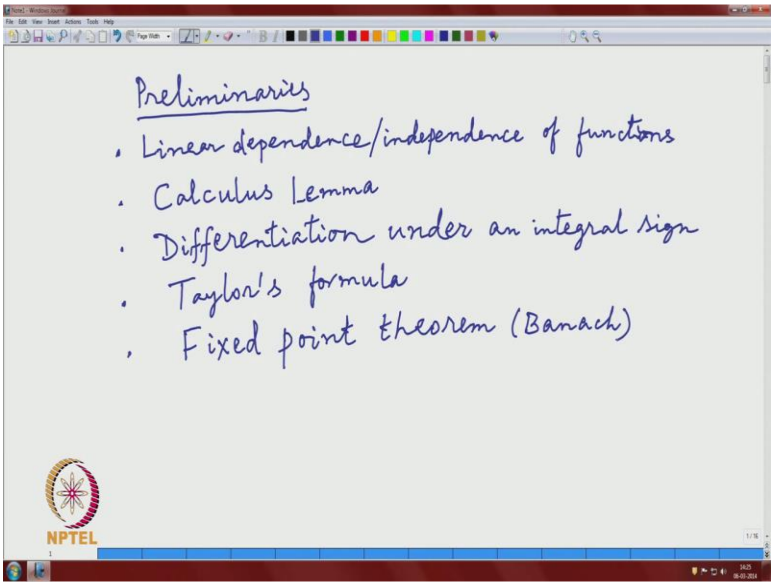

Today we are going to discuss some topics in preliminaries. So, again the topic includes some from analysis, and some from linear algebra etcetera, so specifically. So, let me mention what I am going to do in this one hour; first discuss linear dependence and independence of f functions. So, that is very much used in this study of linear homogenous equations later. So, linear dependence and independence of functions. And then we will also discuss the calculus lemma, which is going to be used in a qualitative theory of differential equations. So, this is one specific calculus lemma. There are many things in calculus,, but this what I am going to state about this lemma. And then an important formula; this is referred to a differentiation and their integral sign. So, this is going to be used; for example, in the study of boundary value problems, using green function and an integral sign. And then we will also learn something about Taylor’s formula which will be used, in the linearization theory of non-linear equation. And finally, we will also discuss some fixed point theorem and some related issues. So, this is referred to Banach fixed point theorem. So, we are going to learn about that.

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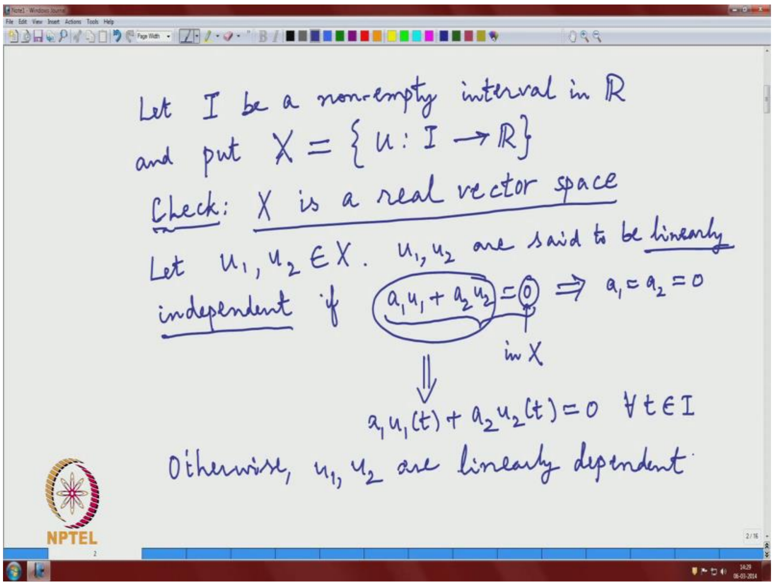

So, let me start with linear dependence and independence of solution. So, let I be a non empty interval in r. So, this can be finite infinite, does not matter, and put x is equal to. So, this is collections of all functions, real valued functions defined on the interval. As of now I am not putting any structure on these solutions, continuity or differentiability anything. So, they are just functions, and it is easy to think that x. So, check x is a real vector space. So, if we are given two functions in this set x u 1 u 2, then I can define their addition and multiplication by a real number. And if u belongs to x minus u also belongs to x and minus u is the ((Refer Time: 05: 30)) of u, and the identically 0 function is the 0 in this vector space. So, this is not difficult to check that x is the real vector space. So, we are going to discuss what is meant by linear dependence or independence of 2 elements in this vector space x.

So, let u 1 u 2 belongs to x. So, this is general definition unit vector space. So, u 1 and u 2 are said to be linearly independent. If a 1 u 1 plus a 2 u 2 equal to 0 implies a 1 is equal to a 2 equal to 0. The only thing we have to remember here is, what is meant by these. So, this is 0 in x. So, a 1 u 1 plus a 2 u 2 is again a function from the interval I into real numbers. So, left hand side is function, and this is the 0 function. So, this implies a 1 u 1 t plus a 2 u 2 t equal to 0. Now, these are real numbers, for all t in the interval I. So, whenever this happens, if that implies a 1 equal to a 2 equal to 0 then u 1 and u 2 are said to be linearly independent. So, otherwise u 1 u 2 are linearly dependent. So, now, we are going to get some sufficient conditions on u 1 and u 2, for them to ne linearly independent.

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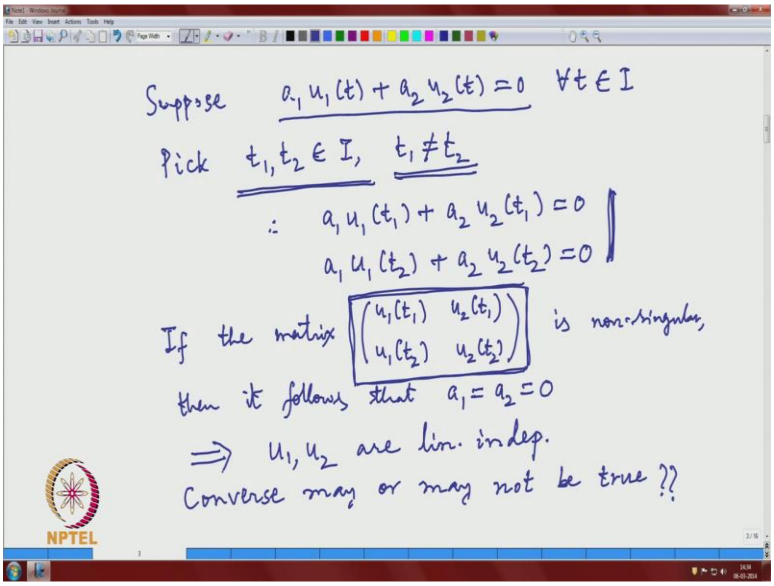

So, that next thing. So, suppose a 1 u 1 t a 2 u 2 t is 0 for all t in I. So, u 1 and u 2, I am picking two functions, and suppose their linear combination is 0. So, when can I conclude a 1 equal to 0 a 2 equal to 0; that is my question. So, for this purpose pick t 1 t 2 in I and t 1 different from t. So, pick any two distinct points. and from these, now if you substitute t equal to t 1 and t equal to t 2, because that is valid for all ((Refer Time: 09:22)). So, therefore, I have a 1 u 1 t t 1 plus a 2 u 2 t 2 t 1 equal to 0 and a 1 u 1 t 2 plus a 2 u 2 t 2 equal to 0. and now look at these two linear equations; they are homogenous, because right hand side is 0, and consider that matrix u 1 t 1 u 2 t 1 u 1 t 2 u 2. So, in a sufficient condition if the matrix u 1 t 1 u 2 t 1 u 1 t 2 u 2 t 2. So, this is coefficient matrix, is nonsingular, then it follows that, because these are two homogeneous equations, and the determinant is non zero, that matrix is nonsingular. It follows that it has only trivial solutions a 1 is equal to a 2 equal to 0. So, with this condition, so if you are able to pick two distinct points in the interval I, such that if this matrix is nonsingular, then we are getting a 1 equal to a 2 equal to 0.

So, that implies u 1 u 2 are linearly independence, so let me shorten it. So, that is only a sufficient condition. It does not appear to be necessary, but in this class x, if x a huge class of functions. So, it is difficult to find an example you can try that. So, converse may not be true. So, I just make a remark, so you may check that, may converse, may or may not be true. So, I am not sure about that, because that class is too big to; may not be true. So, what I am trying to say is, suppose you pick any two distinct points in the interval I, and suppose for all the choices if this matrix is singular, and yet the functions u 1 and u 2 are linearly independent. So, that you can... So, this question, I put a question mark, but when you take special functions. So, we are going to now take a subset of x, then we can say much more; that is the next thing will go.

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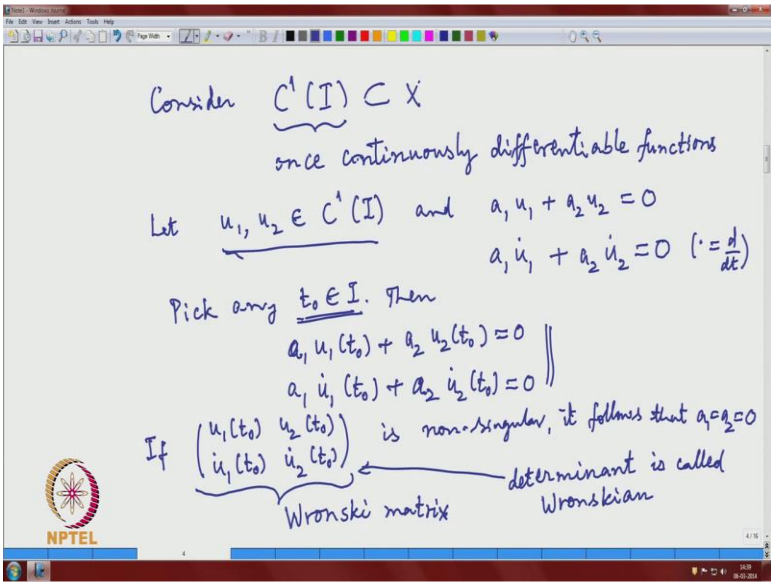

So, consider c 1 I c 1 I. So, this is a subset of x. So, these are once continuously differentiable functions. So, this class is much smaller than x. So, now again lets study when two functions are linearly dependent or independent. In previous discussion we just took two functions. We can also consider infinite number of functions, and discuss their linear dependence or independence, and that the algebra will be more complicated, but the same idea. So, instead of two distinct points you have to consider n distinct points and do that. So, again let me just restrict the discussion to two functions. So, let u 1 u 2 belong to c 1 I and a 1 u 1 plus a 2 u 2 equal to 0.

So, I would like to see under this condition when I am imposing, some conditions are u 1 u 2 if necessary. when does it follows that a 1 equal to a 2 equal to 0; that means, u 1 and u 2 are linearly independent. In this situation when they are both differentiable, so yet the second equation automatically. So, just differentiate this. So, my differential notation is dot a 1 a 2 are just constants. So, when I differentiate the first equation I get that. So, dot is remember d by d t. now you fix 1 t naught, pick any t naught in I, then we have this a 1 u 1 t 0 plus a 2 u 2 t 0 equal to 0 and a 1 u 1 dot t 0 plus a 2 u 2 dot t 0. So, unlike in the previous situation, now just one point will do, and you are getting again, getting two equations, and from this we would like to conclude a 1 equal to a 2 equal to 0, and for that this coefficient matrix should be now 0, so this matrix. So, if again this u 1 t 0 u 2 t 0 u 1 dot t 0 u 2 dot t 0 is non singular, it follows that a 1 equal to a 2 equal to 0, and this matrix is called Wronski matrix of u 1 u 2 at the point t 0 and its determinant. Let me write here determinant is called Wronski. So, we will introduce some notation; Wronskian of u 1 and u 2 at t equal to 0. So let me just again write that thing.

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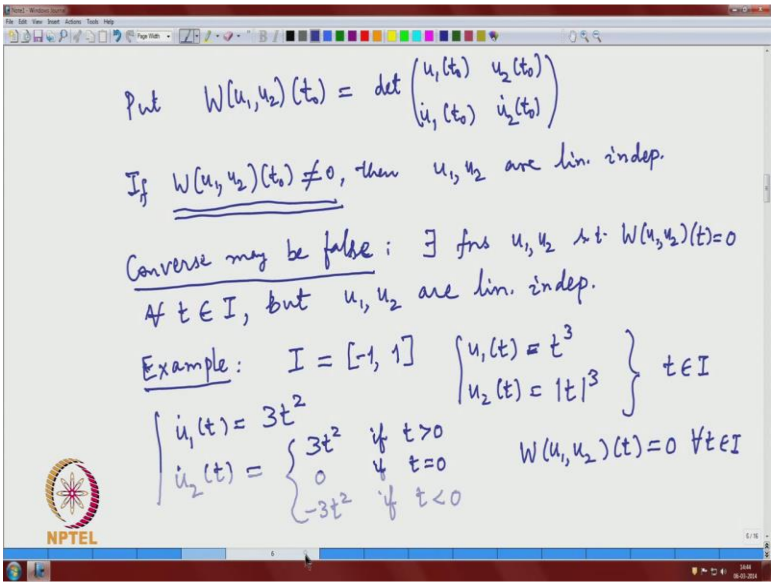

So, put w u 1 u 2 at t 0 equal to determinant of this matrix u 1 t 0 u 2 t 0 u 1 dot t 0 and u 2 dot t 0. So, what we saw in the previous slide, is that if w u 1 u 2 t 0 is not 0 then u 1 u 2 are linearly independent. So, when you put more structure into the function. For example, here you have taken differentiable functions. So, we see a simple condition and that too just at one point. So, even if the Wronskian is non zero at one single point, those two functions u 1 and u 2 are going to be linearly independent; however, converse may false. So, there exist functions u 1 u 2 such that; w of u 1 u 2 at t is 0 0 for all t in the interval, whichever interval we are considering, but u 1 u 2 are linearly independent. So, this I was taking examples. So, let me just mention that. So, example; so, you work out the details, so I is some interval containing both positive and negative real numbers; say for example, minus 1 and 1 and u 1 t is t cube and u 2 t is mod t cube. This is for t belongs to I. So, you readily see that u 1 is differentiable. So, you might have some trouble seeing that u 2 is also differentiable. So, let me just mention here that u 1 dot t that you easily do it. So, this is just 3 t square. So, u 2 dot t simple exercise, so this is 3 t square. If t is positive just you break this into several regions. So, you will see that if t equal to 0 and minus 3 t square if t is negative. So, it is something 3 t square, but sign changes, that is important. And using this definition and this computation, you see that Wronskian of u 1 u 2 t is 0 for all t in I.

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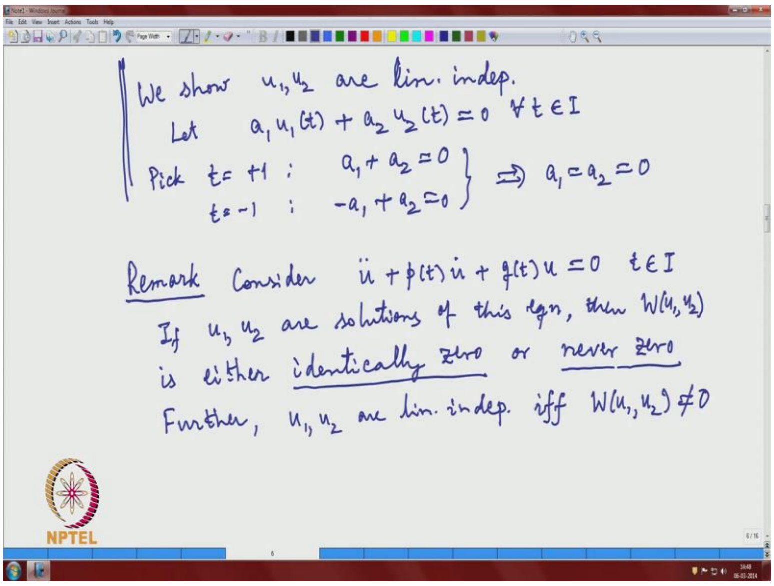

However we show that u 1 u 2 are linearly independent. So, for that, let a 1 u 1 t plus a 2 u 2 t equal to 0 for all t in I. So, in particular you pick t equal to plus 1, you get a 1 plus a 2 equal to 0. And if you pick t equal to minus 1 you get minus a 1 plus a 2 equal to 0 and this will imply a 1 equal to a 2 equal to 0. So, such examples do not occur for some special solutions again, some special functions. So, this remark. So, you are going to see this in the study of linear second order equations.

So, consider this second order equation; so u double dot plus p t u dot plus q t u equal to 0. So, linear homogeneous equation of second order, so in some interval, let me again just write any interval. So, if u 1 u 2 are solution of this equation, then they are Wronskian u 1 u 2 at any t, they are Wronskian; that is a function e is either identically zero or never zero. So, if it is zero at one point then it has to be zero everywhere, and if it is not zero at one point, then it has to remain non zero. And this happens only for this, u 1 u 2 are solution of this homogeneous, so that is special functions. And further u 1 u 2 are linearly independent, if and only if, see the earlier we proved, if that Wronskian is not zero even at one point, then u 1 u 2 are linearly independent, now here its if and only if. So, we saw through this examples this converse may not be true,, but when u 1 and u 2 are solutions of this homogenous second order equation, then converse is also 2 if and only if the Wronskian is not zero. So, that is an important thing you are going to learn, in the study of second order linear equations. So, now we move on to the calculus lemma; that is the next.

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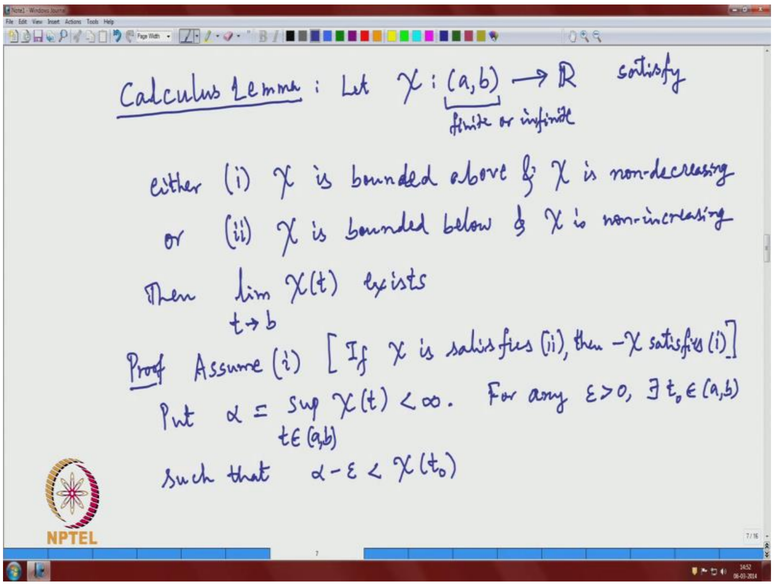

So, let me state it, so this is what I have in mind. So, this we are going to need in the study of qualitative analysis of non-linear system. So, let chi be real value function define on the interval a b. And again this may be finite infinite, it does not matter, this interval finite or infinite satisfy. So, these are the assumptions on the function chi. Either one of these two; chi is bounded above in that interval, and chi is non decreasing or bounded below, and chi is non increasing, then limit chi t s t tends to b, so that could be infinity, so does not matter, adjust. This limit exist means it is a finite number, finite real number. So, with proof… So, assume one, so 2 is similar, and in fact, if chi satisfy 2, then minus chi satisfies 1. So, if limit chi t exist then limit minus chi t also exist. So, it is sufficient to prove the lemma in one case. Since chi is bounded above. So, put alpha is equal to supreme chi t, t belongs a. and since we are assuming chi is bounded above, so this is finite, so for any epsilon positive. So, now we are going to show that the limit exist, for any epsilon. So, if you consider alpha minus epsilon, alpha minus epsilon is not supreme of chi. So, therefore, they at least 1 point in the interval where that alpha minus epsilon, will be strictly less then chi of t 0. So, for any epsilon there exists t 0 in the interval a b; such that alpha minus epsilon is no more supremum, so they should be less than a chi t 0.

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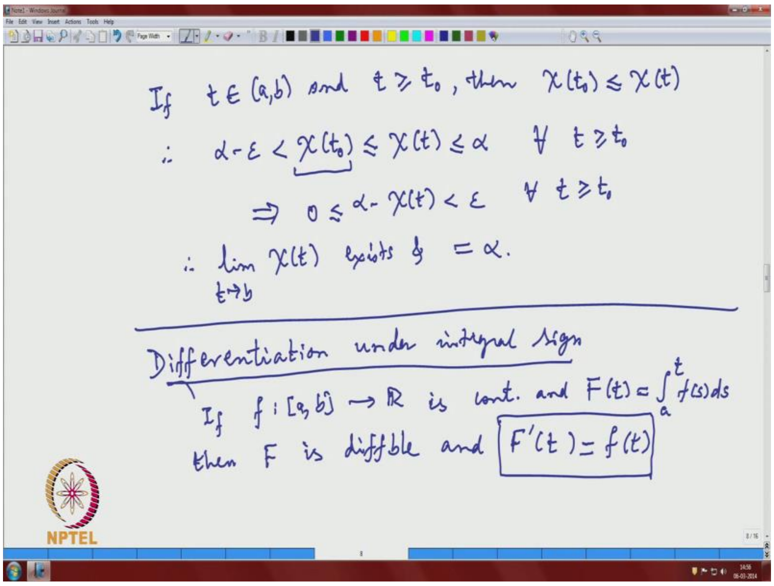

Now the second hypothesis, so if t belongs to a b and t bigger then equal to t 0, then. Since chi is non decreasing, so chi o t not is less then equal to chi t. So, if you put the previous in equality. So, therefore, we have alpha minus epsilon less then chi t 0, and that is less then equal to chi t and is supreme of all chi t. Chi t belongs to a b, so this is automatically infinite equal to alpha, for all t bigger then equal to t. So, if you rewrite these things. So, just remove this I t 0. So, this implies alpha minus chi t, which is always bigger than equal to 0 less then epsilon for all t bigger than equal to t, and this is same as saying that, so therefore, limit chi t as t tends to b adjust and equals to alpha. So, that is very simple thing, but very useful one, so this we will see in many situations. And next whether I am going to discuss about, this differential equation under integral sign. So for this recall, again from fundamental theorem of calculus. So, if f from a b to r is continuous. And if you define f of t, the indefinite integral a to t f s d s, then f is differentiable and f prime t is equal to. So, what I am going to state is a generalization of this, so this you remember, a generalization of this.

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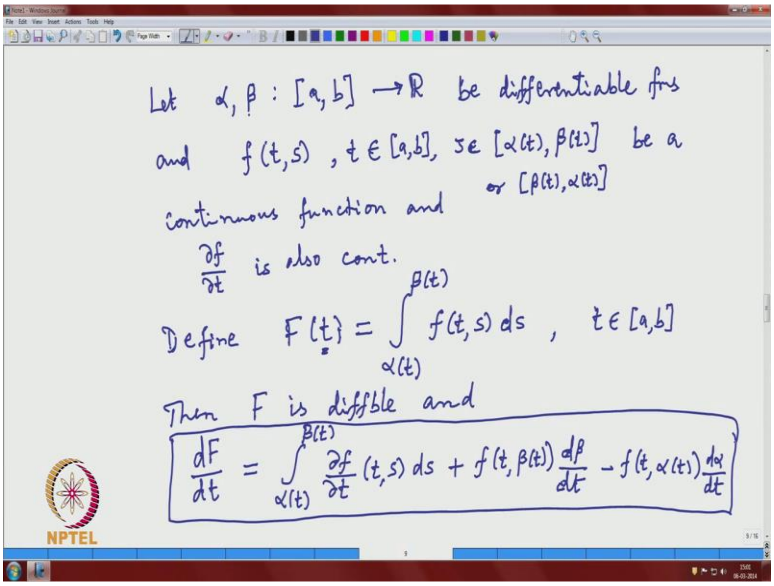

So, again let alpha beta from some interval a b to r be differentiable functions, and f is a function of two variables, so t s. So, t belonging to a b, and s belonging to this interval alpha t beta t. In this notation usually you assume that alpha t is, less than beta t. So, if beta t happens to be less than alpha t then you interchange it, or beta t alpha t. So, there is no restriction on the values of alpha and beta. So, some t alpha t can be less than beta t, at some other t beta t can be less than alpha t; that does not matter. Be a continuous function of both the variables, and I am just taking partial derivative of f with respect to the first variable is also continuous. So, since we are essentially using Riemann integration. So, these hypothesis are needed is also continuous. So, these are the hypothesis and these functions alpha beta and small f. So, now I am going to define a function of t l o f of t just like in the previous case, but now with all these things.

So, integration is also. The integral limits are now variables alpha t beta t, and I am going to integrate with respect to s. So, remember that t is also, t is just fixed. So, once you fix this t, so this you define or t in a. So, then f is differentiable and. So, this is the formula. It is not really difficult to prove it, just like you do in the previous case. You write f of t plus h minus f t by h, and then you manipulate the integral limits, you will get it. So, it is not at all hard. So, this is alpha t beta t d f by d t. So, now, I am assuming that is continuous. So, this integral makes sense, and now since the limits are themselves variables. So, we get some addition terms and these are like this t beta t d b by d t. And since this in the upper limit, it comes with a plus sign and alpha t is in the lower limit that comes with a negative sign d b d alpha by d t. So, just and this is very useful when you study boundary value problems.

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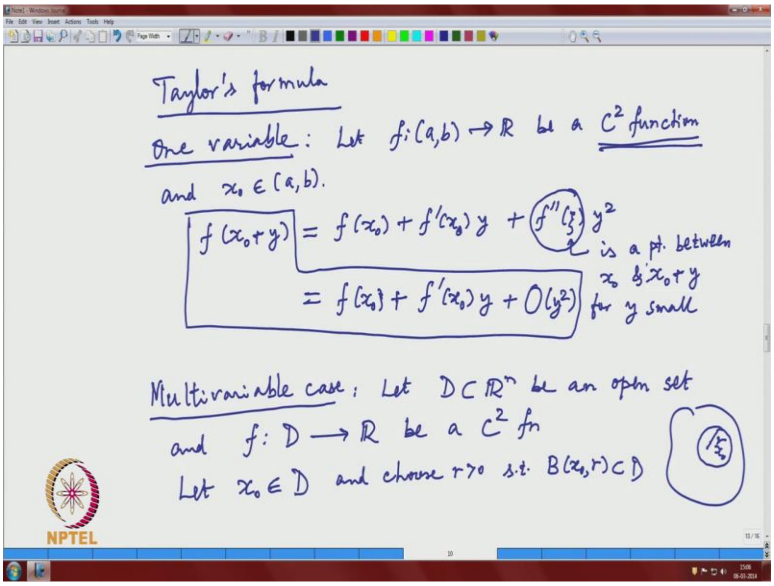

And the next topic of discussion is Taylor's formula. So, let me again start with one variable; that is very familiar with you, to all of you. So, let f from some interval a b to r b a c 2 function. So, I am assuming now f is twice differentiable and that second derivative, is also continuous function and you fix some x 0 in a using mean value theorem and other things. So, it is not difficult to see that f of x 0 plus y is equal to f of x 0 plus f prime x 0 y plus f double prime let me just write psi y square. So, this psi is a point between x 0 and y. So, because we are assuming it is a c 2 function.

So, when y is small. So, this quantity is bounded and usually we write that as f of x 0 plus f prime x 0 y plus o of y square for y small, and this is very much used in linear linearization of non-linear equations. And now once you have this for one variable case we will see how to extend it for multivariable case. So, again let me start with a real valued function multi variable case in act to reduce it to the one variable case. So, let d belonging to r n be an open set and f are same as that f. now it is defined on this d to r b a c 2 functions. So, in this case c 2 function means. So, the all the first order partial derivates and second order partial derivatives with all the variables n, there will be n variables, they all exist and they are continuous. So, again let x 0 belong to x and not x d and you choose a small ball. So, sin we are assuming it is open and choose r positive such that b x 0, r is also continuity d. So, d will be something like that x 0 is here and I am just taking a ball of radius of the ball; that is one. So, now, we are just picking y there and trying to.

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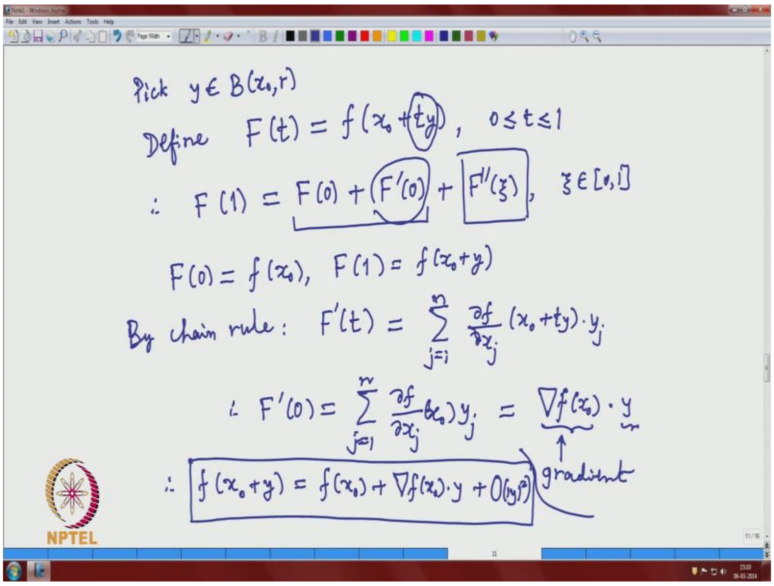

So, pick any y, pick y in b x 0. So, define. So, now f is from d to r. So, now, I am going to define a one variable function. So, this is f of x 0 plus t y. So, t is between 0 and 1. So, now if you apply the one variable case, so with y equal to 1 and x 0 equal to 0. So, therefore, we have f of 1 is equal to f of 0 plus f prime 0 into 1, so that is fine. And now double prime psi and psi is between 0 and 1 f double prime. So, just try to. I mean ignore this second order derivative you are not interested again; that is bounded. So, you are going to. So, we are just interested in seeing the linear.

So, let us calculate what f 0 is and f prime 0. So, f 0 if you look at the definition of the capital f functions. So, f 0, I put t equal to 0 there. So, this is just f of x 0, and what is f 1; f 1 is f of x 0 plus y, and by chain rule f prime t. Let me just write f prime t. So, this is a function of n variables. So, by chain rule this is just del f by del x j evaluated at x 0 plus t y into the differentiation of this argument with respect to t, and there is only dependence is through this t y. So, we just get y j; remember y is a vector. So, this is just y j j equal to 1 to n. So, since we are in n variable case that happens. So we are interested in f prime 0, so therefore, f prime 0 is del f by del f g t equal to 0, means just x 0 y 0 and this you can write in the notation of gradients. So, this is a gradient of f at f 0 dot y. So, this is a vector, and that is a vector, so this is scalar product, so this is a gradient. So, therefore, if you put back x 0 plus y f of x 0 plus gradient of f at x 0 dot y plus. Let me just write that as y square. So, we are interested only in the linear terms.

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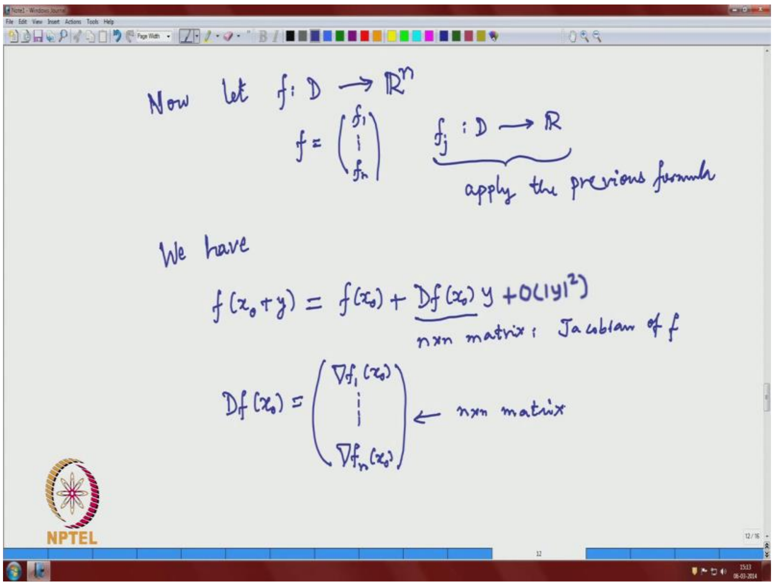

Now, you can just extend it to (( )) case. Let d b as before, and now let f from d to r n. So, you can also do it for r n. So, let me just for simplicity do just for r n, and then f will have n components. So, f 1 f 2 f n, and each f j is now from d to r. and now we can apply the previous discussion with this, apply the previous formula in this for f j, and if you do for all f j then. So, we have this, then you put together we have; that is f of x 0 plus y. Now this is a vector, so this is also vector. So, you should get a vector here, but y is only a vector, so that even if I want to get a vector out of that, so you should multiply by a matrix, and this is just y. So, this is n by n matrix, and is called Jacobean of f, and if you look at the previous formula and apply to each f j. So, this d f x 0 is nothing, but (()) matrix, and each row is gradient f 1 at x 0, and we have gradient f n x 0. So, you write the gradient as a vector, so the first vector coming from f 1, second vector from f 2 etcetera. So, this is a matrix. So, this is n by n matrix.

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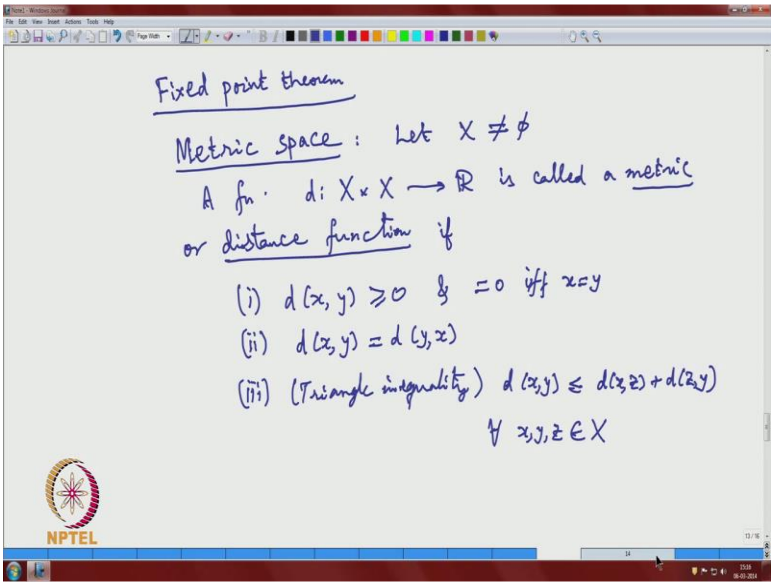

So, finally, let me quickly discuss this fixed point theorem. For this we need the notion of a metric space. So, let p be a non empty set a function d from the Cartesian product to r, is called a metric or distance function, if it satisfy the following three properties. So, if you take any two points in x. So, the existence is always non negative and equal to 0, if and only if x equal to y. This property is referred to as positive definiteness, and second one is symmetry. So, distance from x to y is same as distance from y to x symmetric, and third one is called triangle inequality. So, d of x y is less than or equal to d x z plus d z y. So, these three properties should hold for all x y z in x. So, again this notion comes from the equilibrium distance.

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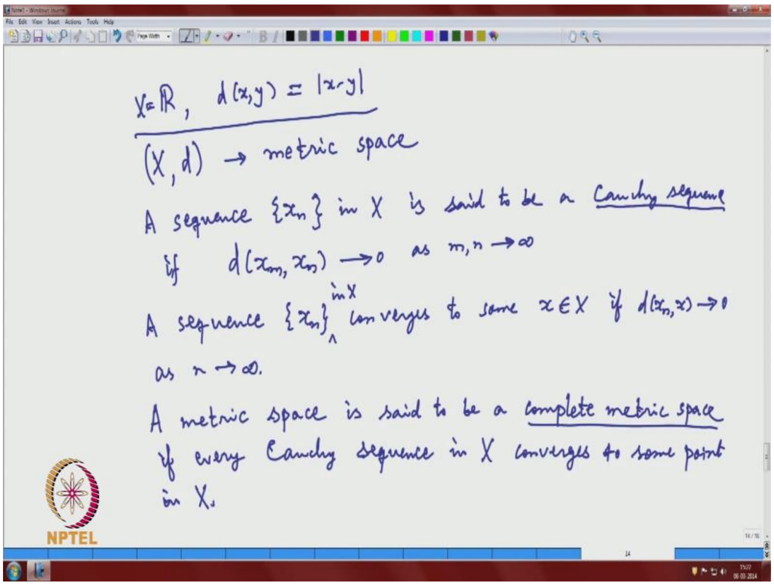

So, this is for example in r if you take x equal to r, and the distance defined by the usual Euclidean distance, so that is a metric. And in r n, again the standard Euclidean distance, so these are examples of metric spaces. So, x d; x is a non empty set and d is a metric ton x, is referred to as a metric space. So, we have another few minutes. So, let just a sequence x n in x; is said to be a Cauchy sequence if limit. So, remember this when you put d. So, these are real number now x m x n, and this should go to 0 as m n goes to infinity. You can also write it in form of epsilons. A sequence x n converges x in n x converges to some x in x, if d x n x tends to 0 as n tends to infinity. So, a metric space is said to be a complete metric space, if every Cauchy sequence in x converges to some point in x. So, it is easy to check if a sequence converges to some x, that x is unique. So, it is a sequence cannot converge to two elements in x. So, that limit is unique.

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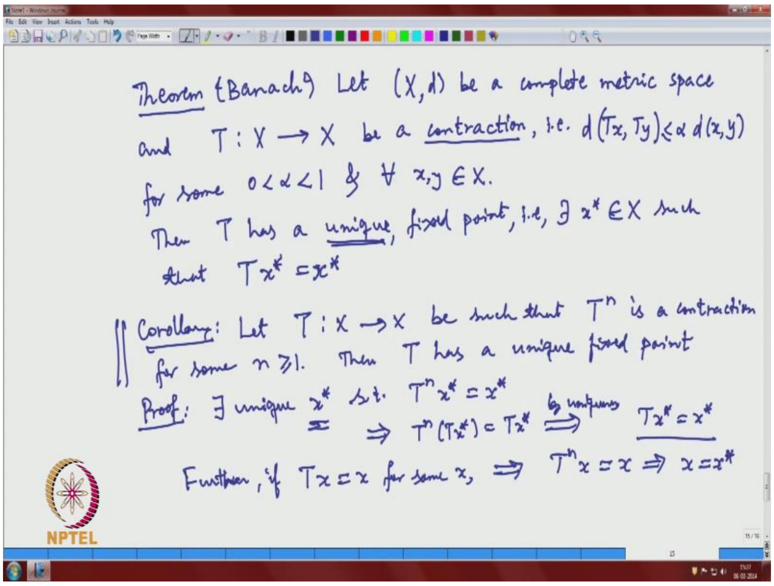

So, now we are going to state this fixed point theorem, Banach fixed point theorem, so let x d be a complete matrix space. So, completeness plays an important role. So, without completeness, the conclusion of this theorem may be false. And t from mapping from x into x itself, be a contraction. So, that is... So, I am going to define what is contraction? So, if you take any two points x and y in x, and now you take their images under t. So, t x t y that their images, and you compute the distance between them, and this should be less than or equal to alpha distance x y. So, for some 0 less than alpha less than 1, and all x y in x. the conclusion is, then t has a unique fixed point that is there exists x star in x, such that t x star is equal to x start and x star is unique in with this property. So, that is meant by that unique fixed point a, its corollary. So, let again the same settings, so let t from x to x be such that. So, t need not be a contraction, but what we are assuming is, t to the n is a contraction. So, t to the n means, you compose t with itself n times. So, t square is t composite t etcetera, is a contraction, for some n bigger than equal to 1.

Then t has same conclusion; unique fixed point. Let just me indicate how this corollary follows. So, proof of the corollary. So, since t n is a contraction we apply Banach's theorem, so there exists unique x star such that t n x star is x star, because t n is a contraction, we are assuming t n is a contraction. And this implies if I again compose with t, so t n of t x star is equal to t x star, and by uniqueness, so this is by uniqueness t x star is equal to x star, because x star is unique. Now we are showing that t x star is also a fixed point for t n, and that is why you should have this t x star equal to t x x star. And further if t x is equal to x for some x. So, if t has another. So, that implies t n x, is again x, and that again by uniqueness implies x equal to x star. And especially with this hypothesis of the corollary is very useful, that you will be going to see in the existence of solution of differential equations; that is very much useful. So, with this thing we will conclude this lecture on some preliminaries.

Thank you.