Ordinary Differential Equations
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Module - 2
Lecture - 6
Linear Algebra Continued

Welcome back. So, yesterday we were discussing about similarity matrix and let me begin with the advantage of this similarity matrix. So, this is our, so a is an introduced notation yesterday.

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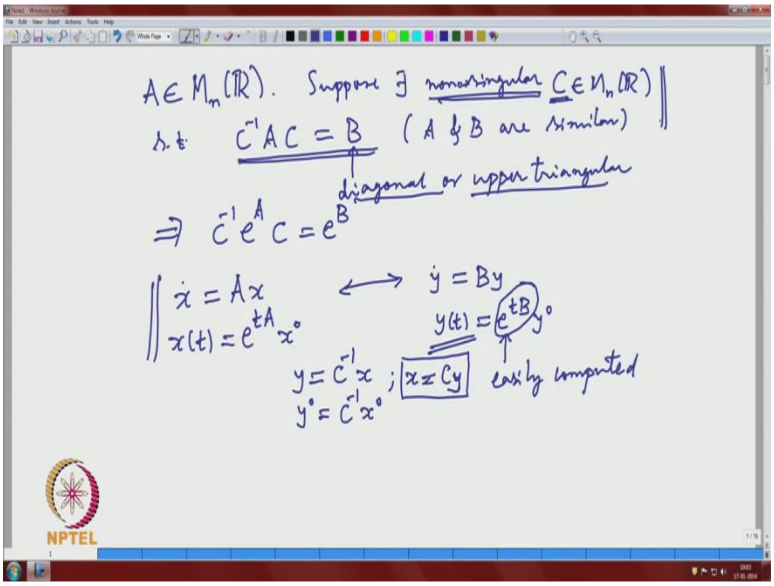

So, this is an m by N real matrix ok we call a is similar to b, so suppose there exists non-singular C belonging to m N r such that C inverse a C equal to b. So, this is we say that a and b are similar. So, we are aiming at finding suitable such that this B is either diagonal are in the worst scenario an upper triangular matrix we will come to this little later and we also saw that, so this implies C inverse E a C equal to E to the b. So, the exponentials of a and B are also similar in the context of a d, so let us consider this x dot equal to a x suppose this is the given system and remove that the general solution it is given by E to the t a and x naught x naught is an arbitrary vector in r m and now through similarity transformation we transfer this given system into y dot equal to B y and similarly y is t is given by E to the t B sum y naught and when B is this simple either diagonal or upper triangular this is very easily computed.

Not only that, we can easily analyse the qualitative behaviour of this y t and how x and y are connected it is very easy, so you just you put y is equal to C inverse x and. So, this y naught will be just C inverse x naught and once you know y, so it is easily we can also analyse x since x is C y, so that is the advantage.

So, that is why this all effort is being done to find the suitable non-singular C such that this C inverse is equal to B and B is quite simple. So, that we can compute the exponential of E B very easily and then we analyse the solution of this ready system and then we get information about the original system that is the advantage ok. So, let us now begin with this thing, so before going to that thing, so just let me again make a few remarks.

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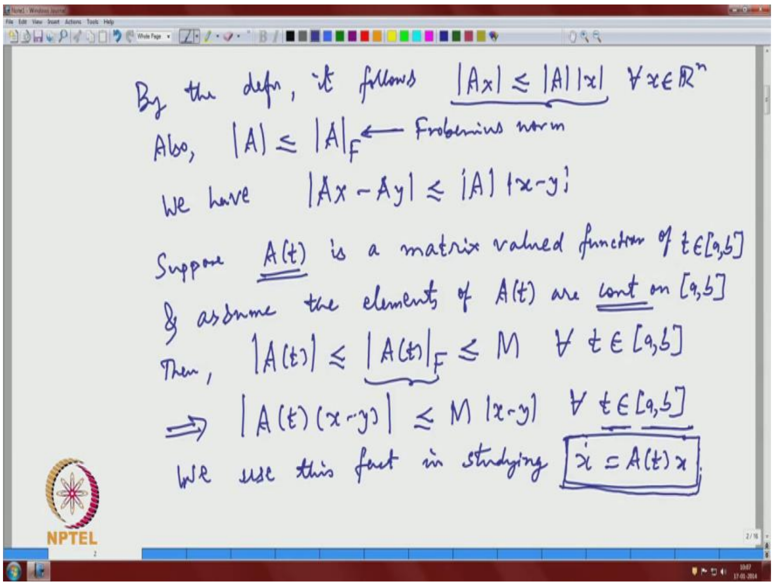

So, by the definition of the norm it follow that norm of a x is less than and equal to norm a norm x for all x in r m we also, saw that also we saw this a is less than or equal to this frobenius norm ok, so this is frobenius norm and from this if I replace x by x minus y and by linearity of a, so we have a x minus a y less than or equal to norm E x minus y and that proves this function x going to a x is Lipchitz continuous.

More is true. So, suppose A t is a mat matrix valued function of t in sum interval, so that means, for each t in A b A t is in m N r and suppose and assume the elements of A t are continuous on a b and then, this for each t I have this thing and since this frobenius norm is sum of all squares of the elements of A t and since I am assuming they are all continuous, so this is just a some let me put m for all t a using continuity of the elements of A t ok.

So, this implies, so we have this useful inequality. So, less than equal to m x minus y for all t in A b, so that means, this matrix the x going to A t x is uniformly Lipchitz in this interval and this is we use this fact in studying the linear system x dot equal to A t x and because of this global existence the solutions exist in any interval where A t is continuous.

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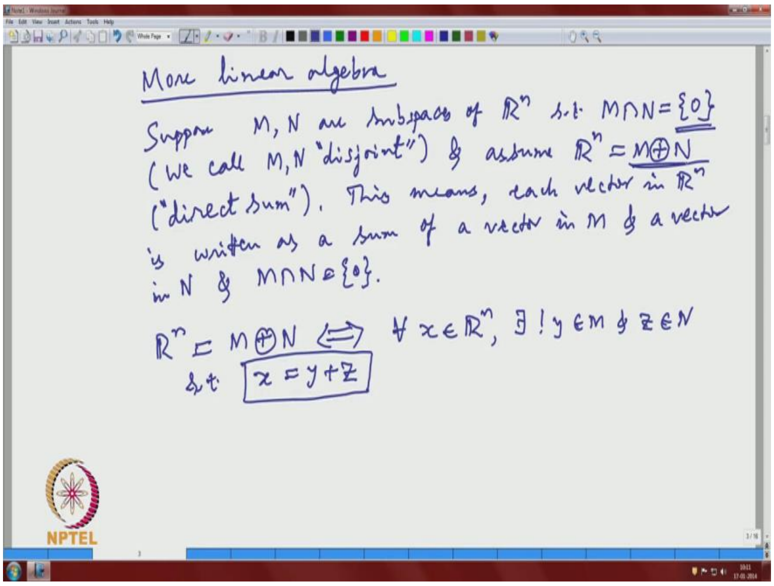

So, with this little remark, so let us again go back to linear algebra more linear algebra as we have said yesterday, this is not a course on linear algebra , so I am just recalling certain facts that are used in our study of differential equations. So, suppose m and N are subspaces of r N such that, so since they are subspaces 0 element is always there and I just want their intersection to be just 0 we concept such spaces m N disjoint. So, though the intersection is not empty, but it is the trivial subspace namely the 0 subspace we still call it disjoint and assume r N is m plus N ok.

Let me explain that little bit. So, this is called direct sum of m N this means each vector in r N is written as a sum of a vector in m and a vector in N and this condition m N are discharged ok. So, we can easily check that r N is equal to m plus N m direct sum m if and only if for all x in r N there exists unique y in m and z in N such that a x equal to y plus z ok, that should happen and this is easily extendable to any finite number of subspaces and that is what we are going to do ok.

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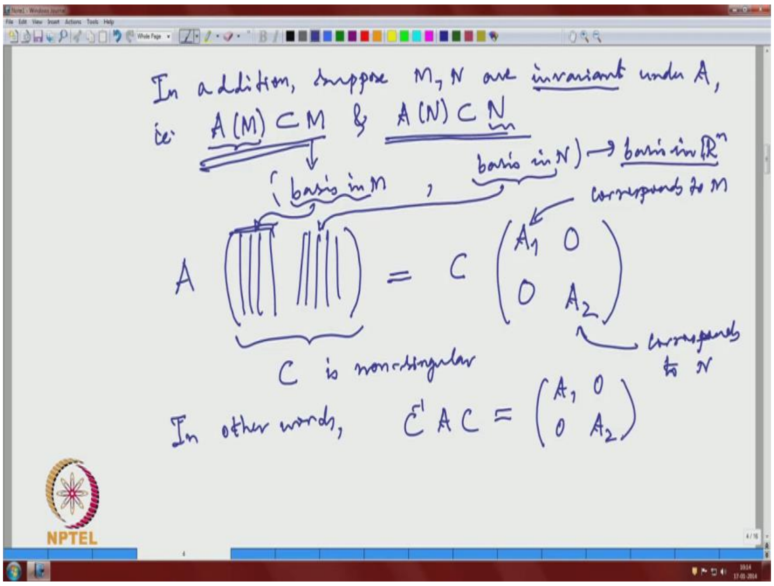

So, in addition suppose m N are invariant under a, so remember a is given matrix N by N matrix, so that is again definition a m, so you at a on every element of m and again that should be in m and similarly for n. So, this means this means you act a on every element of m and again the result should be again back in m in this situation, suppose we choose a basis in m and a basis in N and if you put together that will be a, so I will I am just try to this and this will form a basis in r m now you take those basis elements and form this matrix C ok, so this is C and these are you put the basis elements of m here and then you put basis elements here ok and then you add a. So, since these are all basis vectors C is non-singular and by this suggestion that m and N are invariant under a you can easily check that this C is here A 1 A 2, so this is sub matrix, so this is corresponds to m and this correspond to n because if I act a on this basis elements which are in m and the result is again they are back in m, so each action is again a linear combination of vector of these things and that is expressed like that.

So, in other words, so if you have two subspaces invariant under m and whose direction is r m then the matrix with respect to that basis decomposition to a block matrix. So, in other words what we have is C inverse a C is A 1 0 0 A 2 and this is easily extendable to make finely many subspaces.

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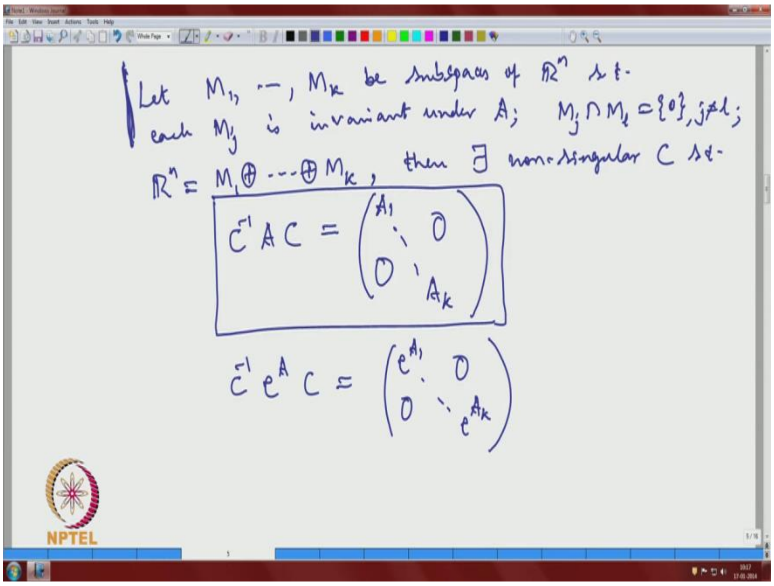

So, let me just state that thing, so let m 1 m k be subspaces of r N such that each m J is invariant under a that is 1 condition and I have m J intersection m l is just 0 J different from l, so they are mutually disjoint and the third condition, so r N is direct sum of m k with these conditions then there exists non-singular C such that C inverse a C is A 1 a k.

So, in other words in such a situation we can find a similar matrix which is a block diagonal matrix satisfying this condition, so this is important in this immediately we see that C inverse E a C is E A 1 etcetera E a this we saw yesterday ok.

Once the matrix is block diagonal it is very easily computed and now we proceed to find some special sub spacious of r N which are related to a and we satisfy this hypothesis of this thing, so that we have a similar block matrix which is similar to a. So, that is our next aim.

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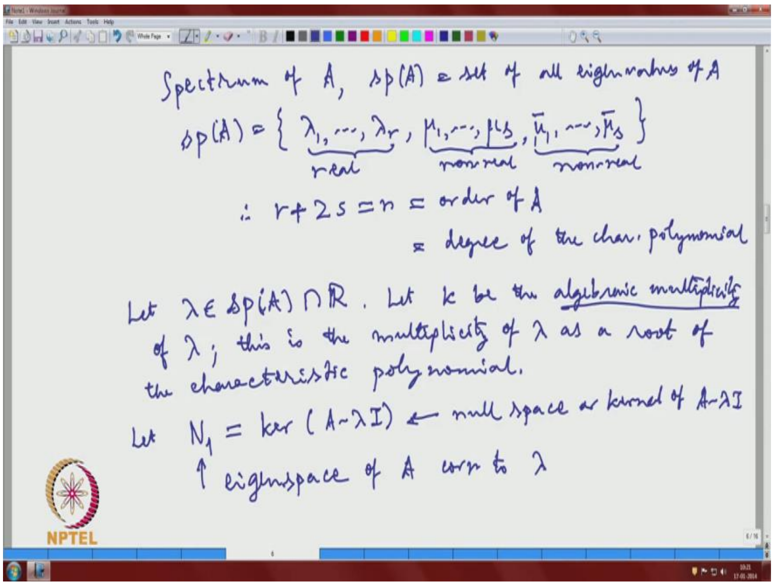

So, yesterday I introduced the concept of spectrum of a, so this is just a spectrum of a, so denoted by s p of a, so this is set of all 8 values of a though a is a real matrix an Eigen value of a could be real complex ok.

So, let us split this into two parts, so let us lambda 1 lambda r, so these are real Eigen values counted with multiplicity and then this we have mu 1 m s, so these are nontrivial and since the characteristic problem well has real coefficients we also have this mu 1 bar mu s bar also Eigen values of a ok, so these are also non, so they always occur in pair, so the total number because we are counting multiplicity, so therefore, we have r plus 2 s is equal to n, so this is the order of a and this is also the degree of the characteristic polynomial right, so now just pick, so let lambda belongs to spectrum of a also real. So, complex is similar.

So, let k be the algebraic multiplicity of lambda, so this is the multiplicity of lambda as a root of the characteristic polynomial that is right, so there is another lotion called geometric multiplicity, so for that we have to introduce a subspace, so let a N 1 B kernel of a minus lambda I ok. So, this is the null space of null space or kernel of this a minus lambda I. So, this is referred as Eigen space of a corresponding to lambda.

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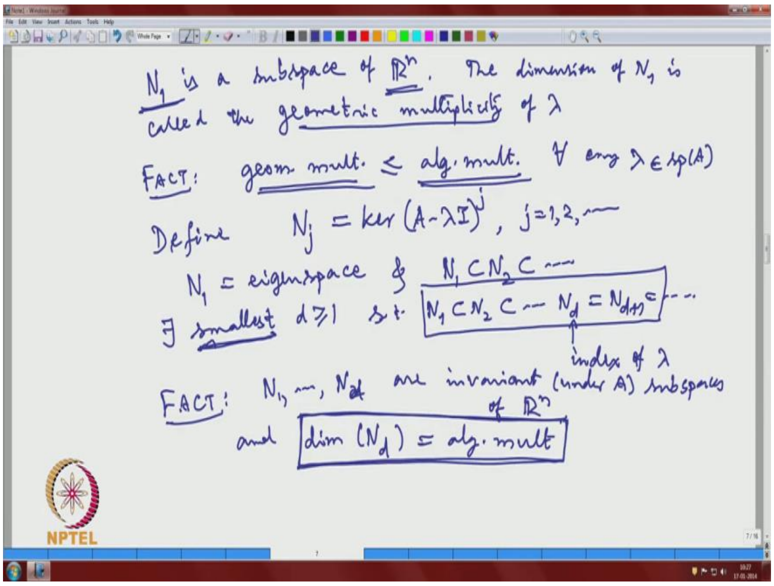

So, if lambda is an Eigen value we just form this and this is a subspace of, so N 1 is s subspace of r n, so in case of complex Eigen value we take the real and imaginary parts of the Eigen vector and we still get a subspace of r N that is important that you keep in mind. So, the dimension as a subspace it has a dimension of N 1 is called the geometric multiplicity of lambda.

So, let me just take a fact that geometric multiplicity is always less than or equal to algebraic multiplicity and the difference is referred to as the difference we refer to as deficiency of lambda for any lambda, so that is always true lambda in spectrum. So, the good case is when geometric multiplicity is algebraic equal to algebraic multiplicity then we will have sufficient number of Eigen vectors spanning this subspace N 1 and a difficult part is when this is strict in equality ok.

So, in that case what we do is, so now define in general, so define N J kernel of a minus lambda I to the power of j, so you do it J equal to 1 to etcetera, so N 1 is the Eigen space that we have defined and we have this J its very easily checked that this is ascending chain of subspaces and since we are in a finite dimension this chain cannot continue for long. So, there exists smallest d bigger than equal to 1 such that N 1 N 2 N d after which they are equal.

So, there are no more additions of new nonzero vectors ok, so this is smallest it has a name and this is called index of lambda. So, everything is we have fixed a real Eigen value lambda and we are just discussing about that, so it again 1 more fact, so these are all subspaces this N 1 N 2 N d are invariant under a, so let me not stress that again and again, so that is invariant subspaces of r N and more importantly and dimension of N d is equal to algebraic materials.

So, when geometric multiplicity of an Eigen value is strictly less than the algebraic multiplicity you have to go for some more linearly independent vectors in order to get the full dimension namely its algebraic you want to reach the algebraic multiplicity and this is the way 1 reaches that day ok.

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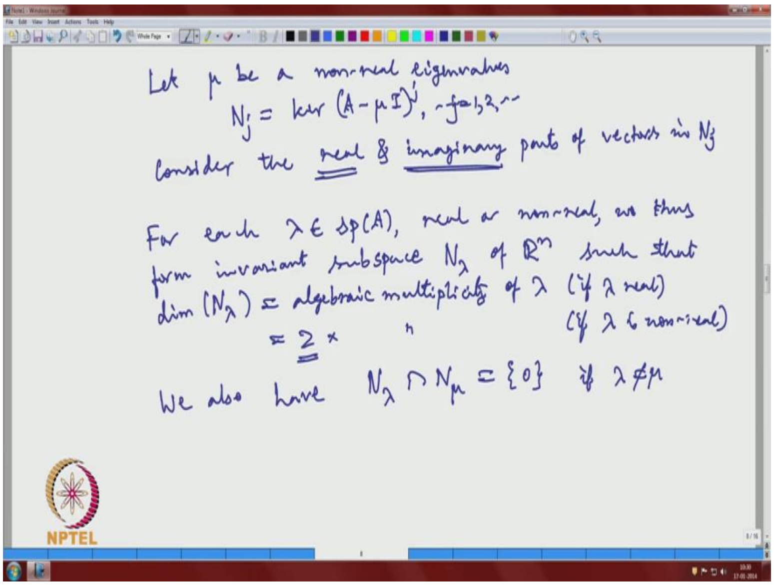

So, let mu because nontrivial Eigen value and again you form the same thing there and there is this 1 there is no problem kernel of a minus mu I J j equal to 1 to N and it will have some index.

Now, you just consider the real and the imaginary parts of vectors in these N j's that is all you have to do, because remember we want to find only a basis for r n, so we want only the real vectors and when there are complex Eigen vectors you take the real and imaginary part they are not Eigen vectors, but they are linearly independent as we saw yesterday and that will do our job, so in this way, so let me now just put this together, so for each lambda in this spectrum of a real or non-real it does not matter we thus form invariant subspace.

So, let me call it N lambda of r N that is important this subspace is r N such that dimension of N lambda is algebraic multiplicity of lambda if lambda is real otherwise, it is two times algebraic multiplicity if lambda is non real because that count has to be proper because when lambda is non real lambda bar is also an Eigen value, so it has to be counted twice. So, that is why this twice and they are all invariant under a and by very choice, so you also have that N lambda intersection N mu is just 0 if lambda is that is by very construction you see that ok.

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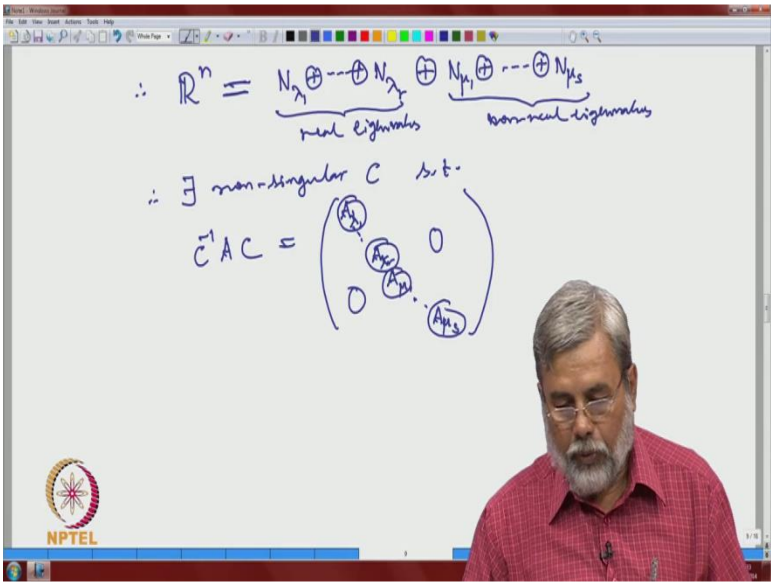

So, therefore, so r N is written as, so let me just write lambda 1 lambda r, so these corresponds to real Eigen values and then I have N mu 1 this N mu s ok. So, I am not writing the mu N bar mu s bar because they are already included here. So, these corresponds to real Eigen values and this correspond to non-real Eigen values and these dimensions perfectly match and that is why the we have this inequality and now you just give the fact we already stated since all these are invariant.

So, these subspaces are invariant, so therefore there exists a non-singular C such that C inverse a C is now A 1 A 2 a r let me write lambda 1 lambda 2 mu 1 mu s and the next task is to find a suitable basis in each of these subspaces, so that this block matrices have a very simple structure that what we are going to do next time next with.

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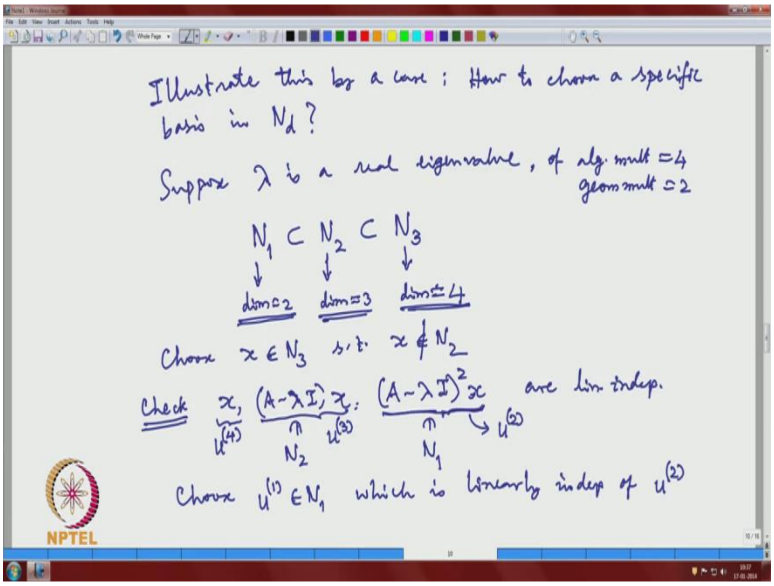

So, instead of doing for a general t, so let me just illustrate by a case how to do that illustrate this by a case. So, how to choose a specific basis in N d. So, suppose lambda is a real Eigen value of algebraic multiplicity four and geometric multiplicity two, so I have N 1 N 2 N 3 say and this dimension is 2 dimension is 3 and this dimension is 4, suppose I have this situation there are other possibilities say let us take 1 such possibility and now I would like to construct a basis where a is simplified to an upper triangular matrix ok.

So, choose x in N 3 such that x is not in N this is possible because the dimension is 3 and this is dimension 4, so there is at least one non 0 vector which is in N 3 which is not in N that is fine now you form this x a minus lambda I x and a minus lambda I square x, so this is in N 2 by very definition and this vector is N 1 and these are all non-zero. In fact, these are are linearly independent check that are linearly independent and now this is a non-zero vector in N1, but dimension of N 1 is 2 , so choose. So, let me call it a u 1 u 1 in N 1 which is linearly independent of, so let me put some name, so this let me call this as u 2 and this 1 u 3 and this 1 u 4, if you change the order the matrix will change, so that is that is what we will see now ok.

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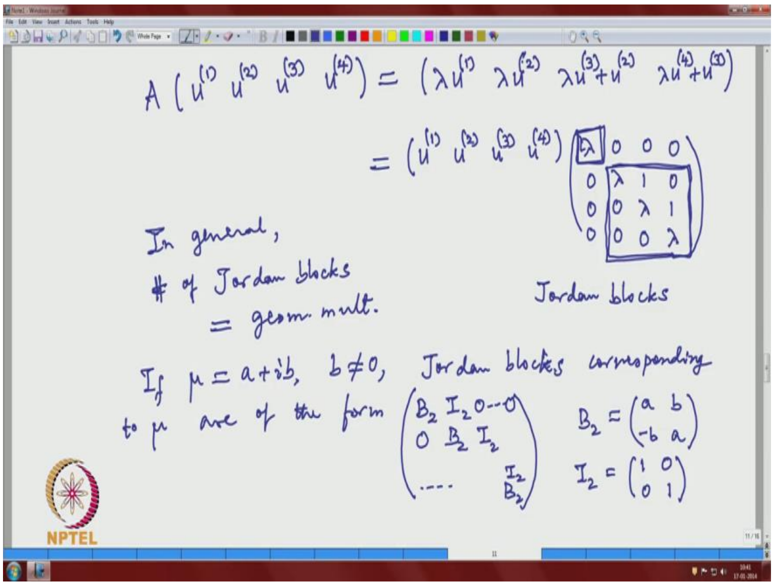

And now you add this u 1 on these vectors u 1 u 2 u 3 u 4 and u 1 is in N1, so this is just becomes lambda u 1 and u 2 is also in N1, so that becomes just lambda u2, but this is not in this u 3 is in N 2, so if you work it out you would see that lambda u 3 plus u 2 and similarly you get lambda u 4 plus u 3, so let me write this this same form u 1 u 2 u 3 u 4 and now I write this matrix lambda 0 0 0 0 lambda 0 0 0 1 lambda 0 0 0 1 lambda so its very easy to check that is what we get just writing this thing and we see significantly here there is 1 block here and another block here see these are referred to as Jordan blocks.

So, in general in general the number of Jordan blocks corresponding to an Eigen value is equal to geometric multiplicity of that particular Eigen value. So, in this case we have the geometric multiplicity 2. So, we get 2 blocks ok.

And if mu is non- real, so just a plus I B b non-zero the Jordan blocks are bit complicated because the Eigen vectors are complex. So, we have to take only real part and a imaginary part the Jordan blocks corresponding to mu are of the form. So, remember they always appear in pairs, so here also we get 2 by 2 blocks, so let me write juts B 2 I 0 B 2 I 2 let me write etcetera. So, just diagonal blocks, so B 2 is 2 by 2 matrix this A B minus B a and I 2 is just 1 0 0 the identity 2 by 2. So, here in case of real thing we have just scalars on the diagonal, but in the complex case you have this 2 by 2 blocks. So, this is the only difference.

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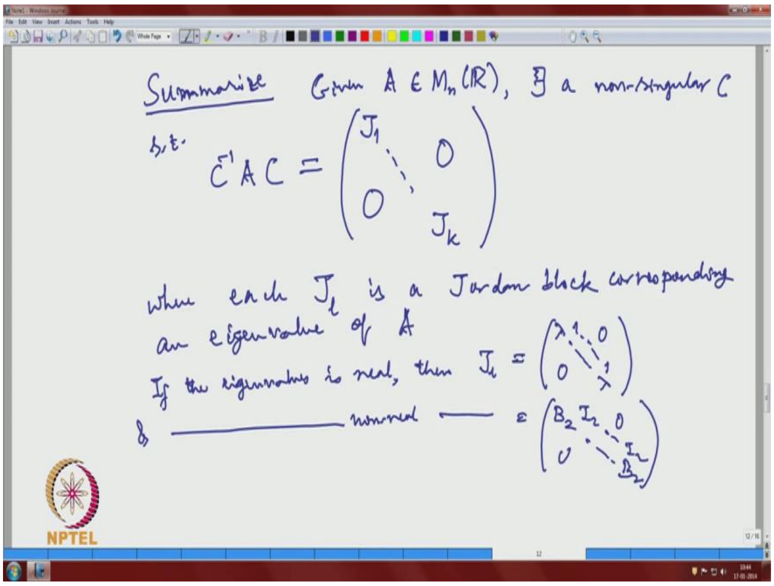

So, let me summarize in detail now, so given a matrix m and r there exists a non-singular C such that C inverse a c, so let me just write J 1 J 2 J k where each J l is a Jordan block corresponding to an Eigen value of a and we have. So, if the Eigen value is real then J l would be very simple lambda 1 and each J l might contain several blocks. So, 1 of the I am just this is not the only possibility there will be several blocks and if the Eigen value is nontrivial then this will be just B 2 B 2 I 2 I that is the only difference ok

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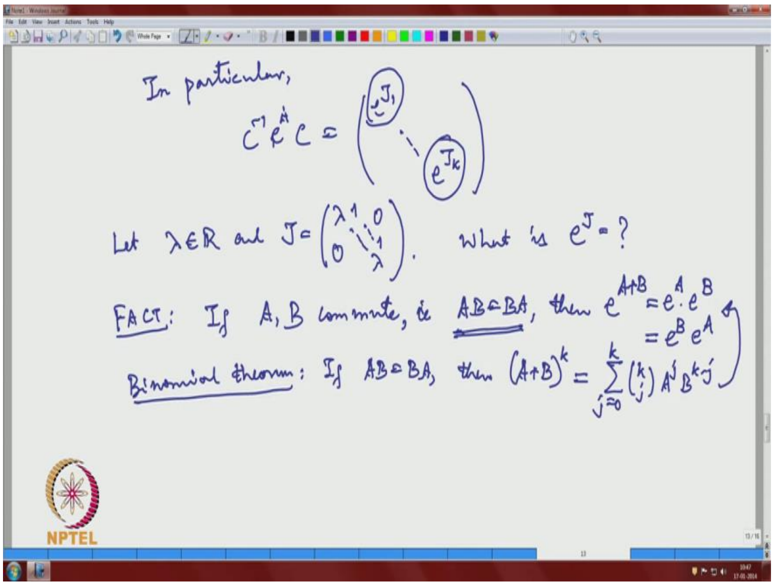

So, in particular we have this C inverse a C not C inverse a C exponential E to the J 1 etcetera and in the next ten minutes I will just show you how it easy it is to compute the exponential of each of these blocks. So, let me just again concentrate on that, so let lambda B real and J is equal to. So, I take one of the blocks.

So, this is a square matrix of some order. So, what is E j. So, that is the main question. So, again a fact. So, if a and B commute that is A B equal to B a then exponential of a plus B is equal to exponential of a exponential of. So, just like exponential of a real number and in general if you have non commutativity then you may not have equality in this E to the a plus B may be different from E to the a and E to the b, so since addition is commutative we also have this ok and this usually follows from this binomial theorem just like again real numbers. So, if A B equal to B a then a plus B to the k is summation k J a J B k minus J equal to zero. So, this you can easily prove by induction and which in turn give this ok.

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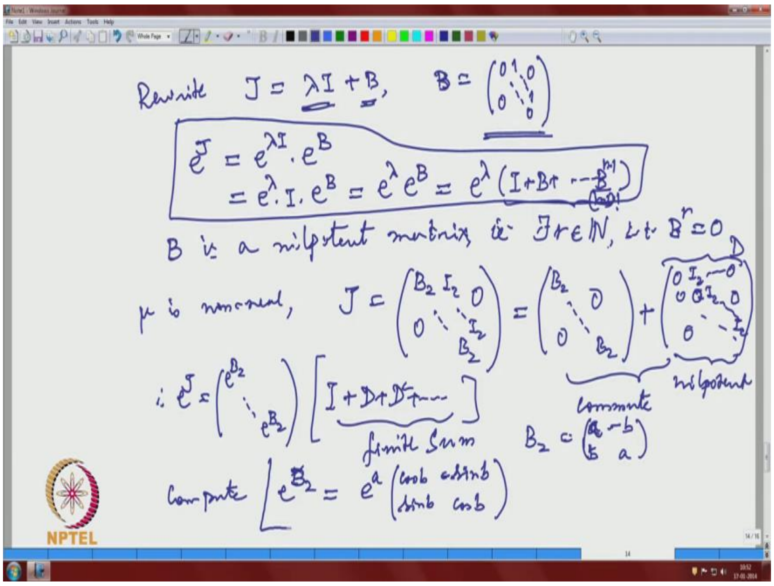

So, that is the only thing involved. So, now we apply this to j. So, rewrite J as lambda I plus b, so B is this matrix 1 1 1 1. So, everywhere 0 except this super diagram which is just all ones now since this is identity. So, these 2 matrices commute. So, we have this E to the J is equal to E to the lambda I E to the b and this you can easily check E to the lambda is nothing, but E to the lambda into I into E to the b. So, this is identity, so we have just E to the lambda E to the B and now let me just expand this, so this is by definition I plus B plus etcetera the important thing about B is B is a nilpotent matrix that is there is exists some r integer r such that B r is 0, so this is a finite sum.

So, it only goes up to B r minus 1 by r minus 1 factorial and it is very easy to compute even the powers, so this the diagonal containing ones will be just shifted above and above. So, it is very easy to compute. So, and we have this thing and in case lambda is non real. So, mu suppose mu is non real then we have this J as block I 2 I 2 etcetera still we can use, so this we will write it as this B 2 B 2 plus again same thing 0 I 2 0 0 I 0 I 2 etcetera.

So, again this is nilpotent and these two come out, so it is very easy to find therefore, E to the J is again E to the B 2 B 2 into I plus, so let me call this matrix as some d. So, I have d plus d square etcetera. So, this is finite sum, so there is no problem about that finite some and I leave as an exercise. So, remember B 2 is a minus B b a just 2 by 2 matrix, so it is very easy to compute that, so compute E to the B 2 is equal to E to the a minus let me just write that E to the a cos B minus sin B sin B cos this is not difficult to compute that ok.

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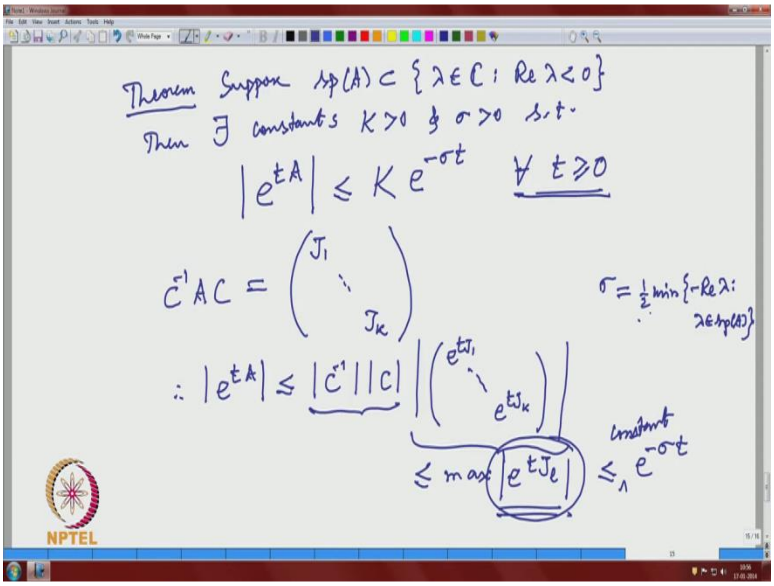

So, finally, let me just take this theorem, so you just combine all these things, so suppose spectrum of a is subset of lambda in C such that real lambda is strictly less than 0, so then there exists constants k positive and sigma positive such that the matrix norm of E to the t a is less than or equal to k E to the minus sigma t for all t. So, this is important this is not valid for all t only for negative and just you have to use is you have this Jordan form J 1 J 2 J k.

So, therefore, you have E to the t a is less than or equal to norm of C inverse norm of C and then you have norm of this matrix E to the t J 1 E to the t J k and this is just a confront you leave it and this was already seen that this is less than or equal to maximum of E to the t J l, because these are all block diagonal and now if you use the previous representation you see that sigma can be taken as minus half maximum of let me minus real lambda in the spectrum of ok. So, this is an exercise you can just find that this can be estimated less than or equal to E to the minus sigma t some constant and put another constant here constants will come.

So, there will be some polynomials here polynomials in t coming from this pair and that will be killed by half of this another half. So, that is why you get only half any fraction will do with that we conclude these preliminaries on linear algebra you can work out several things in the some good text on linear algebra and fill the details that are left out in these two hours.

Thank you.