Ordinary Differential Equations
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Lecture - 5
Linear Algebra

Welcome back. In this lecture and the next one, we will be spending some time in reviewing some parts of linear algebra that are very much used, especially in linear theory and given in qualitative theory of non-linear differential equation. Let me make one point clear, that this is not a course in linear algebra. Our main aim in these two lectures is to give an explanation for exponential of a matrix and some estimates involving matrix norms. So, let, let me just begin with some notation and what we are going to do.

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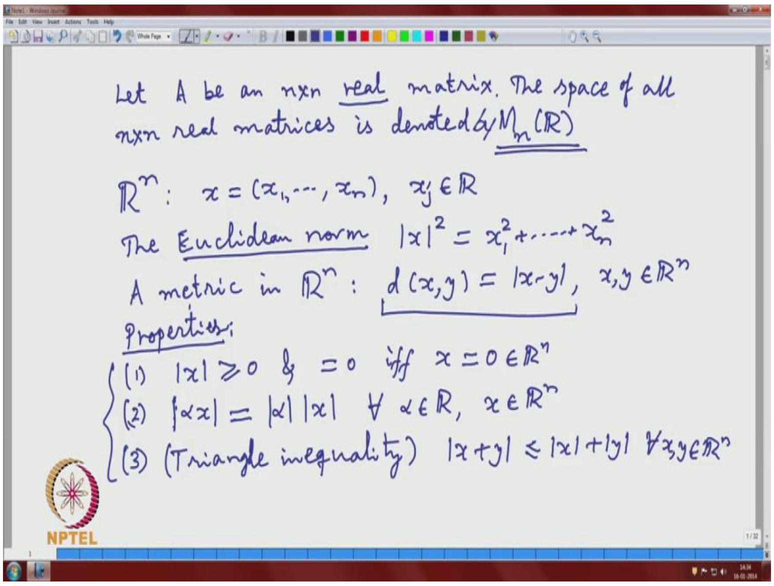

So, let A be an n by n real matrix. So, that means, all the entries of A are real numbers. The space of all n by n real matrix is denoted by M n R. So, soon we are going to put a matrix on this thing, this space, M n R. So, let us begin with the definition of Euclidean matrix R n.

So, this R n, we know this. So, this is the space of all n triples x n. So, x j in R. So, the Euclidean norm, norm x, so let us define in terms of square. So, this is nothing but x n square, so the Euclidean norm. So, this immediately gives rise to a metric in R n; a metric in R n. So, d of (x, y) is nothing, but x y. So, x and y are in R n. So, you can easily check the following properties. The first one, norm of x is bigger than equal to 0, and equal to 0 if and only if x is 0 vector. So, this is 0 vector in R n. So, 0 is n triple consisting of all 0s and alpha x mod alpha norm x.

So, do not get confused with because I am using just single line, so that is, do not get confused with that thing. So, this is all alpha in R and x in R n; just get used to this notation. So, soon we are going to use the same notation even for matrix.

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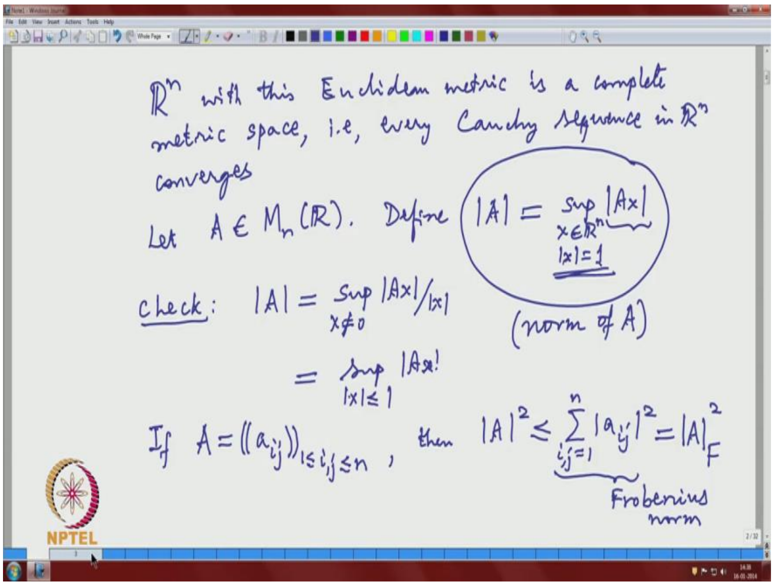

So, the third one, this important property called triangle in equality. And this in turn, these three properties in turn, they prove that d is a metric R n. Not only that, R n with this Euclidean metric coming from the Euclidean norm is a complete metric space, that is, every Cauchy sequence in R n converts. So, that is, and now this Euclidean metric induces a norm in M and n. So, let, you start with a M by n real metric and define again. I am saying, norm of A, this is supremum of Ax x in R n and norm x equal to 1. So, this is well defined. This is a vector in R n.

So, I am taking the Euclidean norm and now I am taking supremum over all x subject to this norm x is equal to 1. So, that is, so you can easily check. So, check norm of A is also equal to supremum… x not 0 and this is also supremum Ax norm x less than equal to 1. So, moreover you can also check that. So, if A is the matrix with entries aij, aij are real, so 1 less than or equal to ij less than or equal to 1, n. So, aij are the entries. Then, norm A square is less than or equal to aij square ij equal to 1 to n. This is not very difficult to see that.

And this one, the right hand side is called the Frobenius norm and usually denoted by A F square. So, this is Frobenius norm. So, different from this norm, but it is this, this norm satisfies this in equality. So, that is, so it is a finite number. So, for any matrix A and we have this in equality. So, that is no problem regarding the finiteness of norm A. So, this is norm, a norm of A, the matrix, the norm of the matrix A.

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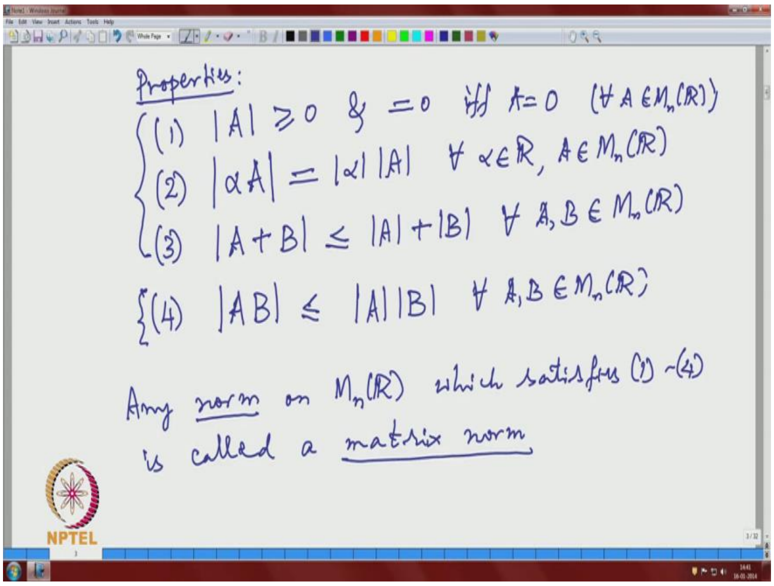

And just like in the previous case. So, you have, can easily verify the following properties. So, again the first one. So, this A is, norm is greater than or equal to 0 and equal to 0 if and only if A is the 0 matrix. So, this is true for all A in M n R.

And second one is again that homogeneity. So, if I multiply the matrix by a real number. I take the matrix. So, this is, so for all alpha n real numbers and A in M n R. So, a real m by n matrix. And third one, again triangle in equality. So, if I have two matrices, A and B. So, A plus B is also in M n R and this is, so for all A, B in M n R. So, you already know, that M n R itself is a vector space over the real numbers. So, it is more than that.

So, we also have the concept of matrix multiplication. So, in case of vectors we did not have this. So, if A and B are two matrices, also have this product of those two matrices, that is again a matrix and this is and this is an important property of this matrix norm for all A, B in M n. So, any, this is the definition any norm, norm is just on M n R, which satisfies 1 to 4 above is called a matrix norm. Norm, by definition, is just usually these three things are included and for matrices we have this special thing four and which we call it a matrix norm. This is only special for matrices or in general linear transformations, ok.

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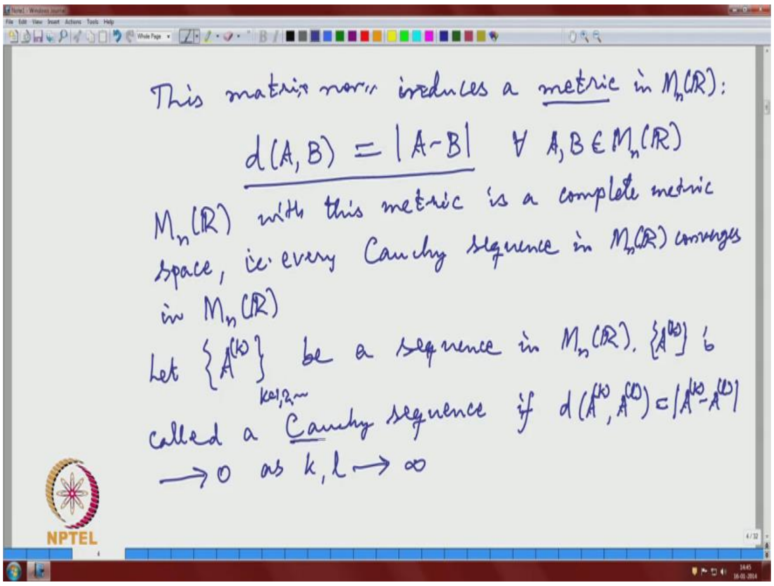

So, with this thing we can just use, so this, so M n now we can define. So, this matrix norm induces a metric in M n R. So, I have to define distance between two matrices and this is the definition A minus B. So, for all A, B in M n R and using this define, that R n with Euclidean matrix is a complete matrix phase. It is not difficult to show, that M n R with this metric is a complete matrix space, that is, let me stress, that every Cauchy sequence in M n R converges in M n R.

So, let me just explain this fact little more. So, let A k be a sequence in M n R, that means, each k equal to 1, etcetera. A k is a real n by n matrix. So, A k is called a Cauchy sequence. I am just writing a general definition in any matrix way. This is true if d of A k A l and this by definition, just look at here, this by definition is norm of A k minus A l tends to 0 as k l tends to infinity, that is a Cauchy sequence.

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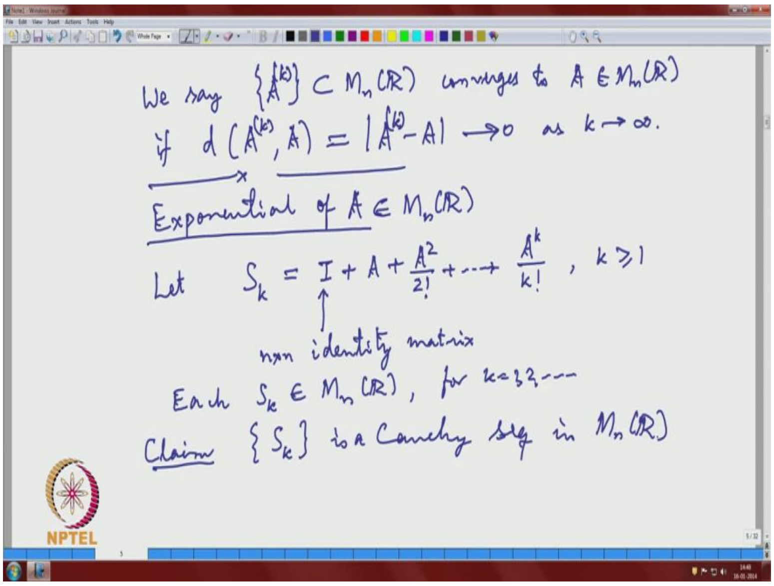

So, what is a Converge sequence, ok? We say, A k, again another sequence in M n R, converges to A in M n R. If distance A k, A, again let me say, that this is just norm of, matrix norm of A k minus A and that goes to 0, as k tends to… So, the complete ((Refer Time: 18:06)) says, that every Cauchy sequence in M n R converges to a matrix in M n R. So, this fact we used to define the exponential of a matrix. So, exponential of A, this is in M n R.

So, let, you define this thing, S k is equal to I plus A plus A square by 2 factorial, etcetera, sum A k by k. So, k bigger than equal to 1, ok. So, I is the identity, n by n identity matrix, so which will have all 1s in the diagonal and 0 elsewhere. So, this is a finite sum. So, each S k is a, each S k, each S k belongs to M n R for k equal to 1, 2, etcetera. We claim that this sequence x, this sequence S k is a Cauchy sequence in M n R, ok.

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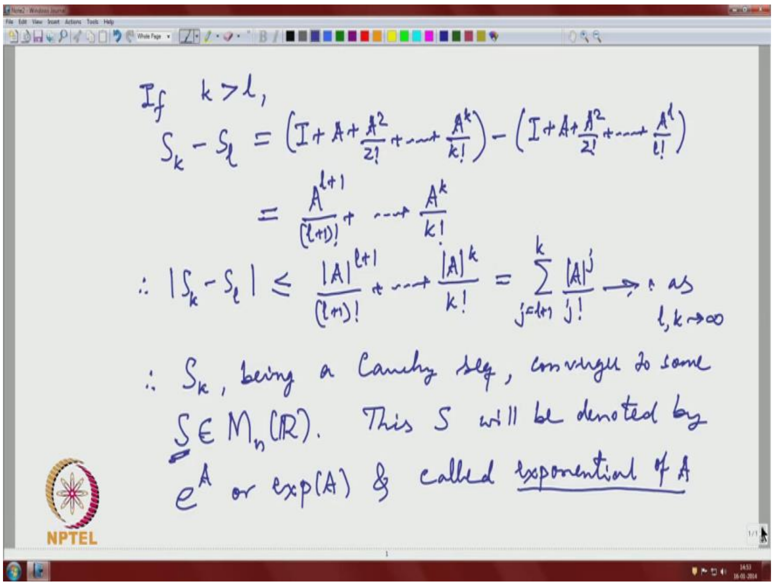

So, if k is bigger than l, S k minus S l. So, let us compute this. So, this is I plus A plus A square by 2 factorial plus A k by k factorial minus I plus A plus A square by 2 factorial A l by l factorial. So, up to l they get canceled. So, we have just A l plus 1. No, this is just power, A l plus 1 by l plus 1 factorial plus etcetera, A k by k factorial.

So, therefore, S k minus S l. So, use properties 1 to 4. So, you get repeatedly, but this is just detail of the usual numerical exponential function. So, this is just, let me write this A j matrix norm j equal to l plus 1 to k and this certainly goes to 0 as l k tends to infinity.

So, therefore, S k j, so therefore S k being a Cauchy sequence converges to some and that is unique S in M n R, ok. And this S would be denoted by this limit, this limit S. S will be denoted by e to the A or exponential A and called exponential of A. So, that is the definition, is certainly no problem, but the computation. So, the computation may be difficult. So, there is no, absolutely no problem with the definition of the exponential of a matrix and our, one of our main aim is how to simplify the computation of the exponential of a matrix and also matrix norm.

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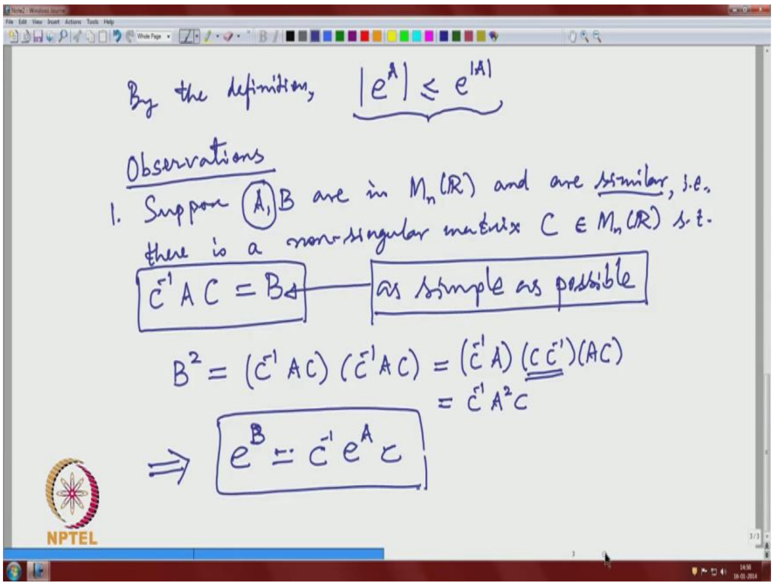

So, it is very easy to know, see, so by the definition, by the definition it is not very difficult. So, e to the A is also a matrix now. So, this is just less than or equal to E to the mod A, but this is quite true and it did not bring in any special property of the matrix A. So, that is what we want to improve upon and obtain a better estimate on this matrix norm of exponential of A. So, that is what we are going to do.

So, before we proceed further, so let us make some observations, ok, observations. Suppose, so this is first one, A B are in M n R and are similar, that is, there is a non-singular matrix, matrix C in M n R. So, everything is in M n R. So, we are not going outside the real field. So, we are always in the real field such that C inverse A C is equal to B. So, we are repeatedly using this thing. So, just remember that. So, A and B n by n matrices are called similar.

If there is a non-singular matrix C in M n R such that C inverse A B is C and let us see how their exponentials are related. So, you observe that. So, B square. Let us compute B square. So, this is C inverse A C. So, in general, the matrix multiplication is not commutative, so you have to be bit careful, but is associative, that is one good thing about matrix multiplication, it is associative. So, you just write like that and now this is identity. So, what you get is just C inverse is A square C. So, this is true for B square. So, by induction it is true for any k. So, what you can see that e to the B is C inverse e to the A C. So, that means, if A and B are similar, exponential of A and exponential of B are also similar, that is what it says.

And our main aim is try to find, given a matrix A, given a matrix a try to find a non-singular matrix C such that C inverse is equal to B and this is as simple as possible, as simple as possible and this is a vague statement. So, let me make it, let me explain what simplicity we want. So, let us take one example, ok.

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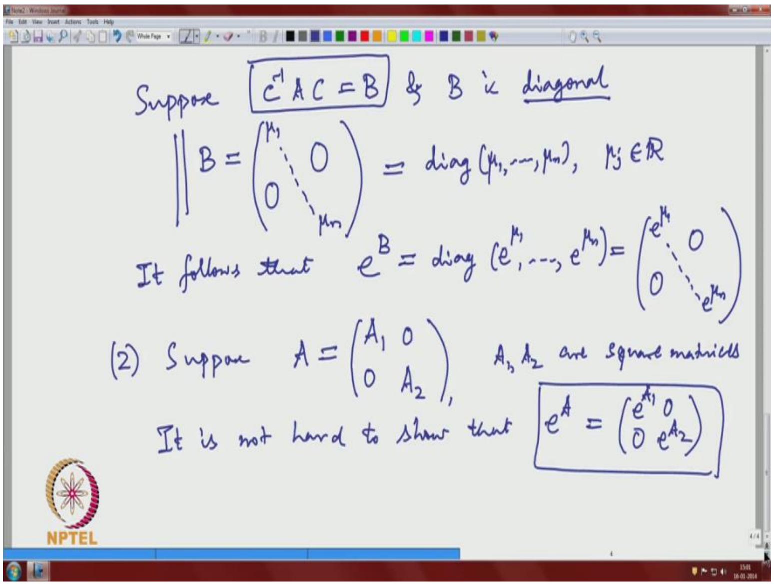

So, suppose C inverse A C is equal to B and B is diagonal, so what does that mean? So, we are again using this concept again and again. So, this means, B is some, let me write mu 1 mu n and off diagonals are all 0, ok. So, this is sometimes also written as diagonal mu 1 m n, all mu's are real numbers and let us try to compute the exponential of B. That is again very easy. So, you just, so it follows that, it follow, that e to the B is diagonal e to the mu 1 e to the… Very simple, because if you take powers of a diagonal matrix you just get powers of the diagonal elements. So, once you add, you get back your exponential. So, that is very easy, ok. Unfortunately, this will not be the case all the time. So, what best one can do, that is the next question.

So, second observation. Suppose, A can be written as a diagonal block matrix A 1 A 2. A 1 A 2 are square matrix and this we will see, will be the case in many instances and then just following this previous example it is not hard to show, hard to show, that e to the A is simply e to the A 1 and e to the A 2. That is also very nice and this easily extends to any number of block matrices, ok.

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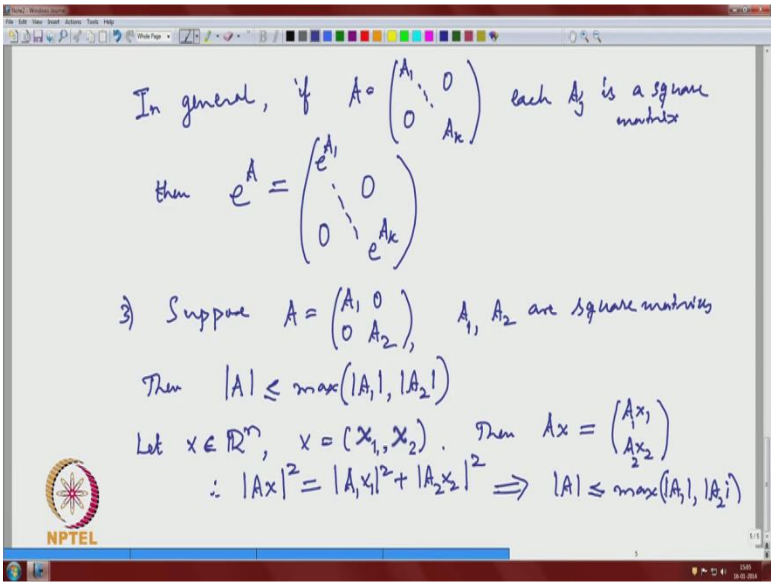

So, so in general, if A is equal to A 1 A 2 A k 0, each A j is a square matrix, then exponential of A is the block diagonal matrix e to the A 1, etcetera, e to the A k, fine, ok. And next thing, again I would like to see is, again is one of the observations is, so suppose, again let me start with two block matrices. So, this is, so A 1 A 2 are square matrices.

Then, now I come to the norm. So, this is less than or equal to maximum of A 1 A 2. So, this is straight forward ((Refer Time: 33:52)) and this will be. So, let me just sketch a proof of this thing, this is not very difficult. So, you let x belongs to R n and you write x is equal to x 1, x 2 and x 1, x 2 corresponds to the order of A 1 and A 2. So, let me not write everything in detail. So, you can just see that.

So, then A x is simply this Ax 1 and Ax 2 A 1 x 1 A 2 x 2. So, therefore, if you work out A x is equal to A x square equal to, A x square equal to, A 1 x 1 square plus A 2 x 2 square and then you restrict to norm x equal to 1 and take the supremum and immediately you see, that this is no problem, maximum of A 1 A 2. So, this you use for the exponential in particular.

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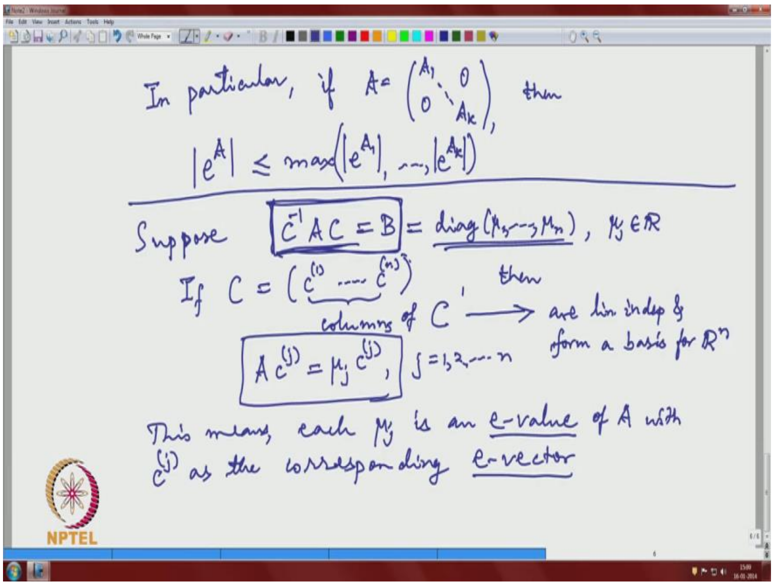

So, in particular, if A is equal to A 1 A 2 A k, then norm of exponential of A is less than or equal to maximum of e to the A 1, etcetera, e to the A 2 and this is what we are going to use in order to derive a final estimate for the exponential.

So, in the remaining 15 - 20 minutes, so let me just explain the plan we are going to do. Again, go back, so suppose C inverse A C is equal to B and this is, suppose, diagonal mu 1 mu 2 mu n, so mu j is in r E. In that case, let us try to see how these mu 1 mu 2 mu n are related to A. So, that is, let us see that thing. So, if you write C in the, as a column, so let me write them as C 1 C n. So, these are columns of C, the matrix C, columns of C.

Then, so you expand this thing, just to expand this thing you see, that A C j is equal to mu j C j. So, j equal to 1 2 n and this you have already seen what does this mean. This means that, this means each mu j is an Eigen value. Let me write E value of A with C j as the corresponding Eigen vector. So, Eigen value, Eigen vector ok.

So, if at all we are going to, if we wish to have such a thing and then, what you should have is this Eigen values of A, have Eigen vectors that generate the entire space R n. So, it is, since this is non-singular, the columns of C are linearly independent and form. So, since there number is n, form a basis for R n and we see immediately examples where this is not true.

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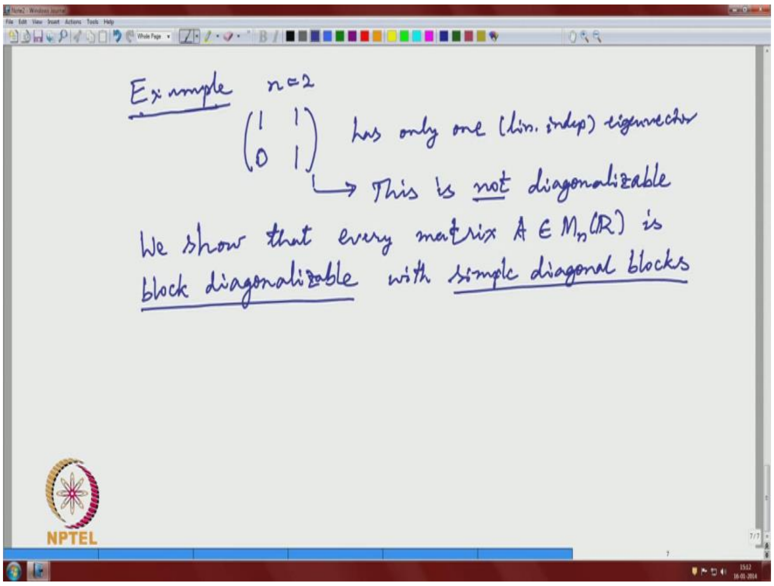

So, a simple example, n equal to 2, this matrix 1 1 0 1, ((Refer Time: 40:43)) matrix has only one linearly independent Eigen vectors. So, you can easily check that, ok. And since this is in R 2, we need two linearly independent Eigen vectors to span this space, but this is not possible. So, what we call is this matrix. This is not diagonalizable. So, this is, so this is the terminology we use. This is not diagonalizable and however, we show that, we show that, we show, that every matrix A in M n R is block diagonalizable.

So, we will explain what this means is, block diagonalizable with simple diagonal blocks. This is what our aim is and this leads to the so called Jordan canonical form. And we will explain now several steps that lead to this block diagonalization. So, let me recall that. So, ok…

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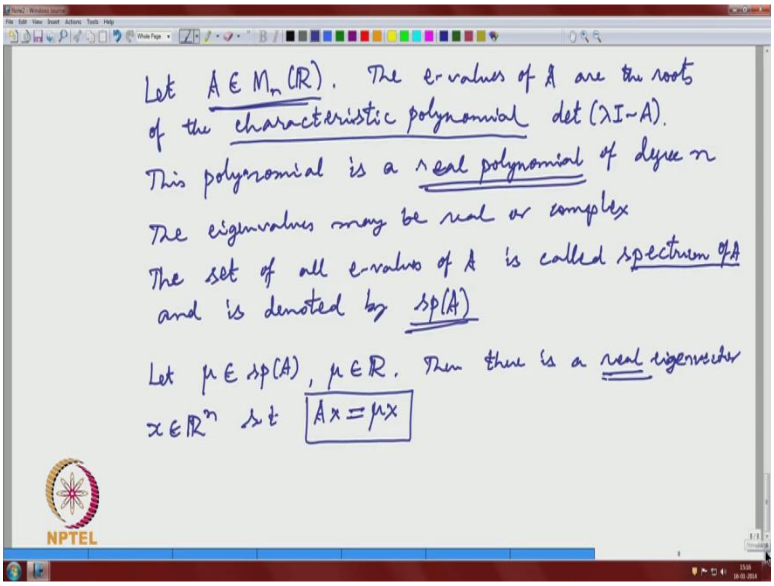

So, let again A belongs to M n R. So, the Eigen values of A are the roots of the characteristic polynomial, which is determinant of lambda I minus A. So, this polynomial is a real polynomial because we are taking A in M n R. So, real polynomial of degree n. Though this is a real polynomial, this, excuse me, see the Eigen values may be real or complex, ok. So, that is another problem we have to deal with.

So, the set of all Eigen values of A is called spectrum of A and is denoted by spectrum of A. So, just remember this notation, spectrum of A. Let us start with something and see what are the problems? So, let mu belongs to spectrum of A and mu real, ok. So, then, there is a real Eigen vector. I stress that real Eigen vector, call it x in R n such that A x equal to mu x. That is no problem, so what if mu is complex, ok.

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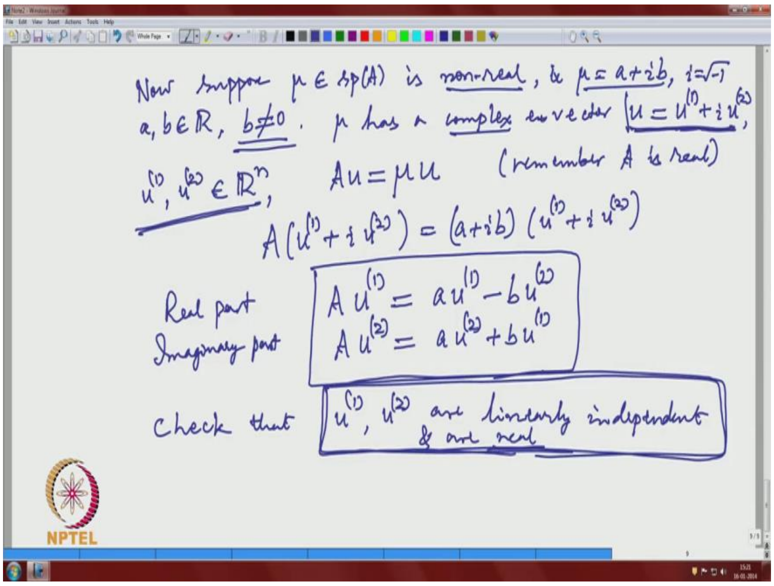

And now, suppose mu belongs to sp A, spectrum of A, is non real. So, that is, mu is equal to a plus i b, i is square root of minus 1 is a complex one. a, b are real and importantly, b is not 0. So, in this case also there is an Eigen vector. By definition, every Eigen value will have an Eigen vector, but in this case it will be a complex Eigen vector. So, mu has, mu has a complex Eigen vector.

So, let me call it again, just like here we separate the real and imaginary parts, let me write that. So, u 1 and u 2 are real vectors. And we have this A u is equal to mu u and remember A is real. So, remember this, this is important, A is real. So, now we separate the real and imaginary parts. So, here you have u is equal to u 1 plus i u 2 and u equal to a plus i d. So, let me write, that u 1 plus i u 2 is equal to a plus i b u 1 plus i u 2.

Now, you write separately the real part. So, real part. So, A u 1 on the left side and on the right side I have a u 1 plus rather minus b u 2. And imaginary part, A u 2 is equal to a u 2 plus b u 1. You see, u 1 and u 2, they are not Eigen vectors, ok, but they do satisfy this relations and you can check that, check that u 1 u 2. Now, there are two vectors, are real vector; they are real vectors, are linearly independent. So, this is what we want. This is important; this is important.

So, in case when mu is a real Eigen value, we automatically get a real Eigen vector, that is fine and in case of a complex Eigen value we get two linearly independent vectors.

They are not Eigen vectors, but related to the Eigen vector, but they are real and are real, that is important.

So, with this observation we will continue next time and our main aim is to construct a basis for R n using the Eigen vectors and some vectors like this and some more vectors that we will come up in the next class. So, that is our main aim and then, we will see how to utilize that basis in order estimate the matrix norm.

Thank you.